Matrix Theory

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Lecture 44

Fundamental properties of normal matrices

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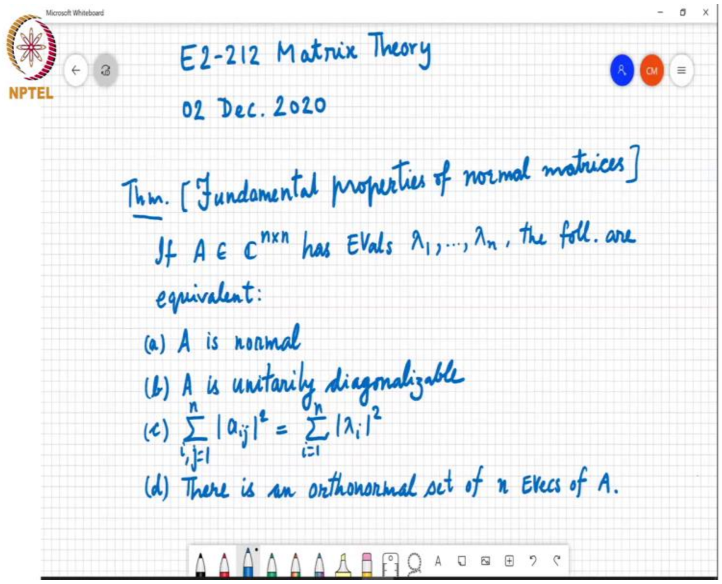

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Recall that matrix is normal if A A Hermitian equals A Hermitian A. It commutes with its Hermitian and we saw that it is a generalization of unitary, real symmetric and Hermitian matrices. And we saw a simple 2 cross 2 example of a matrix that is normal but not unitary, Hermitian or skew Hermitian. We also defined unitary diagonalizability. So, a matrix is said to be unitarily diagonalizable if it is unitarily equivalent to a diagonal matrix. So, the similarity transform that will take the matrix to a diagonal matrix is in fact a unitary matrix.

So, this is the theorem which gives us some fundamental properties of normal matrices. So, if the matrix A which is of size n cross n has eigenvalues lambda 1 to lambda n then these four properties are equivalent. First is that A is a normal matrix. Second is that A is unitarily diagonalizable. And the third is that the sum of the squares of all the entries of A which is also the Frobenius norm square of the matrix A is equal to the sum of the squares of its eigenvalues. And the fourth is that there is an orthonormal set of n eigenvectors of A that is n has a full set of eigenvectors which are orthonormal to each other.

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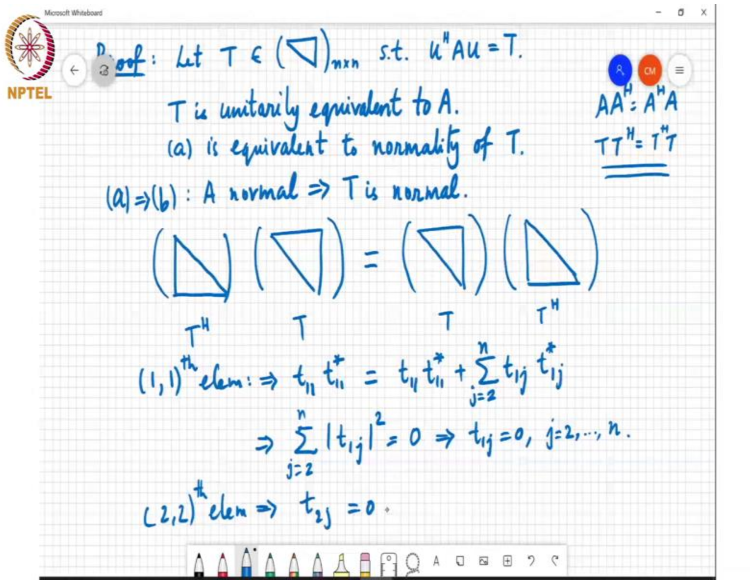

