Matrix Theory
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Lecture 47
Determining the Jordan Form of a Matrix

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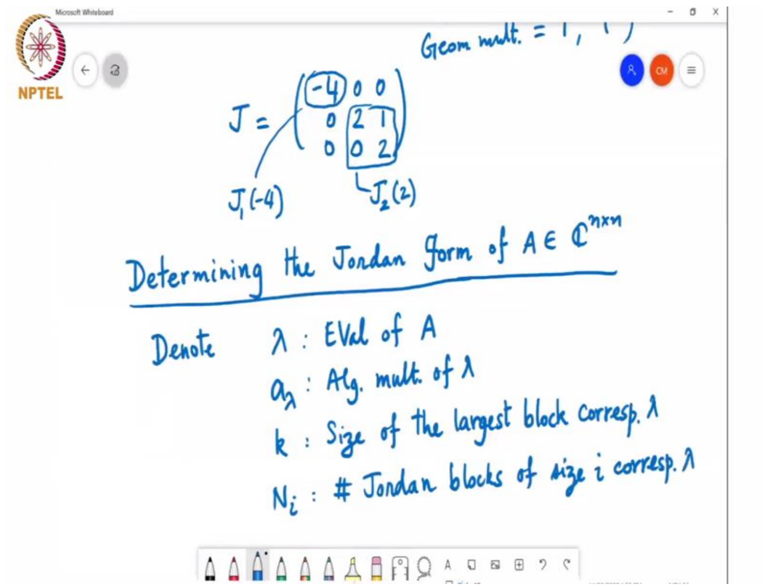

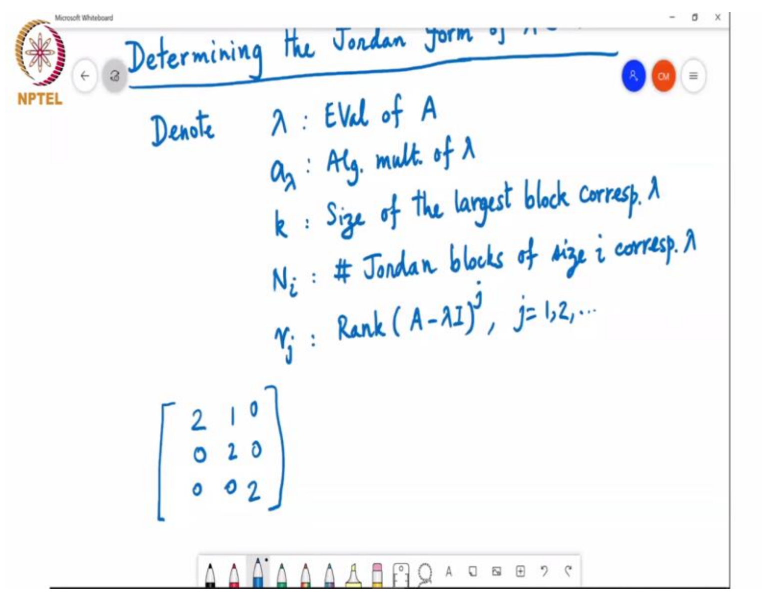

So, in need some notation so, lambda is an eigenvalue of A, A lambda is the algebraic multiplicity of lambda. k is the size of the largest block corresponding to lambda. Ni is the number of Jordan blocks corresponding to a block of size i corresponding to lambda. And rj is rank of A minus lambda i power j and so for j equal to 1, 2, etc. So, this is some notation and for the moment, just bear with me, I will outline the procedure and then you will see why we need all this notation. So, the following proposition,

Student: Sir can a particular eigenvalue have different sizes, Jordan blocks?

Professor Chandra R. Murthy: Yes... You could have multiple Jordan blocks associated with the same for the given eigenvalue. And it is not necessary that all the blocks associated with that eigenvalue should be of the same size. The easiest way to see things like this is to actually write out some Jordan matrices. It is already in Jordan form, you know that that is the, so that is the Jordan form of that matrix. So, it is already similar to a Jordan matrix.

So, for example, if I were to write 2120, this is a 2 cross 2 Jordan block associated with eigenvalue 2 and there could be one more block here and then I just fill in 0s everywhere else. So, this is a 3 cross 3 matrix which is already in the Jordan form, it has only 1 eigenvalue equal to 1 distinct eigenvalue and that eigenvalue equals 2 and corresponding to eigenvalue 2, there are 2 Jordan blocks.

The first Jordan blocks is of size block is a size 2 cross 2, the second Jordan block is of size 1 cross 1. And so, basically the algebraic multiplicity of the eigenvalue 2 is 3, it occurs 3 times as the root of the characteristic polynomial and the geometric multiplicity of the eigenvalue 2 is going to be 2, you can find 2 linearly independent eigenvectors corresponding to the eigenvalue 2. So, you could take this matrix and try to find a basis for the eigenspace of this eigenvalue equal to 2. And the nice thing about these Jordan blocks is that you can actually, if you just try it for a couple of matrices, you will realize that you can actually write it out quite easily.

