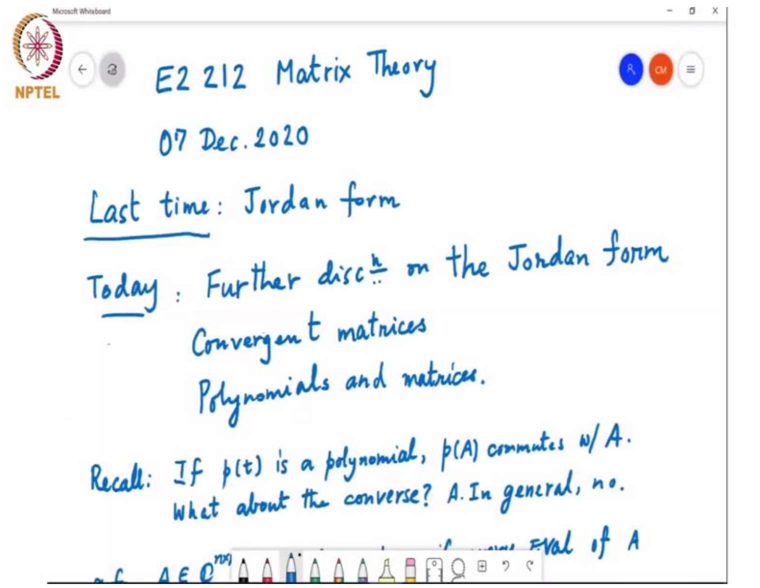
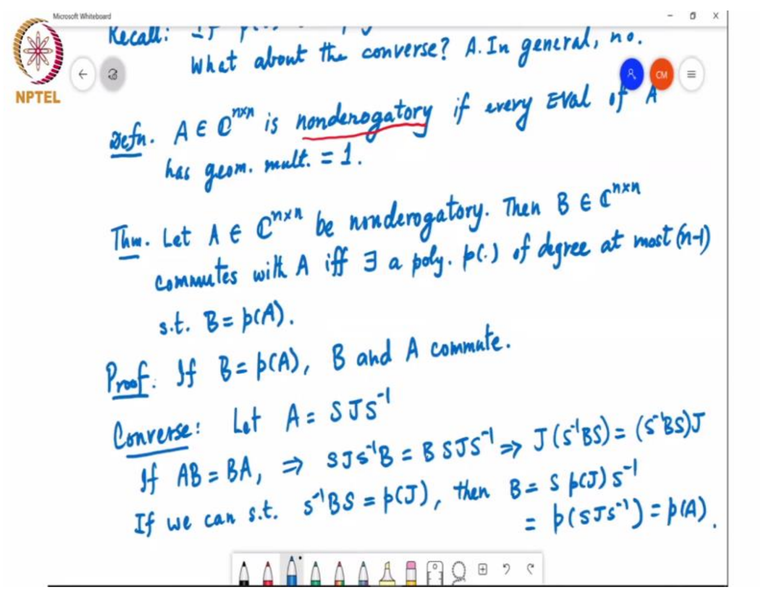
**Matrix Theory**
**Professor Chandra R. Murthy**
**Department of Electrical Communication Engineering**
**Indian Institute of Science, Bangalore**
**Lecture 48**
**Properties of the Jordan Canonical Form (part 1)**

(Refer Slide Time: 00:13)




So, the last time we were looking at the Jordan form, and we discussed some properties of the Jordan form, for example, that it can be used to show that any matrix is similar to its own transpose. So today, we will discuss just a few more properties of the Jordan form, and then link it up to Convergence of matrices, and also start some discussion on Polynomials and matrices. I briefly mentioned things like the minimal polynomial of a matrix and so on, we will discuss that some more today.

So, at the end of the previous class, we were discussing the following point, that if p of t is any polynomial, then if I compute p of A, I get a matrix that commutes with A, that is a trivial but useful fact. Now, what about the converse, that is, if I am given a matrix B that commutes with A can I write B as some polynomial function of A and the answer we saw that is that in general, it is no, we took an example of the identity matrix and showed that it will generally not be possible.

So, but we can give a more refined answer to the question by considering the following definition. So, a matrix A is called non derogatory if every eigenvalue of the matrix A has a geometric multiplicity equal to 1, what that means is that each distinct eigenvalue has only one Jordan block involving it, so, with this definition, we have the following result. Let A in C to the n cross n be non-derogatory.

Then matrix B and C to the n cross n commutes with A if and only if, there exists a polynomial p of degree at most n minus 1 such that p equals p of A. So, one way of the proof is very simple. If there exists a polynomial p such that B equals p of A then it is clear that B and A will commute that we saw already, that is just a consequence of this thing here that if p of t is a polynomial that p of A commutes with A.