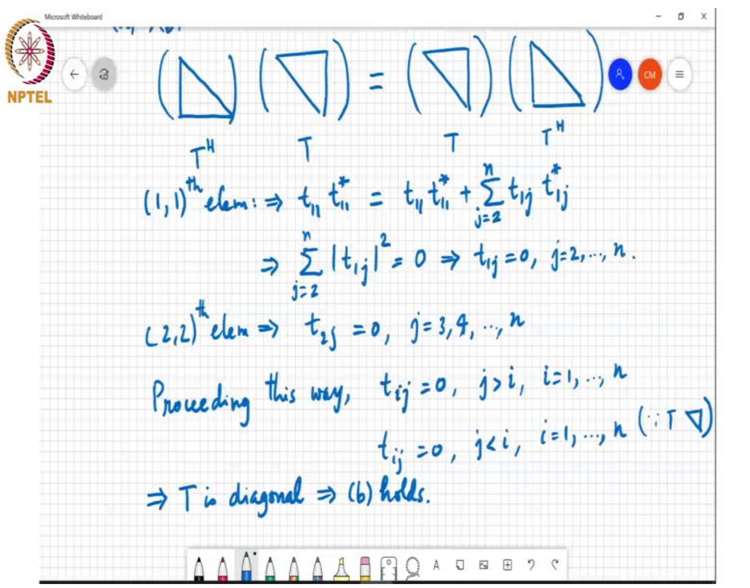

So, we will see have to show this. So, if, so the starting point is again Schur's unitary triangularization theorem. So, let T be upper triangular of size n cross n such that u Hermitian A u equals T. So, this means that T is unitarily equivalent to A. That means that this when we say A is normal then that is the same as saying that T must be normal. So, that is something that you can immediately verify that. So, A, normality of A is equivalent to normality of T. So, you can easily check that if A is normal then T is normal. And this is an if and only if condition, so…

Student: Sir.

Professor: Yes, go ahead please

Student: Sir what is normality of A and normality of T?

Professor: Normality is the property that a matrix commutes with its conjugate transpose. So, normality of A is the property that A A Hermitian equals A Hermitian A. The normality of T is the property that T T Hermitian equals T Hermitian T. So, if A is normal then T is normal. But T is upper triangular. So, if I look at upper triangular matrix and I write out this condition T T Hermitian equals T Hermitian T. That looks like this.

So, I will write T T Hermitian. So, T Hermitian would be a lower triangular matrix and T is a upper triangular matrix. T Hermitian, so this is T Hermitian, this is T and that is the same as T T Hermitian. And now if you see what happens when you equate the entries of these two products. What I am going to argue is that if T is normal and upper triangular then it must be diagonal.

So, if I take the 1 1 th element of both the left and side and the right hand side here then what we get is the 1 comma 1 th element of this is going to t 11 times t 11 star. And so that will be t 1, t11 star and the 1 comma 1 th element here would be this column times this row which is t11, t11 star plus the summation j equal to 2 to n t1j, t1j star.

So, now this is mod t11 square and it cancels with this mod t11 square. And this is mod t1j square. So, this means that summation j equal to 2 to n mod t1j square equals 0, which in turn means that, these are all non negative quantities, so if you add up all of these and you are still getting 0 every one of them must be 0. So, t1j a equals 0, j equals 2 through n.

So, basically other than the 1 comma 1 th element all other entries in the first row must be equal to 0. Similarly if you equate the 2 comma 2 th entry what you will get is that t2j equals 0 for j equal to 3, 4 up to n. And so on. So, just proceeding this way we get tij equals 0 for j greater than i and i going from 1 to n. And of course tij equals 0 for j less than i and i going from 1 to n because T is upper triangular. So, for all j less than i the entries are always equal to 0. So, this means that T is diagonal, which means that A is unitarily diagonalizable.

