Matrix Theory
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Lecture 46
Jordan Canonical Form

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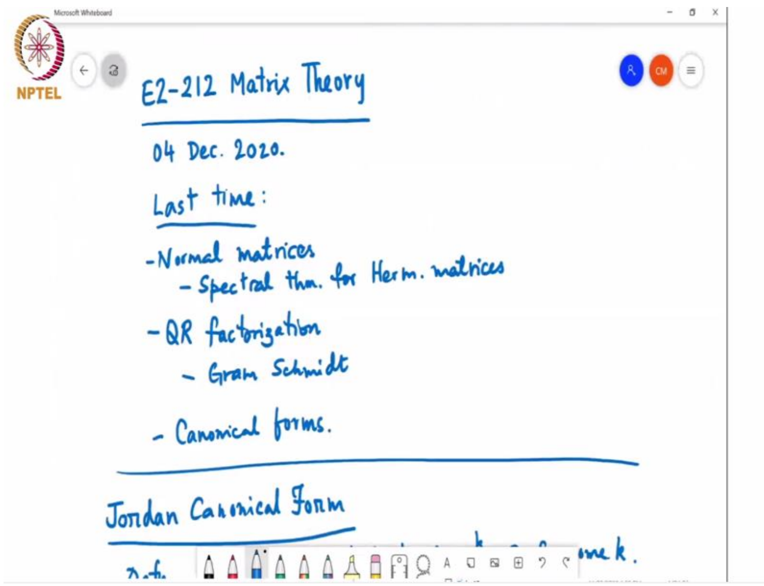

So Good Afternoon. We will begin. So, the last time we were looking at Normal Matrices and Spectral Theorem for Hermitian symmetric matrices. We also discussed the QR factorization, which is based on the Gram-Schmidt orthogonalization procedure, and we started discussing these Canonical forms.

Canonical forms are a way to reduce a matrix to a simpler form, which will allow you to compare matrices and see whether they have the same canonical form or not, which in turn allows you to conclude whether those matrices are going to be similar or not. And other there are many other uses, which we will discuss in the course of these lectures.

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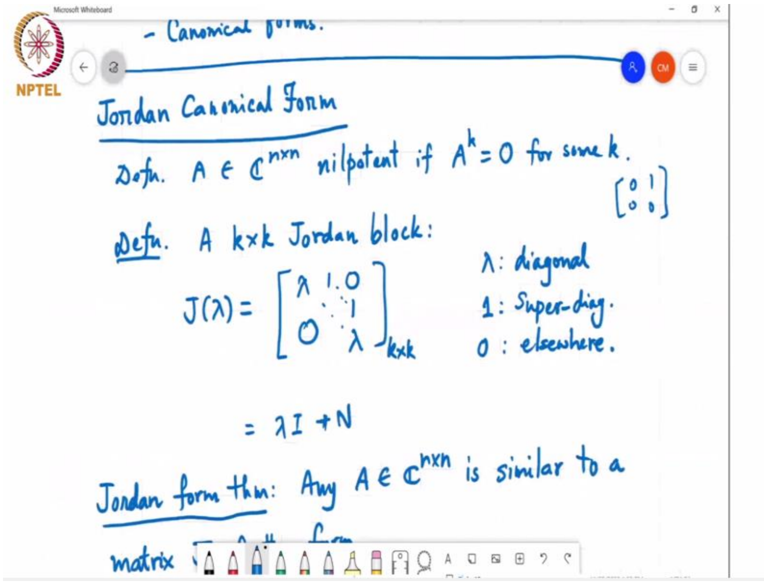

Now, the specific form that we will start with as we started discussing is what is called the Jordan Canonical Form. So, just to recall a matrix A is called is said to be nilpotent of order k or index k if A power k equals 0 for some, for some value of k and typically, the index is what we what we call the index is the smallest dimensional smallest power you need to raise it to so that you get the 0 matrix.