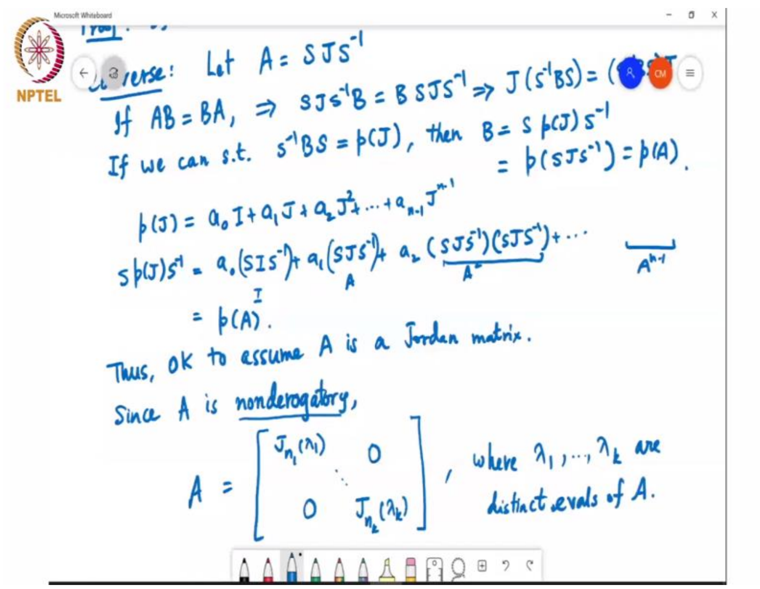
So, what we need to prove is really just the converse that if B commutes with a then there must exist a polynomial p of degree at most n minus 1 such that B equals p of A. So, let us do that. So, just for the sake of completeness, say that if B equals p of A, then B commutes with A so, we need to show the converse. So, again start with the Jordan canonical form. So, let A equals SJS inverse, where J is the Jordan canonical form of this matrix A.

Then what we need to show is that if B commutes with A, then there exists a polynomial of p degree at most n minus 1 less than B equals p of A. Now, so, basically the starting point is that AB equals BA then this implies I just substitute for A so SJS inverse B equals B SJS inverse which in turn implies S is an invertible matrix.

So, I can multiply by S inverse and then I can on the left and I can multiply by S on the right and what I will get is J times S inverse BS is equal to S inverse BS times J where I am just putting brackets to show that S inverse BS and J now need if A and B commute then S inverse BS commutes with J.

So, basically if we can show that S inverse BS equals p of J then. So, to write that So, if we can show that S inverse BS this is some other matrix and if we can show that this is equal to p of J then if I see so, then B will be equal to S p of J times S inverse and this if you consider the polynomial expansion, you can see that this S and S inverse can be pulled inside this polynomial function. And so, we can write this as p of SJS inverse which is equal to p of A. So, whatever polynomial we find, which connects S inverse BS to J is the same polynomial that will kind of connect B to A.

(Refer Slide Time: 07:35)




So, basically the problem then reduces to assuming that now, assuming that this matrix A that we had considered earlier is actually this Jordan matrix, and the matrix B we had considered was this S inverse BS matrix here.

Student: Sir, could you repeat how we went from B equal to S pJ S inverse to down?

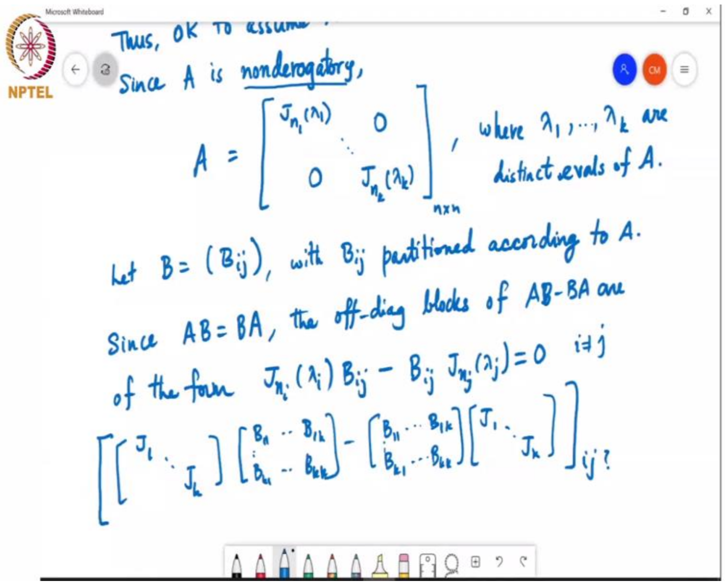
Professor Chandra R. Murthy: So, see pJ in general and so you do not have to pay attention, I mean, I am just giving you an illustration here. So, I can write it as some a0 times say the identity plus say a1 times J plus a2 J square, plus etc up to we have said that it is a degree at most n minus 1. So, I will write it as a n minus 1 J n minus 1 something like this. So, if I did if I do S p of J S inverse, that is going to be equal to a naught times SIS inverse plus a1 SJS inverse plus a2 SJS inverse SJS inverse plus etc right.