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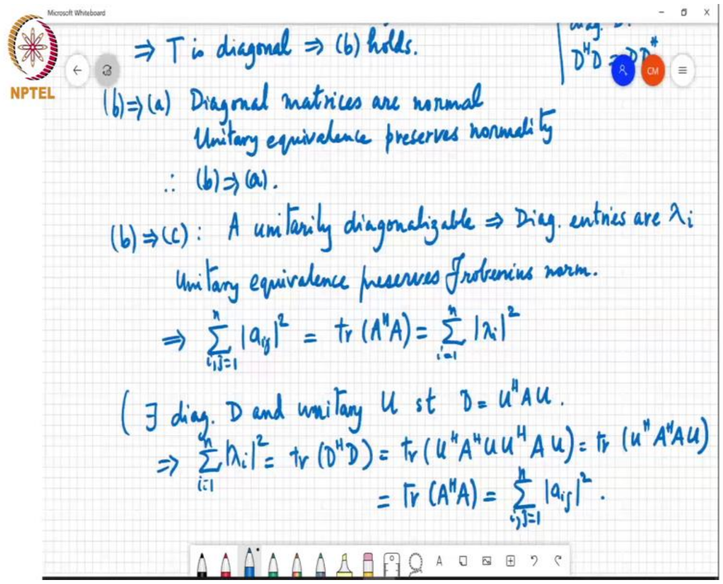

The other way is actually much simpler. So, T implies A. So, what we need to show is if A is unitarily diagonalizable then A is a normal matrix. So, basically any diagonal matrix is already normal. For any normal matrix D, D Hermitian D just contains mod d i squared around the diagonals. And so that is equal to D D Hermitian for any diagonal. So, all diagonal matrices are normal. And unitary equivalence preserves normality. So, basically therefore b implies a. So, a and b are done.

Now, if you want to show b implies c what we need to show is that if A is unitarily diagonalizable then summation of mod a ij square equals the summation of lambda i square. If A is unitarily diagonalizable then the resulting diagonal matrix will have the eigenvalues lambda i on the diagonal, because I have told it is a similarity transform and if you are using a similarity transform to get a diagonal form then the diagonal entries must be lambda i.

So, and further, I will just show this in a second, unitary equivalence is a property that preserves Frobenius norm. So, that means that sigma ij equal to 1 to n mod a ij square is equal to the trace of A Hermitian A which is equal to sigma i is equal to n mod lambda i square. So, to see this, I mean this is simple.

Basically if A is unitarily diagonalizable then there exists a diagonal D containing eigenvalues of A and unitary u such that D equals u Hermitian A u. And then so then summation of lambda i squared, which is equal to the trace of D Hermitian D which is equal to the trace of, I will substitute for D, u Hermitian A Hermitian u u Hermitian A u which is equal to the trace of, this is just the identity matrix, so u Hermitian A Hermitian A u. And this trace is the similarity invariant and this u Hermitian times u is a similarity transform. And so you have trace of A Hermitian A which is equal to sigma ij 1 to n mod a ij square.

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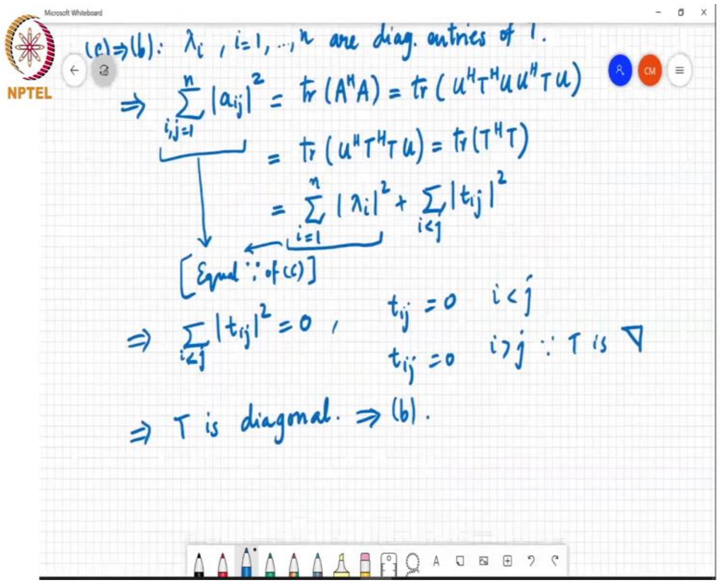

To go the other way, so to show the other way, so specifically that we want to show that if summation of a ij squared equals summation of lambda i square then A is unitarily diagonalizable. So, when we use the Schur's triangularization theorem to find a unitary transform such that u Hermitian A u equals T, lambda I, i going from 1 to n are the diagonal elements of T or will be the diagonal elements of T. So, that is basically how the eigenvalues are related to the matrix A. So, there exist a u such that u Hermitian A u equals T and the diagonal entries of T are these lambda is.