So, for example, if I take the 2 cross 2 matrix 0100 this is nilpotent of index 2, because when I take the square of this matrix, I get the all 0 matrix. So, basically the Jordan canonical form theorem will say that every matrix is similar to a matrix of the form D plus n where D is a diagonal matrix and n is a nilpotent matrix.

So, so, one other definition is that of a Jordan block, a Jordan block J of lambda is of size some k cross k, where basically you have that you have lambdas on the diagonal, once on the first super diagonal and 0s everywhere else in the matrix. This you can see that this is of the form lambda times the identity matrix of size k cross k plus a nilpotent matrix because it has only once on the first super diagonal.

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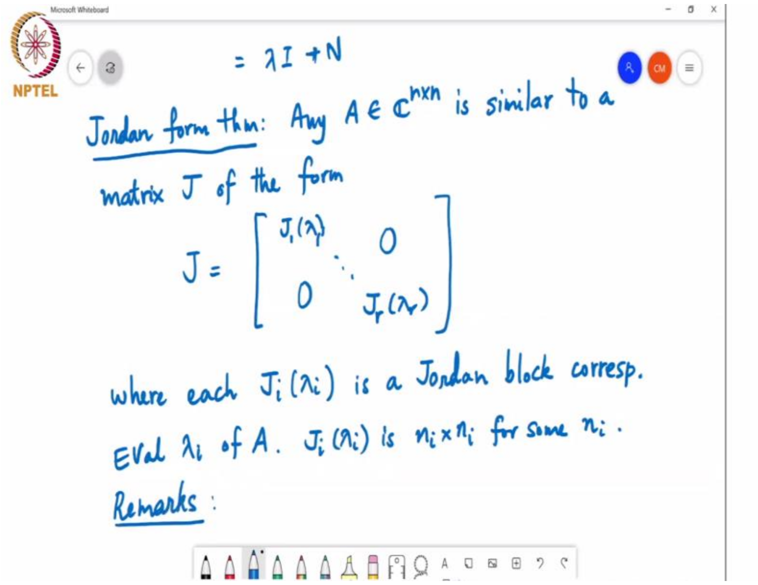

So, the Jordan form theorem basically says that any matrix A of size n cross n is similar to a matrix J of this form. So, it is a block diagonal matrix with Jordan blocks along the diagonal and there are such Jordan blocks and each Ji of lambda i this is the ith block is the Jordan block corresponding to eigenvalue lambda i of A. So, the same value of the same eigenvalue could be repeated in multiple blocks. This block is of size ni cross ni for some value of ni. So, for example, ni could even be equal to 1. So, just to clarify what this theorem is saying here are a few remarks.

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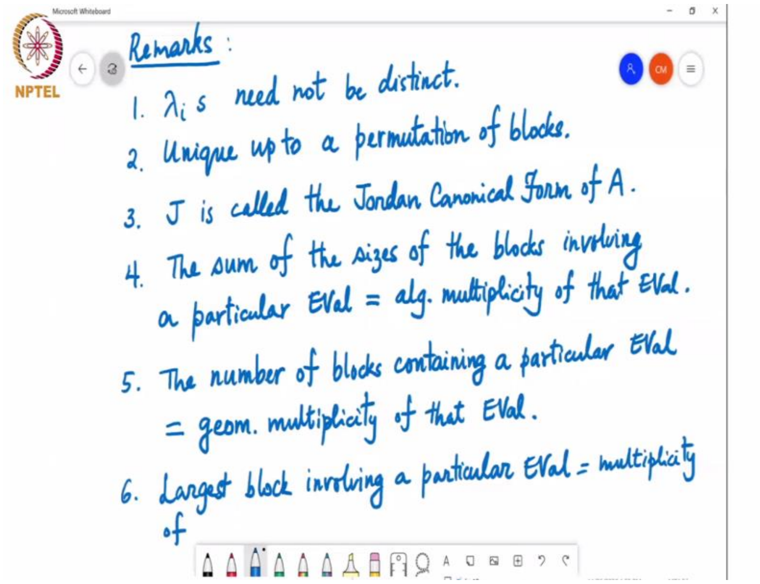