Student: Hello, sir. Previously, you told that once we know the algebraic and geometric multiplicity, we can directly write the Jordan form.

Professor Chandra R. Murthy: Yes.

Student: But if, let us say the sizes are not necessarily same, so if the let us say algebraic multiplicity is 4, geometric multiplicity is 2?

Professor Chandra R. Murthy: No, so the thing is that there could be multiple blocks here. So, in the algebraic multiplicity is 4…

Student: Like in this case, 1, 1 block can be 3 cross 3 and other can be 1 cross 1 or other case,

Professor Chandra R. Murthy: Other case can have 2 blocks were, which are both equal to of size 2 cross 2. So, this procedure that I am going to tell you will help you figure out exactly which case it is. So, I agree with you that it is not sufficient to know the algebraic and geometric multiplicity of every eigenvalue, you also need to know the sizes of those blocks.

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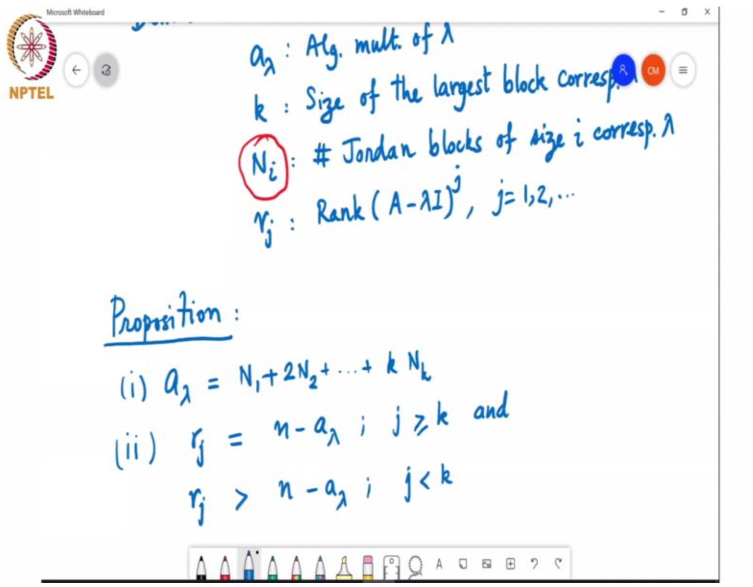

And so that is actually where this Ni will enter into the picture. So, you need to know all of these actually to find to write out the, the Jordan canonical form, and we will figure out we will we will outline a procedure to determine all these things.

Student: Okay.

Professor: So, here is a proposition which will actually tell us how to determine the Jordan canonical form. So, it has several parts to it. Point 1 is that a lambda equals N1 plus 2N2 plus etc plus k Nk. So, now I must point out that all these definitions are for a particular eigenvalue. So, I am fixing an eigenvalue lambda of a, for that eigenvalue a lambda denotes the algebraic multiplicity of that lambda, k is the size of the largest block corresponding to lambda, writing k lambda here, but just to keep the notation light, I am just calling it k, but k is going to be different for different eigenvalues of A.

Similarly, Ni is the number of Jordan blocks of size i, corresponding to lambda. So ideally, I should be writing Ni comma lambda or lambda comma i. But just to keep the notation light, I am just calling it a Ni, but keep in mind that it is associated with a particular eigenvalue. Similarly, rj is the rank of a minus lambda i power j, j equal to 1, 2 etc. And this also depends on the eigenvalue lambda that I am fixing here.

So, ideally, I should be rj lambda, but to keep the notation light again, I am omitting the lambda from this, so a lambda. So, basically, this is not difficult to see there is N1 blocks of size 1 corresponding to lambda, there are N2 blocks of size 2 corresponding to lambda, etc, up to there are there this is k is the size of the largest blocks there is, so their k, k times Nk is the number of blocks of size k.

So, if you take the sum of all these things, that must equal the algebraic multiplicity of lambda. The second point is rj is equal to n minus a lambda for j greater than or equal to k and rj is strictly greater than n minus a lambda for j less than k. What that means is that, if I start at j equals 1, and I look at rank of A minus lambda I, I get some number which is going to be strictly bigger than n minus a lambda.

And I take j equals 2, again, I will get a number which is strictly bigger than a n minus a lambda. But when I hit k, this rj will be equal to n minus a lambda. So, it will start with a number that is bigger than n minus a lambda. And it will keep decreasing as I take higher and higher powers here and at j equal to k, it will hit n minus a lambda, and then it will stay there. So, we will discuss this more later. But for now, just keep in mind that rj is a decreasing sequence that will start somewhere and keep decreasing down until it hits n minus a lambda and it will stay equal to n minus a lambda for all j bigger than or equal to k.

