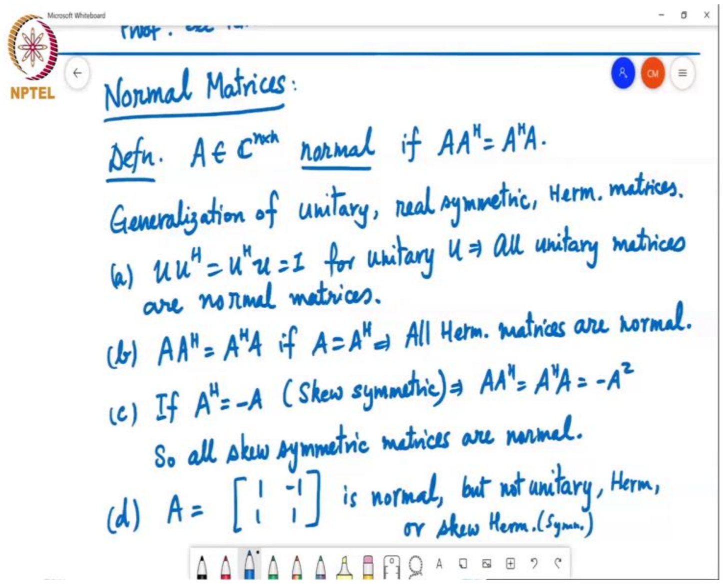
Matrix Theory
Professor Chandra R. Murthy
Department of Electrical Communication Engineering
Indian Institute of Science, Bangalore
Lecture 43
Normal Matrices: Definition and fundamental properties

(Refer Slide Time: 00:14)



So, normal matrices are, I mean do not get fooled by the name. It is just the property. It has nothing to do with in some sense the matrix being normal or in any way related to the Gaussian distribution. So, the definition is like this. A in C to the n cross n is normal if it commutes with its conjugate transpose. Any matrix for which this is true is called a normal matrix. Normal matrix is a generalization of unitary, symmetric and Hermitian matrices.

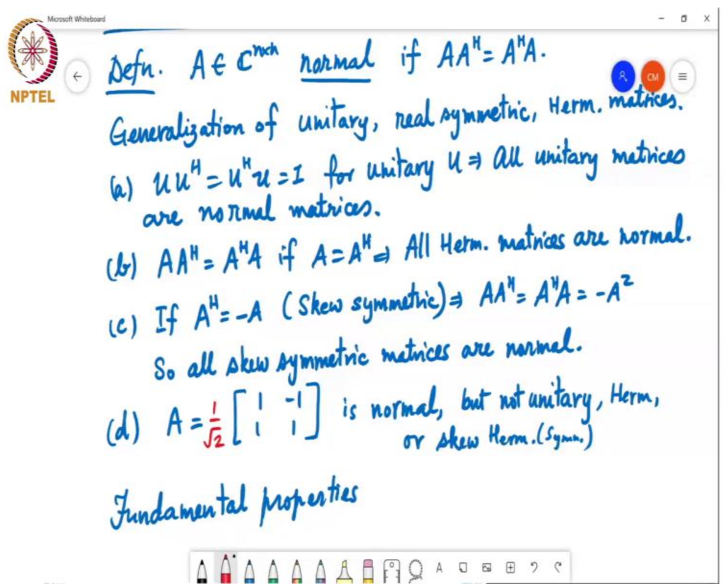
For example, for a real symmetric matrix symmetry such that A transpose equals A. And if it is real symmetric then A transpose equals A then A A Hermitian is same as A A transpose. But A transpose equals A. So, this is equal to A square. This is also equal to A square. So, it holds for a real symmetric matrix. Similarly for the Hermitian matrix this equality holds. And as a consequence all such matrices unity matrix, symmetric matrix or Hermitian matrices are all normal matrices.

So, just to illustrate that it if A, if u is unitary then u u Hermitian equals u Hermitian u, which is equal to the identity matrix, implies unitary matrices are normal matrices.. Similarly, A A Hermitian equals A Hermitian A if A equals A Hermitian. So, all Hermitian matrices are normal.