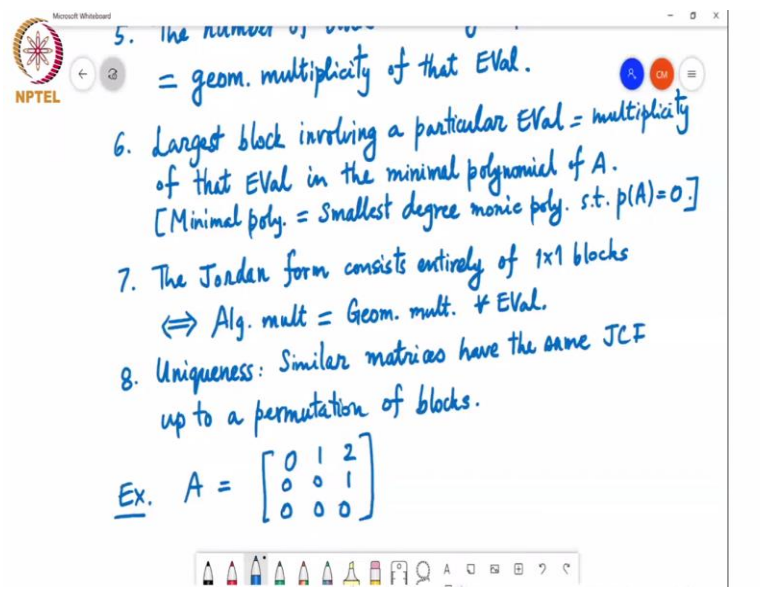

First, these lambda s need not be distinct. That is the same lambda i could occur in multiple blocks. The second is that, this decomposition into this canonical form is unique up to a permutation of blocks. Third point this matrix J is called the Jordan canonical form of A. So, the sum of the sizes of the blocks involving a particular eigenvalue is the algebraic multiplicity of that eigenvalue.

So, what is the algebraic multiplicity? It is the number of times that the particular eigenvalue occurs as root of the canonical. So, basically the Jordan canonical form will if you are able to compute it will reveal the algebraic multiplicity of all the eigenvalues of the matrix or it contains in it the information about what the algebraic multiplicity of every eigenvalue of the matrix is. The number of blocks involving a particular eigenvalue is the geometric multiplicity of that eigenvalue.

So, what is the geometric multiplicity? It is the dimension of the eigen space corresponding to the eigenvalue or in other words, the number of linearly independent vectors in the, in the linearly independent eigenvectors that you can find corresponding to the that eigenvalue. That is the geometric multiplicity and that equals the number of blocks containing a particular eigenvalue.

So, if the algebraic multiple multiplicity is equal to the geometric multiplicity for every eigenvalue, it means that all the Jordan blocks will be of size 1 cross 1. And for every Jordan block, you will have one linearly independent eigenvalue, eigenvector. So, this point is a little it is something good to know, but I would not elaborate on it because it requires me to explain about the concept of minimal polynomials, which I will come to a bit later. But for now, I will just make a note here, and we maybe we will do go, go back and look at this when we discuss minimal polynomials. So, the largest block involving a particular eigenvalue is the multiplicity of, that eigenvalue in the minimal polynomial of A.

So basically, this minimal polynomial of A is essentially so, for now, just for the sake of completeness, I will write this here. So, the minimal polynomial is the smallest degree monic polynomial such that p of A equals 0. So, we certainly know that the characteristic polynomial is a polynomial satisfying p of A equals 0. But it is possible that there is a lower degree polynomial also satisfying p of A equals 0. And that smallest degree polynomial that you can find such that p of A equals 0 is called the minimal polynomial of A.

And the largest block is basically the multiplicity of that eigenvalue in such a minimal polynomial of A. So, if the Jordan block content consists entirely of 1 cross 1 blocks, then basically there are no super diagonal elements in the Jordan canonical form of the matrix. And that implies that the algebraic multiplicity equals the geometric multiplicity for every eigenvalue. The Jordan form is actually a diagonal matrix.