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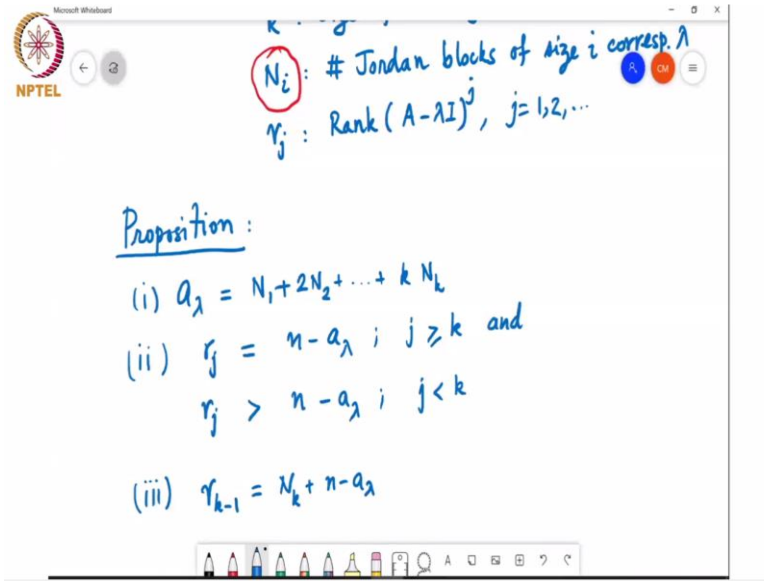

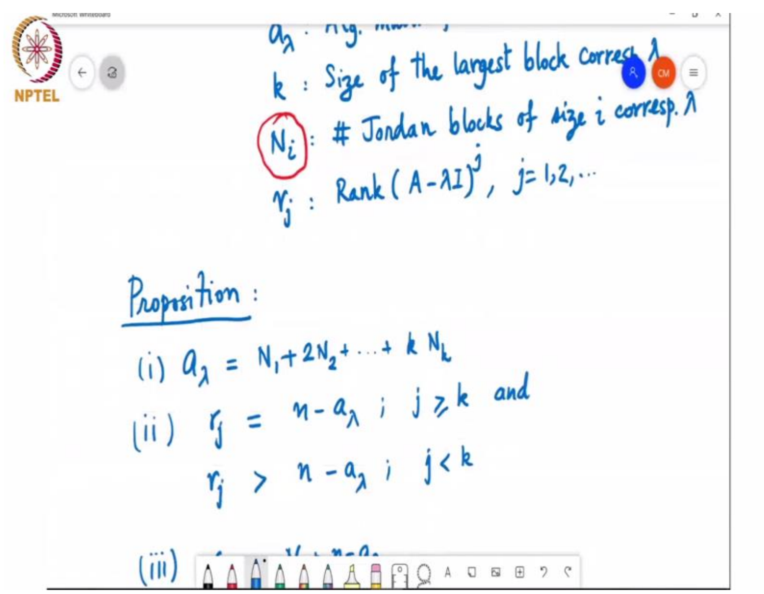

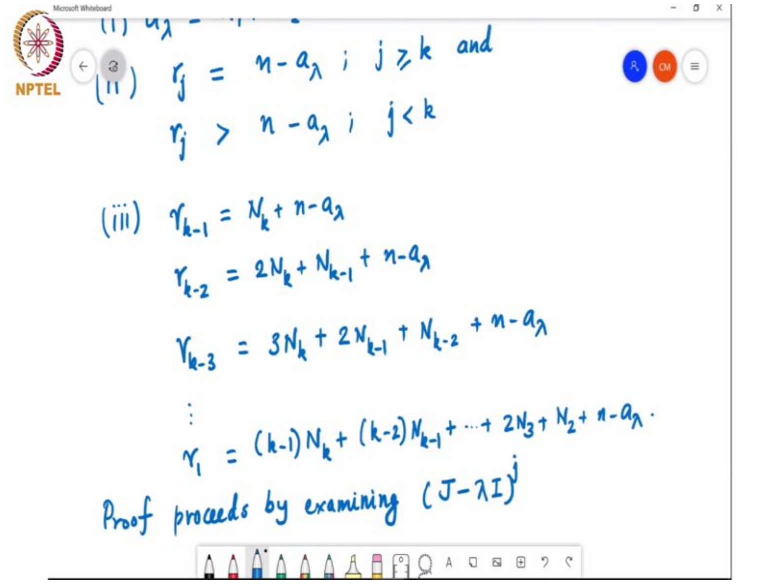

So, 3, you can actually say exactly what rj will be for j less than k and that is this third point here so rk minus 1. So, rk equals n minus a lambda, rk minus 1 will be equal to Nk plus n minus a lambda. So, Nk is the size of the is the number of Jordan blocks of size k and k is the large size of the largest block corresponding to lambda and so rk minus 1 will be equal Nk plus n minus a lambda.