So, basically if somebody told us that summation, so i is equal to 1, ij both, ij equal to 1 to n a ij square. So, if we were to try to compute this, this is equal to the trace of A Hermitian A which is in turn equal to, I will just write it. so that its clear…Trace of A Hermitian A and I will substitute for u and I will write this as trace of u Hermitian T Hermitian u, u Hermitian T u, which is equal to trace of u Hermitian T Hermitian T u, which is equal to trace of, because trace is unitarily invariant that are invariant to this kind of a similarity transform.

This is a similarity transform on T Hermitian T, and the trace is invariant to similarity transformation, so this is the same as trace of T Hermitian T, which I can write since the diagonal entries are lambda i. So, the trace of T Hermitian T can be written as sigma, it is the sum of the mod of all the entries in T, mod square of all the entries in T. And I keep the diagonal entries separate, lambda i square plus sigma i less than j mod T ij square. This, these two, this equality is coming because of c. So, we are assuming c is true and we are trying to show that the matrix must be unitarily diagonalizable.

So, this immediately implies that, or sorry, we put it this way. This, this quantity here and this quantity here are equal because of our assumption that summation ij squared equals summation lambda i square. So, that immediately implies that sigma i less than j mod t ij square equals 0 or t ij equals 0 for i less than j. And of course t ij equals 0 for i greater than j because T is upper triangular.

So, if the row index is bigger than the column index all these entries are always 0. So, T is diagonal. So, that means that the upper triangular matrix we got by applying a unitary transformation of A through the Schur's triangularization theorem is in fact diagonal. So, which implies that a, matrix A is unitarily diagonalizable which is the statement b.

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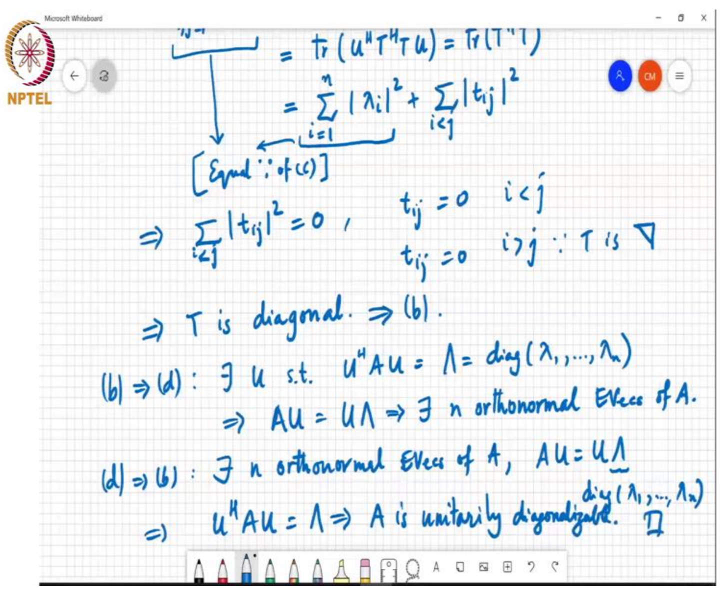

Then b implies d. So, if A is unitary diagonalizable we want to show that there is an orthonormal set of n eigenvectors of A. So, if A is unitarily diagonalizable then it means there exists a u such that u Hermitian A u equals lambda which is equal to this diagonal matrix containing lambda 1 through lambda n. That so if I take u to the other side or rather multiply u on both sides I will get

A u equals u lambda. Or each of these columns of u are actually, so this lambda is a diagonal matrix. So, it means that A ui equals lambda i u i. Or there exists in orthonormal eigenvectors of A.