You can imagine that for a, you can easily see that for a matrix that is diagonalizable the Jordan canonical form will come out to be a diagonal matrix. So, uniqueness, similar matrices will have the same Jordan form up to a permutation of blocks.

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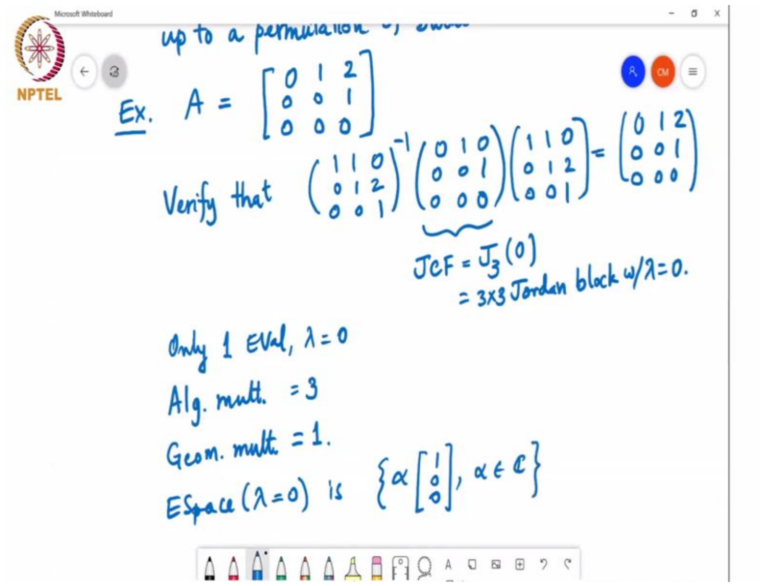

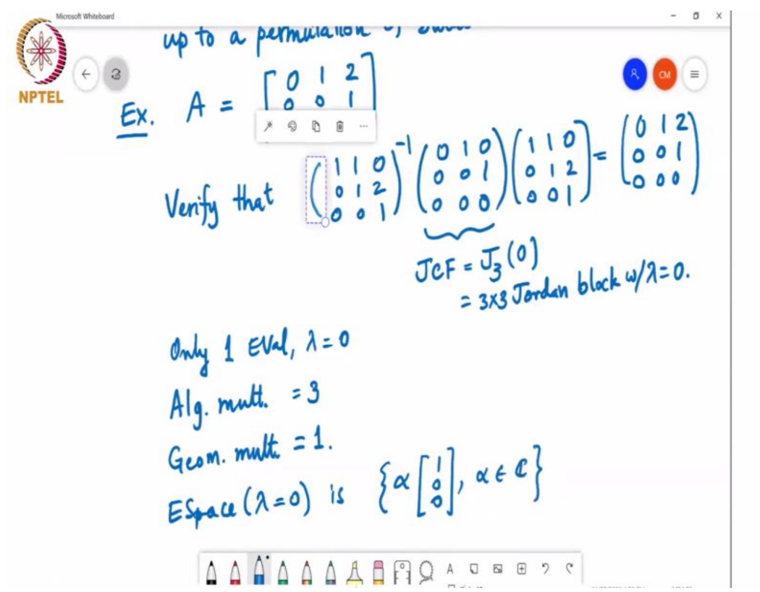

So, let us maybe look at one or two examples and see how this looks like. So, if I take for example, A equals the matrix 012 001 000. Then one can check that if I do 110 012 001 inverse times 010 001 000 times 110 012 001, if I carry out this multiplication, I will get this matrix 012 001 and 000. So basically, this matrix in between here, this is the Jordan canonical form, which is actually equal to J3. It is a 3 cross 3 matrix here. So, again, notational abuse here because I had written J1, J2 etc, where Ji was an ni cross ni block, but nonetheless and I will write it like this.