So, it is strictly bigger than n minus a lambda it is bigger than n minus a lambda by exactly this value Nk, rk minus 2 is equal to 2Nk plus Nk minus 1 plus n minus a lambda. So, Nk is always at least equal to 1 because by definition, when I say k is the size of the largest block, I mean that there must be at least 1 block corresponding to of size k, so, Nk is at least equal to 1.

Now, n k minus 1 need not be equal to 1 it could even be equal to 0, but here I have a 2Nk plus n minus a lambda. So, rk minus 2 is strictly bigger than rk minus 1 and so on. And write one more to show you the pattern rk minus 3 is equal to 3Nk plus 2Nk minus 1 plus Nk minus 2 plus n minus a lambda and so, on down to r1 is equal to k minus 1 Nk plus k minus 2 Nk minus 1 plus etc plus 2N3 plus N2 plus n minus a lambda.

So, the way this the proof of this proposition goes it is a bit detailed I may do that in the next class, but the way it goes is so, the proof proceeds by looking at. So, you look at powers of J minus lambda i power j now, when I do J minus lambda i, j has the all the eigenvalues of the matrix A along its diagonal. So, when I do j minus lambda i it will kill the diagonal components where this particular eigenvalue appears and all others you will get some nonzero value along the diagonal. And wherever the you have killed the eigenvalue those wherever the diagonal entry appears as 0 those are nilpotent Jordon blocks. And when I start taking higher and higher powers, those blocks will start becoming equal to 0.

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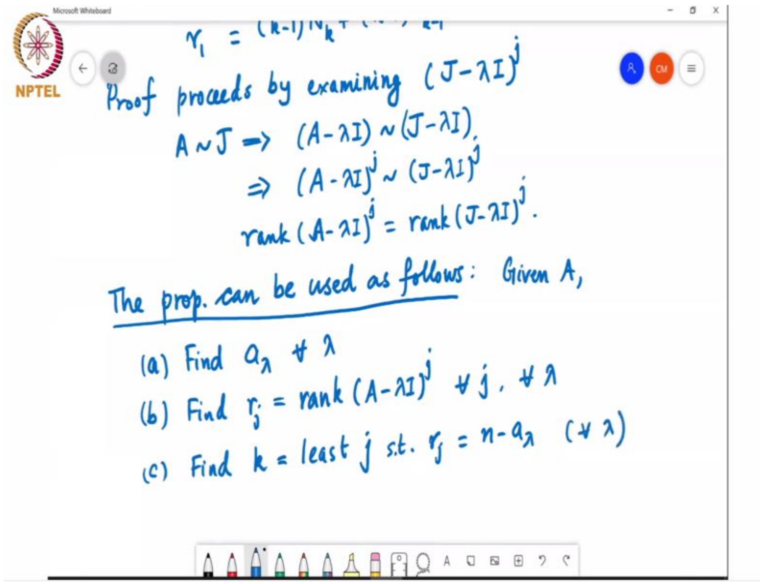