And the other way is also exactly the same. If there exists n orthonormal eigenvectors, so if there exists n orthonormal, all these steps are reversible, so basically it follows… of A, then A u equals u lambda where lambda is a diagonal matrix containing the eigenvectors of A. So, that implies u Hermitian A u, since these eigenvectors are orthonormal u Hermitian equals, or u Hermitian u equals the identity matrix. So, if I do u Hermitian, if I multiply u Hermitian on the left then I get u Hermitian A u equals lambda which is a diagonal matrix, so which means that A is unitary diagonalizable.

Student: Sir

Professor: Yeah

Student: Sir, these i1, lambda i1, lambda 1, lambda 2 up to lambda n, how are they saying that there are distinct always?

Professor: They are not. That is a crucial point actually. So, this is the main, one of the main differences between what we said earlier and what we are saying now. Earlier for diagonalizability one of the conditions was that, one of the sufficient conditions was that the eigenvalues need to be distinct. We do not need that here.

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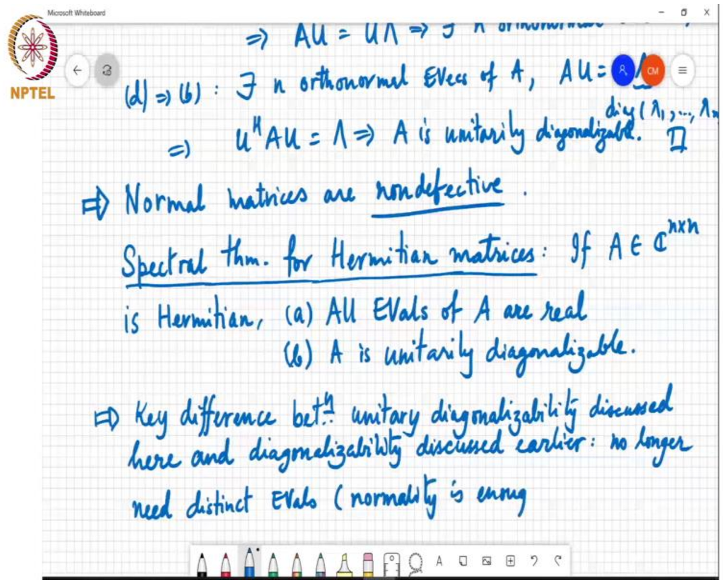

So, the important consequence of this is that normal matrices are non defective. They are always diagonalizable, and the algebraic multiplicity equals the geometric multiplicity of the matrix. So, we have seen that Hermitian matrices are a special case of normal matrices. If A is Hermitian then it means A equals A Hermitian. And therefore A A Hermitian equals A Hermitian A which is both equal to A square.

And so Hermitian matrices are normal and for Hermitian matrices we can say one small extra thing which is known as the Spectral theorem for Hermitian matrices. So, if A in C to the n cross n is Hermitian then all eigenvalues of A are what, what can we say about the eigenvalues of?

Student: Real

Professor: Real, exactly. And b, A is unitarily diagonalizable. See this is useful because all covariance matrices are Hermitian symmetric, by definition. Covariance matrix is the expected value of x x Hermitian where x is a vector. And so since covariance matrices are Hermitian symmetrical matrices all eigenvalues of covariance matrix are real and any covariance matrices unitarily diagonalizable. Now, the statement b here immediately follows from the fact that the matrix is, any Hermitian matrix is a normal and a normal matrix we just showed this, that any normal matrix is unitarily diagonalizable.

Now, the point a that the eigenvalues are real follows because if a matrix, any diagonal and Hermitian symmetric matrix must be real. And so, and unitarily, I mean this unitarily equivalence preserves Hermitian symmetry. And so if I find the matrix that it is unitarily equivalent to matrix A and it is diagonal then the matrix T which is unitarily equivalent to A must be Hermitian symmetric as well.

And if it is Hermitian symmetric and diagonal it is the real valued matrix. So, all the eigenvalues are real. Between unitary diagonalizability here and diagonalizability that we discussed earlier which was through similarity transforms is that we no longer need distinct eigenvalues. In other words normality is enough.