And so, basically, you see that this is actually equal to so, this is this matrix so, that whatever is this a so, if I call this some, if I call SJS inverse as some matrix A, then what I have here is this is just the identity matrix. This is the matrix A this is the matrix A square these two together. And similarly here I will get a to the n minus 1. So, this is nothing but p of A. So, that is all I am trying to say there.

So, what we have shown is that basically so one way to say it is that it is to assume A is Jordan matrix and proceed, because we know that if we can, if we can show that a Jordan, Jordan matrix commutes with B, then I can write B as a polynomial of the Jordan matrix, then I know how to write if a were not a Jordan matrix, I know how to write B as a polynomial of a the same polynomial that will work.

So, now that what we need to show is that if BJ equals JB, then B can be written as a polynomial of this J. So, now we use the fact that A is non derogatory, which is what we assumed in the statement of the theorem. So, since A is non derogatory, it is Jordan form, A is already in Jordan form, we can write A as block diagonal matrix with say J n1 of lambda 1 J say nk of lambda k and 0s everywhere else, and these lambdas are distinct. So, basically what we mean by non derogatory is that each lambda will occur in only one Jordan block the geometric multiplicity of every eigenvalue equals 1 where lambda 1 through lambda k are distinct, distinct just to be clear eigenvalues of A.

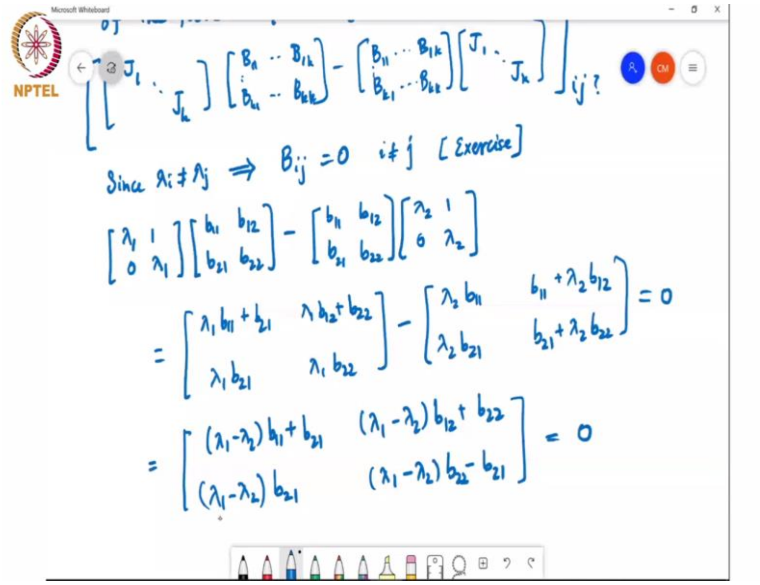
(Refer Slide Time: 12:40)




Now, so, this, this is a certain partition on an n cross n matrix the first one is of size n1 cross n1 the next diagonal block is of size n2 cross n2 the last diagonal block is of size nk cross nk. I will consider the same partition on B with Bij partitioned according to A, then basically so now, AB equals BA that is what we are given B commutes with this Jordan form matrix. And so, if I consider the off-diagonal blocks of AB minus BA AB minus BA equals 0.

So, the off-diagonal blocks of AB minus BA are of the form Jni of lambda i Bij minus Bij Jnj of lambda j. So just all you have to do is to consider this product. There is a matrix like this. So, I will just write this out here, but you have to adjust work out. So, I will just write it in short, I will write it as j1, jk, and I have B11 through B1k, Bk1 through Bkk and then I have to do minus B11, B1k, Bk1 through Bkk times J1 through Jk and then look at the ij th entry, ij th block matrix and that, that is what this thing reduces to. So, you can see that that is the case. So, this is the off-diagonal block, and this is also equal to 0 matrix because AB minus BA equals 0.