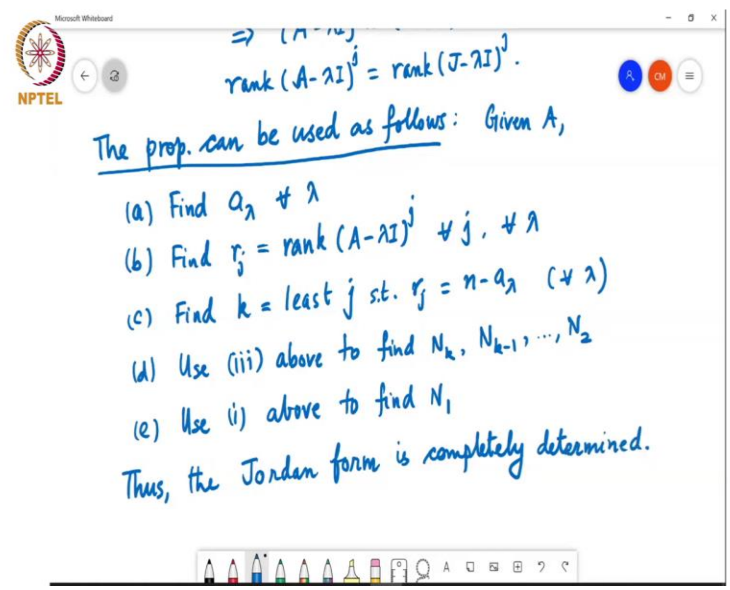

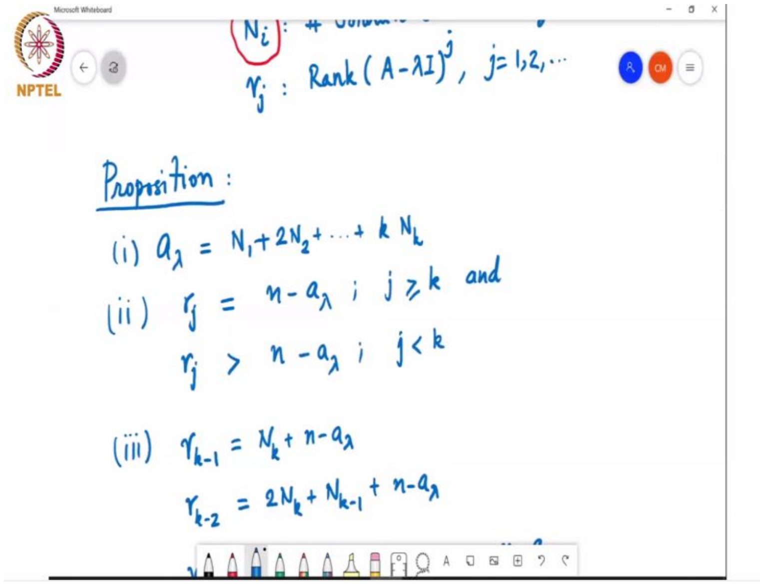

And so basically, we exploit the fact that if A is similar to J, then that means A minus lambda I is similar to J minus lambda I and so, which in, in turn implies that if I raise this to the power j A minus lambda I power j will be similar to J minus lambda I power j. And so, they ranks are, so, these are the essential ideas of the proof, but maybe next time I will walk you through the proof. But for now, I want to say how I want to tell you how this proposition can be used to determine the Jordan canonical form.

So, basically given A, what we do is, the first step is to find a lambda, this is the algebraic multiplicity of every eigenvalue associated with the matrix A. So, you need to solve the characteristic polynomial and then find a lambda. Find rj equal to rank of A minus lambda I power j for every j and for every lambda. So, again, the thing is, this might seem like a lot of work, because you have to go over every j.

But keep in mind that there is some number k, beyond which this rank will stop, it will become n minus a lambda and we will stop there. It would not change after that. So, you just need to keep going till you see that the rank has become equal and it has stopped, stopped decreasing. So, once you do that, it allows you to find k, which is the least j such that rj equals n minus a lambda. So, that is the maximum j to which you need to raise this power. Once rj equals n minus a lambda, any higher power that you raise here and find the rank, the rank will always be equal to n minus a lambda. This is also done for every lambda.

Then use point 3 in the proposition to find Nk, Nk minus 1 etc up to N2. So, if I can scroll up here. So, we know that rk equals n minus a lambda rk minus 1 is what we just determined by finding the rank of A minus lambda I power k minus 1 and that equals Nk plus a lambda So, we know rk minus 1, we know n minus a lambda we can find what Nk is. And then, once we know what Nk is, you can substitute that N here we know rk minus 2 k, we know n minus a lambda, we can determine Nk minus 1, and so on, all the way down to from this equation, we can determine what N2 is.

Then so then you can go back to the point number 1 says a lambda equals all this, I know what N2, N3 up to Nk is. And I know what a lambda is, so I can find out what N1 is. So, then I have determined the number of blocks of each of the sizes for every eigenvalue. So, then the Jordan form is completely determined.

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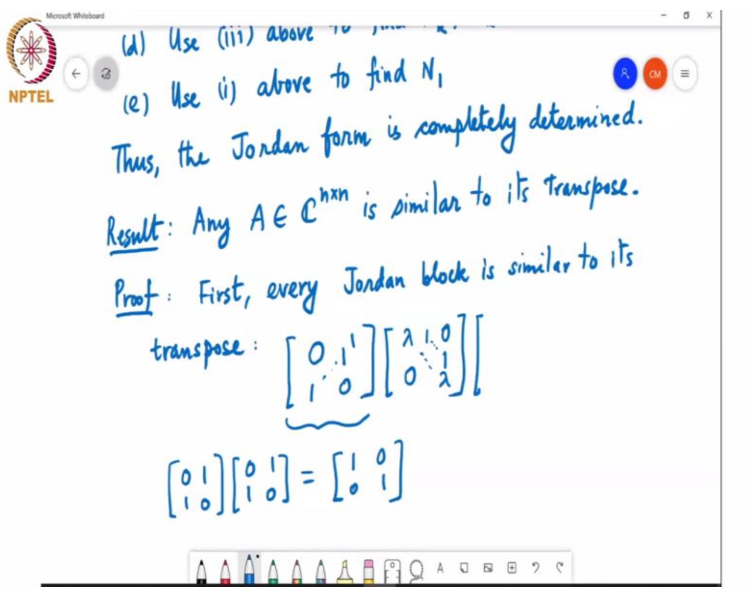

So, in the homework's, I will, so one of the uses of this Jordan canonical form is, as I mentioned long ago, is that you can show that any matrix is similar to its transpose. So, I will call it as result, any A is similar to its transpose. So, how do we use this Jordan form theorem to show this? So, first, we note that every Jordan block…

Student: Hello, sir (())(22:24) compacting the Jordan canonical form first we need to calculate (())(22:29) the value using (())(22:31)

Professor Chandra R. Murthy: Correct.