It is a 3 cross 3 block Jordan block with lambda equals 0. So, this is an upper triangular matrix, its eigenvalues are all equal to 0 that is obvious, but this is the simplest form you can reduce the matrix to which is a, which is the Jordan block of size 3 cross 3 associated with eigenvalue 0. So, basically this matrix has only 1 eigenvalue, 1 distinct eigenvalue lambda equals 0. So, this lambda equals 0 has a algebraic multiplicity 3 geometric multiplicity equal to 1.

The eigenvalue 0 occurs in only 1 block of size 3 cross 3. So, basically if I asked what is the eigenspace of lambda equal to 0 the set of vectors alpha 100 alpha and C. So, you take any vector like this and multiply it with this matrix I will get 0 times this vector 100. So, it has only 1 linearly independent eigenvector corresponding to lambda equals 0. So, it is a defective matrix.

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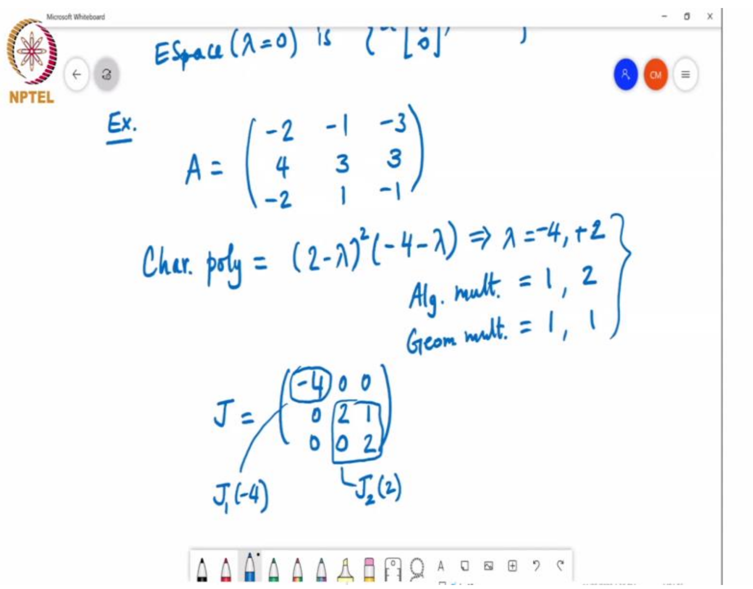

So, here is another example. Now we take a slightly more elaborate example. So, minus 2 minus 1 minus 3, 433 minus 2, 1 minus 1. If I asked what is the characteristic polynomial of this matrix? You can work it out it simplifies to 2 minus lambda the whole square times minus 4 minus lambda. And so, that implies that the eigenvalues are 4 and minus 2, minus 4 and 2 and these have algebraic multiplicity.

So, corresponding to minus 4 the algebraic multiplicity is 1 corresponding to the algebraic multiplicity is 2. So, 2 occurs twice as the solution to the characteristic polynomial. And the geometric multiplicity is at most the algebraic multiplicity and in this case, it turns out to be 1 and 1 is always equal to at least 1, and it is at most equal to the algebraic multiplicity. So, these things you take on faith for now, we will actually see how to compute the Jordan canonical form next, and then you will be able to execute that for this matrix and see that all these are true.

So, the Jordan canonical form for this matrix is minus 4, so, there will be 2 blocks 1 block will correspond to the eigenvalue minus 4 and since the other 1 has a geometric multiplicity of 1 and an algebraic multiplicity of 2, the other block is a 2 cross 2 block with eigenvalue equal to 2. So, that will be 221 and 0 if 2 cross 2 Jordan block corresponding to the eigenvalue 2 and the rest of the elements will be 0.

So, if you know the algebraic and geometric multiplicities you can actually directly write out the Jordan canonical form. So, this thing here there is 1 Jordan block called the J1 of minus 4 and this is another Jordan block called J2 of 2. It is a 2 cross 2 Jordan block associated with eigenvalue 2. So, question is how do you find this, this much? If you can figure out this much then you can write out the Jordan canonical form. So, that is what we will discuss next.