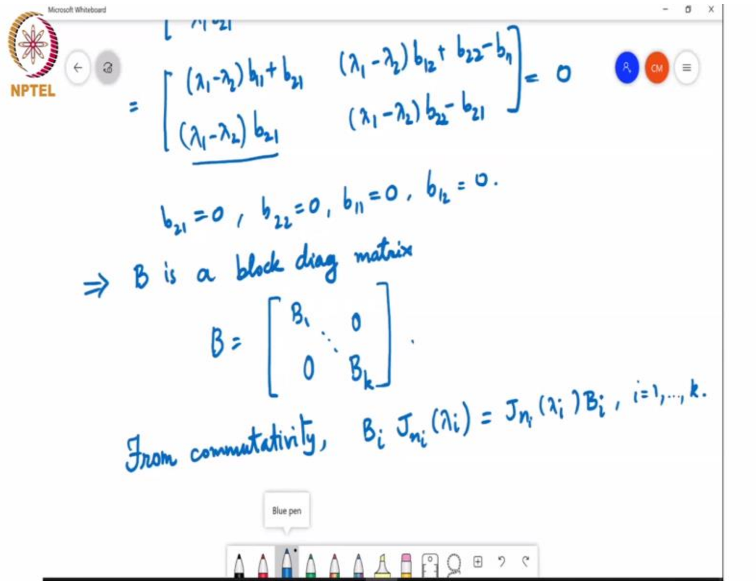
(Refer Slide Time: 16:35)



And since these lambdas are assumed to be distinct, it can be shown that this for the fact that this is equal to 0, this implies that Bij equal to 0 for i not equal to j. This is a little exercise. But I will maybe indicate to you how one arrives at this. So, for example, if I consider just 2 cross 2 block and for ease of notation, I will consider instead of lambda i and lambda j, I will consider lambda 1 and lambda 2 this is the first so lambda 1 here also the first Jordan block these times the corresponding matrix of B, which is b11 say b12, b21, b22 minus the this matrix here, which is again b11, b12, b21, b22 times the second Jordan block, which is some something associated with lambda j, which is lambda 2 1 0 lambda 2.

Lambda 1 equal is different from lambda 2. So, if I expand this out, what I will get is lambda 1 b11 plus b21 lambda 1 b12 plus b22 lambda 1 b21 and lambda 1 b22 minus lambda 2 b11 and b11 plus lambda 2 b12 and lambda 2 b21 and b21 plus lambda 2 b22. And this thing should be equal to 0. And this is equal to lambda 1 b11 minus lambda 2, I will write it this way lambda 1 minus lambda 2, b11 plus b21 and here it is lambda 1 minus lambda 2 b12 plus b22 and here it is lambda 1 minus lambda 2 b21 and here it is lambda 1 minus lambda 2 b22 plus b21 minus b21 equals 0.

(Refer Slide Time: 20:44)



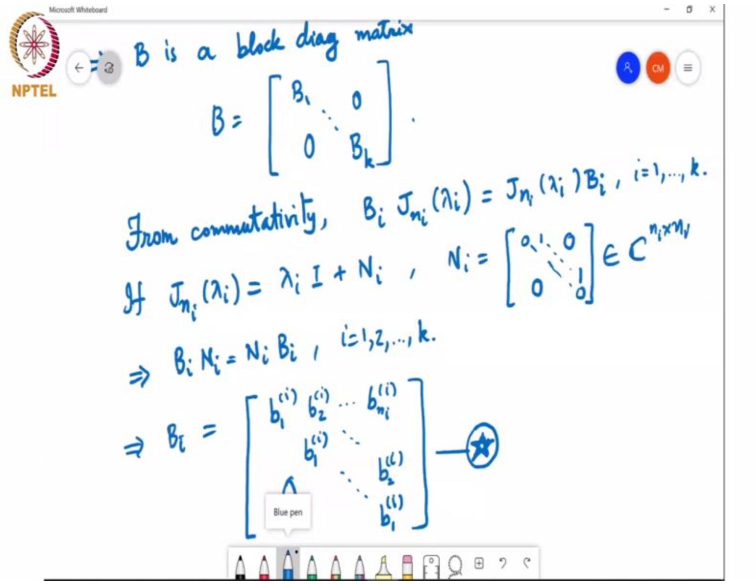

So, if I notice here lambda 1 minus lambda 2 b21 equals 0 and lambda 1 is not equal to lambda 2, so b21 is equal to 0. And then if I plug that in here, lambda 1 is not equal to lambda 2. So, b22 equals 0 and b22 is 0. So, in lambda 1 equals learn is not equal to lambda 2, so b12 equals 0. And finally, here b21 is 0 already, so in this is non 0, so b11 equals 0. So, that implies this matrix is the all 0 matrix I do not have to write that. So, all the entries of this matrix are 0.