Student: Then we can use this proposition into

Professor Chandra R. Murthy: Then we can use?

Student: This proposition,

Professor Chandra R. Murthy: Exactly. The first step in the proposition is to first to find the roots of the characteristic polynomial and from that determine the algebraic multiplicity of every eigenvalue, then corresponding to each eigenvalue, you have to find these rjs you have to find k go to find Nk, all the way up to N2 and N1. And that is it. That is all you need to write out the Jordan canonical form.

Student: Okay sir.

Professor Chandra R. Murthy: So, to see this, basically, if I take this matrix, 0s with 1s along the anti diagonal, and then 0s everywhere else. Now, one interesting thing about this matrix is that what is the inverse of this matrix? So, this this matrix is actually its own inverse.

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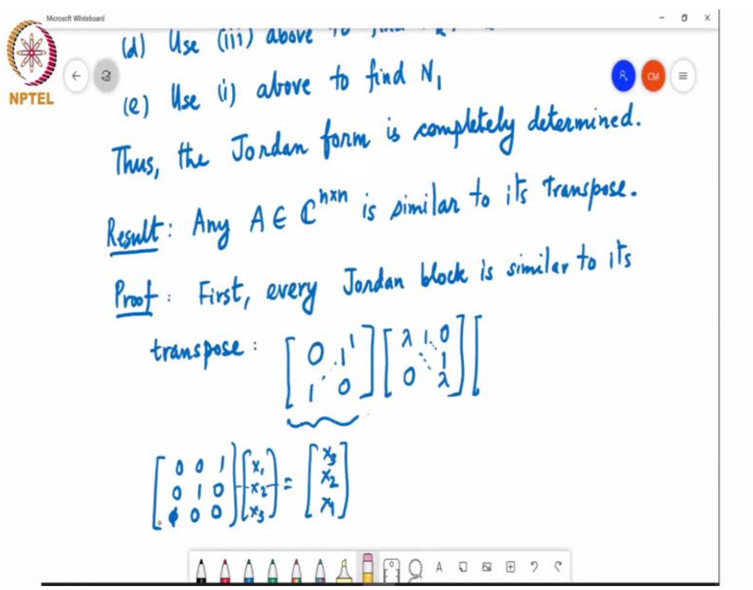