Also if A Hermitian equals minus A, such matrices are called skew-symmetric or skew-Hermitian then A A Hermitian equals A Hermitian A which is equal to minus A square. So, skew-symmetric matrices are normal.

And finally just one more example. If I consider the matrix A equal to 1, minus 1, 1, 1. This matrix is normal. But it is not unitary or Hermitian or skew Hermitian or skew symmetric. So, basically the definition of normal matrices is strict generalization of these other matrices like unitary matrix, matrices earlier symmetric matrices or Hermitian matrices or skew Hermitian matrices. So, here is a very, very interesting result which outlines some properties of normal matrices.

(Refer Slide Time: 05:48)



Student: Sir, It holds for orthonormal also? This A matrix is orthogonal but for orthonormal it should hold.

Professor: Yes

Student: Not generally for orthogonal

Professor: If I had taken an extra 1 over square root of 2 factor here then this would have become an unitary matrix. And then of course since it is unitary it is also normal. But without that also it satisfies the requirement of being normal.

Student: But generally orthogonal are not, right?

Professor: So, again I think you ask this question the last time.

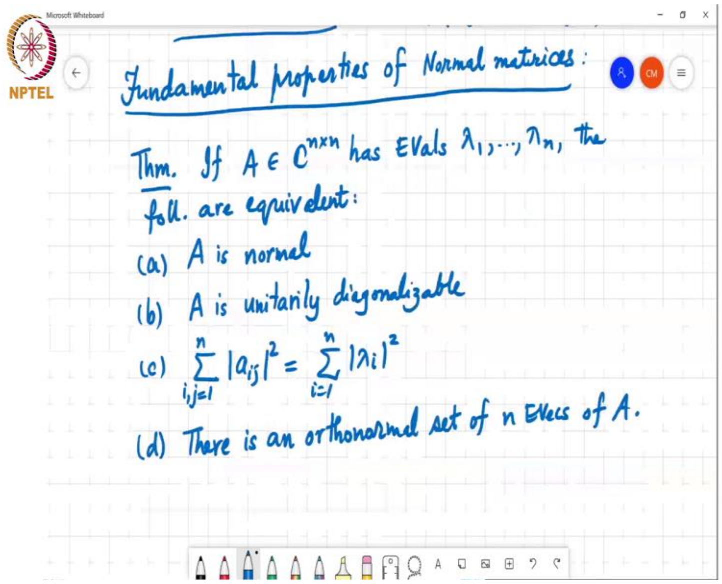
Student: No sir I am asking

Professor: No, I am just… In terms of notation I used two notations. One is a unitary matrix which is potentially complex valued but for which u u Hermitian equals the identity matrix. I also use the notation real orthogonal matrix. To me a real valued matrix satisfying u u transpose equals the identity matrix.

So, I do not have a specific notation for matrices like this A that I have drawn, I have written here where the columns are orthogonal to each other but they are not unit norm. I do not have a specific word for that. But basically a matrix like this whose columns are orthogonal and they have the same norm but not equal to 1 is also a normal matrix.

Because if I take A Hermitian A I will get a diagonal matrix and the values along the diagonal will be equal to a scale, a scaled version of, A Hermitian A will be the scaled version of the identity matrix. And so from that you can see that if I take one over square root of that scaling then that matrix, and apply that to A and A Hermitian, that resulting matrix will become a orthonormal matrix. And so as a consequence multiplying it in the other order will also give identity matrix. So, A A Hermitian will be equal to A Hermitian A.

(Refer Slide Time: 08:43)



So, here is the theorem. So, if matrix A in C to the n cross n has eigenvalues lambda 1 through lambda n, the following are equivalent; a, A is normal; b, A is unitarily diagonalizable; c, sigma i j equal to 1 to n mod a ij square is equal to sigma i equal to 1 to n mod lambda i square. So, this shows that any normal matrix is unitarily diagonalizable, which is a different requirement or a different condition under which you can be assured that matrix is unitary diagonalizable compared to the result we saw earlier where we wanted the matrix to have distinct diagonal values. The eigenvalues lambda 1 to lambda n need not be distinct. If it is normal that is also sufficient. A is going to be unitarily diagonalizable.