So, by similar argument, but applied to slightly I mean, more general cases, when you have n1 and n2, you can show that this thing equal to 0 implies that all the entries of the matrix Bij is equal to 0. So, that in turn means that for i naught equal to j Bij is 0 that implies B is also a block diagonal matrix with the same block structure as J.

Student: Sir in subtraction, in first row in second column, there is an extra term minus b11?

Professor Chandra R. Murthy: There is a minus b11, but right, so then the root is. So, let me do that so that it is clear, but it does not change the conclusion. So, instead of to writing it this way, I will first figure out that b1, b21 equals 0 from here I will figure out b22 is 0. Then I will go here and figure out b11 equals 0 and now b22 b11are 0, so b12 is 0. So, you are right, but it does not change the conclusion. So, B is a block diagonal matrix. And I can write B as, like this and from the commutativity assumption, again, we have not used the i equal to j part of the commutativity. So, this is the commutativity assumption and this is true for i equal to 1 to k

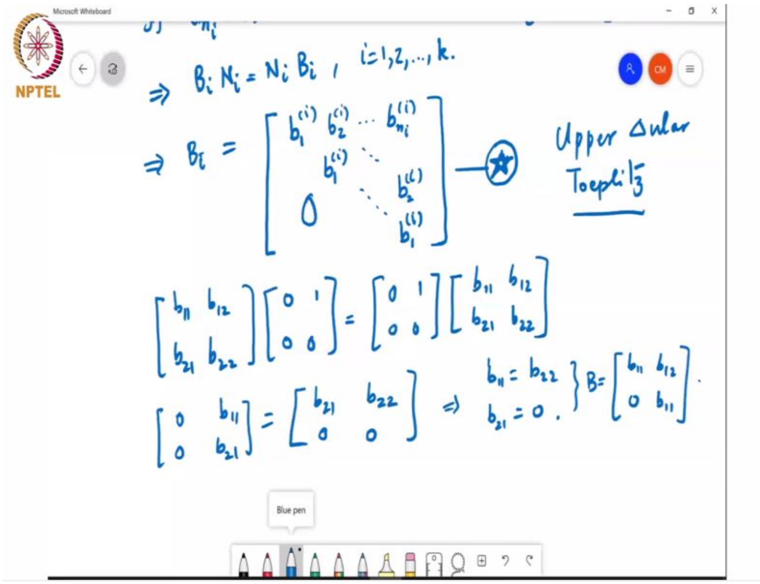
(Refer Slide Time: 24:31)




And now we will use the form of the Jordan blocks. So, if Jni of lambda i equals lambda i times the identity matrix plus Ni where Ni is the nilpotent matrix with 0s on the diagonal, once on the first super diagonal, then 0s everywhere else. So, this is a form of the Jordan block. And this is of size ni cross ni. Then so the identity matrix commutes with anything. So, if I say that Bi times j is equal to j times Bi, it means really that Bi is commuting with this Ni matrix.

So, Bi Ni equals an Ni Bi, i equal to 1 to k. Now, this in turn implies that Bi actually has a specific form, it is not just a non-zero block, but it is actually what is called a Toeplitz, Upper Triangular matrix. This is also something that you can show. This implies Bi is of the form and it has b2 of I here in the first super diagonal, all the way up to bni of i. So, size ni cross ni and 0s down here. So, I will call this form star for later use. So, it has bi’s b1i’s on the first on the diagonal b2 of i on the first super diagonal, b3 of i on the next super diagonal, and bni of i at the top right corner.

(Refer Slide Time: 27:39)




So again, just for illustration purposes, if I consider the 2 cross 2 case, look just so you see that I am not, I am not being unreasonable here. If I take b11 b12 b21 b22 and multiply it with these 2 crosses 2 Jordan block 0100 and this is supposed to be equal to 0100 times the same matrix b11 b12 b21 and b22. What this means is that if I execute this multiplication here, what I get is 0, b 11 0 and b21 this is equal to b21 b22 00 and so if I equate the terms, we see that b11 equals b21, sorry b11 equals b22.

So, the diagnostic terms are equal and b21 equals 0 and b21 equals 0. So basically, B is of the form say b11 and then this is also b11 b21 is 0 and this will be b12 this can be anything. So, bi has this kind of a Toeplitz. So, this is called Upper Triangular Toeplitz form called an Upper Triangular Toeplitz form so bi has this kind of a form.