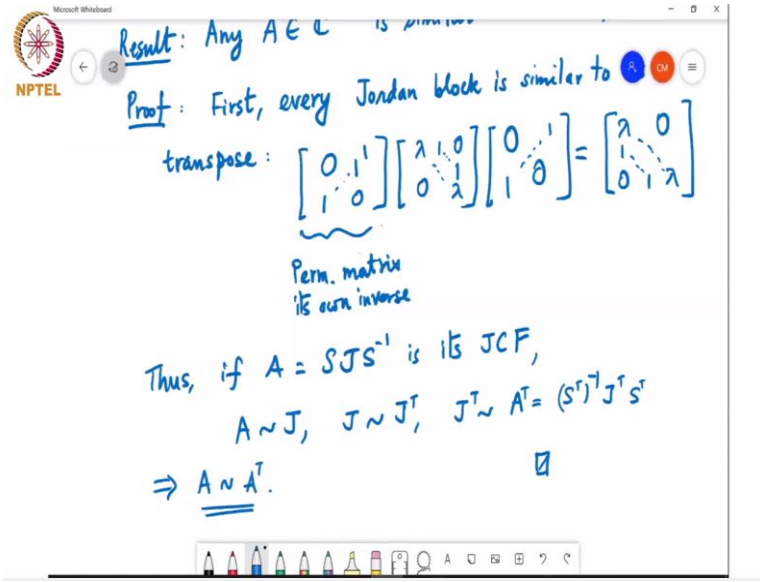

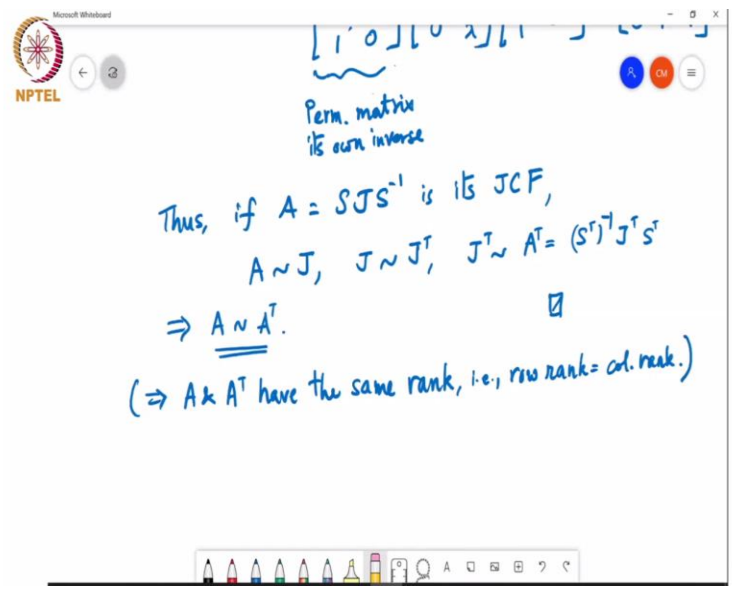

You can check that this same thing applies when, even if you take this matrix of order N or whatever order you like, so this is actually a permutation matrix. It basically flips all the entries of the matrix of a vector. So, if I take 0110 times x1 x2 I will get the vector x2 x1 or even better to make it a little more clear.

If I take the matrix 001 010 100 times x1 x2 x3 what I get is the vector x3 x2 x1, it flips the entries, considering some mirror point in between, if there is an even number of entries, then it will consider. So, if it like x1 x2 x3 x4, you will get x4 x3 x2 x1 like that, it will flip the entries of the thing. So, it is a permutation matrix, it permutes the entries of the vector.

And one property of permutation, this permutation matrix is that it is its own inverse. So, if I multiply this by a matrix, which has 1s along the anti diagonal, what I get is basically this transpose of this matrix. This is something that you can manually verify by multiplying these matrices together.

So, thus if A equals SJS inverse is its Jordan canonical form then basically, we have because it is in this form A is similar to J and J is similar to J transpose and J transpose is similar to A transpose which is equal to S transpose inverse times J transpose times S transpose. So, this is just taking the transpose of this and so J transpose is similar to A transpose.

So, that means that A is similar to A transpose. As a consequence, basically, any matrix is similar to its transpose. And like I mentioned, this is one of those results, which again, it is very difficult to intuitively explain why you should be able to find an invertible matrix such that S inverse AS will give you A transpose and this is possible for any matrix A.

So, A and A transpose have the same rank and that also is the implication of that is basically, similar matrices have the same rank. So, A and A transpose then have the same rank, which is also another way of seeing why the row rank of a matrix must be equal to the column rank of a matrix. So, one of the implications is that A and A transpose have the same rank. So, I mean,

Student: Sir also (())(29:11)

Professor Chandra R. Murthy: I could not hear you very well.

Student: For a matrices row rank equal to column rank was not also quite intuitive in that sense. So, in this case, the non-intuitive things are linked together that way?

Professor Chandra R. Murthy: So, at the time, we did not give a proof for why the row rank must be equal to the column rank. One thing I will point out is that if you go back and carefully look at our development till now, or at least the development of the Jordan canonical form, and the prerequisite needed to determine this Jordan canonical form. The point is that we have not used the fact that row rank equals column rank to come up with the Jordan canonical form.

And as a consequence, it is a valid thing to say that one corollary to this result that we just put down is that the row rank equals the column rank. So, this is one way to prove that the row rank equals the column rank if in our development so far, we had already used the fact that the row rank equals column rank to come up with this Jordan form theorem, then this would not be a proof of the Jordan form theorem, the proof of row rank equals column rank, because you cannot prove something by assuming it is true and then doing a whole bunch of steps and then coming back and showing that it is true. So, this, but, but that is where that that is not that is not the case here. And basically, rank eigenvalues these are all similarity invariant properties and so A and A transpose have the same rank.

Student: So, how to check like formally prove that A and A transpose will have same rank?

Professor Chandra R. Murthy: This is this is I mean, there are other ways to show it also, but this is one way is to say that A and A transpose are similar and because rank is a similarity invariant property, that is any 2 similar matrices have the same eigenvalues and the same rank. And therefore, if A and A transpose since A and A transpose are similar, they must have the same rank.

Student: No sir I am sorry, I mean to say how to show that 2 similar matrices have same rank?

Professor Chandra R. Murthy: So, 2 similar matrices, I mean, these are things we have already discussed, you should just go back and look at your notes, but similar matrices have the same eigenvalues and the number of nonzero eigenvalues is the rank of the matrix. And so, if they have the same eigenvalues, they must have the same set of 0 eigenvalues and the same set of nonzero eigenvalues. So, do similar matrices have the same rank? Now so, there is there is one other result I want to say, which requires another definition, I will maybe state that the result and then the next time we will show it, so, the point is like this.

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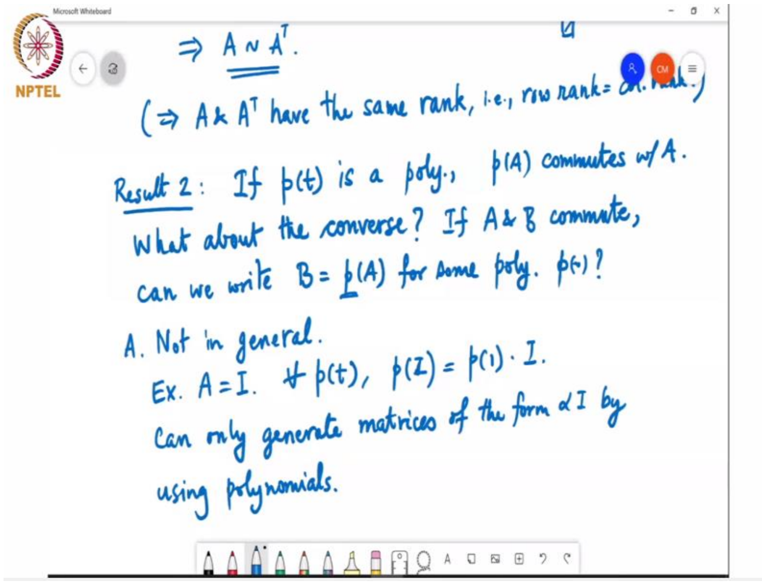

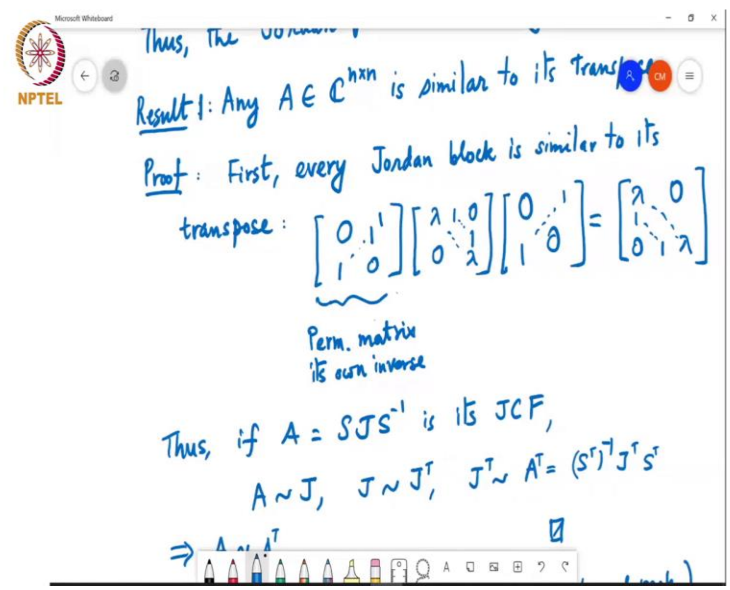

So, if if p of t is a polynomial maybe. Let me do the following. Just to keep it a little more organized. So, I will go up here and I will call this result 1. Now, here is the result 2. You there are some users of the Jordan canonical form. So, now, if p of t is a polynomial, then p of A commutes with A. This is an obvious but useful fact. What about the other way?

So basically, that means if A and B commute. So, can we write B equals p of A for some polynomial? So, that is we have seen that p of A commutes with A, so for any polynomial it is true and so can I write a matrix that commutes with A as a polynomial of A? That is the question? That is the converse of this statement here.

So, the answer is that is not always true not in general. And the Jordan canonical form allows us to answer when it will be possible to write B equals p of A. So, for example, just to show why it is not true, so if I take A equals the identity matrix, now the every matrix can commutes with the identity matrix. So, if I take any other matrix B, B times I is the same as I times B. But if I take any polynomial, then for every p of t, if I compute p of I this is going to be some value p of 1 times the identity matrix.

The polynomial evaluated at 1 times, so basically, it is going to give me a matrix that is proportional to the identity matrix. So, we can only generate matrices of the form alpha times the identity matrix by using polynomials. So, it is not so it is not always possible that you can find a matrix find a polynomial p such that B equals p of A some for some polynomial that the so that is clear, but now the question is when will it be possible to find a polynomial such that a matrix that commutes with A can be written as p of A?

So, we are out of time for today, and we will need to introduce one other definition of what is known as a non-derogatory matrix. And the matrix is non derogatory, every eigenvalue has a geometric multiplicity equal to 1 meaning that each distinct eigenvalue has only 1 Jordan block involving it. And under that condition, we will see what the result about finding a polynomial p such that B can be matrix B that commutes with A can be written as a polynomial of A in the next class. So that is it for today, and we will continue on Monday.