Matrix Theory
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Lecture 39
Unitary Equivalence

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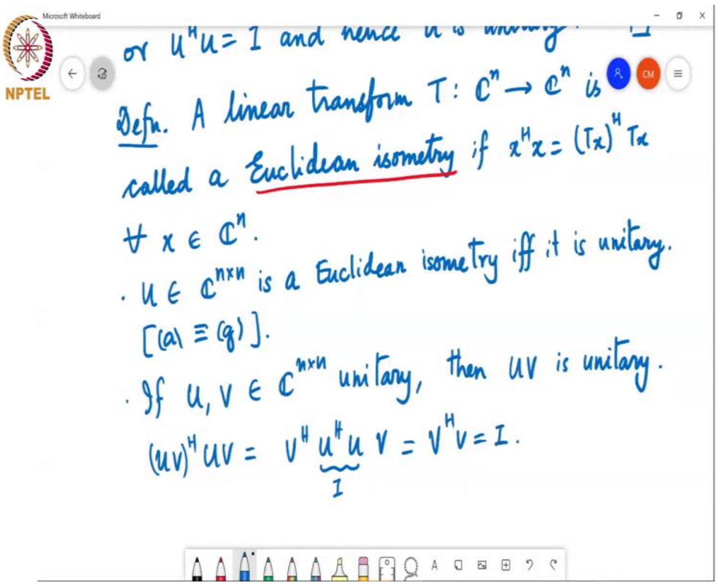

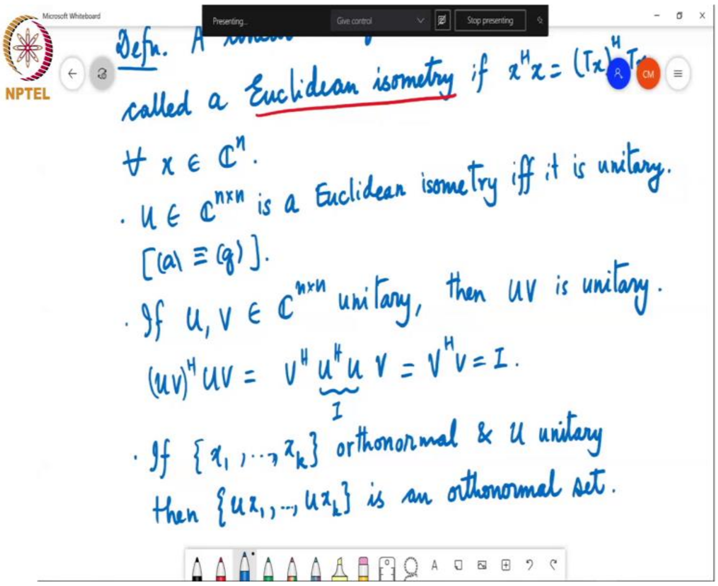

Just one more definition to keep going. So, a linear transformation T which maps from C to the n to C to the n is called a Euclidean isometry. If x Hermitian x equals T x Hermitian Tx for all x and C to the n. So, basically an isometry is something that preserves length the length of a vector. So, it is a transformation any transformation that preserves the length of a vector is called an isometry.

And if it preserves the Euclidean length or the length measured through the Euclidean norm then we call it a Euclidean isometry. So, basically from the previous result we can have that a complex square matrix u in C to the n cross n is Euclidean isometry if and only if it is unitary. This is exactly the same as the statement a is equivalent to g.

One another remark about these unitary matrices is that if u and v are unitary matrices and u v their product is unitary that simply because u v Hermitian u v is equal to v Hermitian u Hermitian u v and u Hermitian u is the identity matrix. So, this is the same as v Hermitian v which is equal to the identity matrix. So, u v is an identity. So, take products of unitary matrices how many products you take.

They will all be unitary matrix another remark related is that if x1 to xk is an orthonormal set and u is a unitary matrix then u x1 through xk is also an orthonormal set. These are all immediate from what we have seen about unitary vectors and orthonormal matrices. x1 through xk are orthonormal and u is unitary then u x1 u xk orthonormal set.

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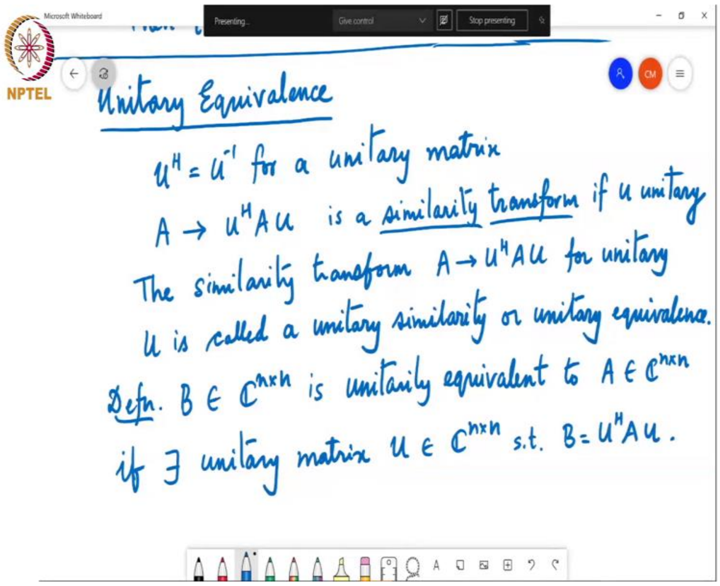

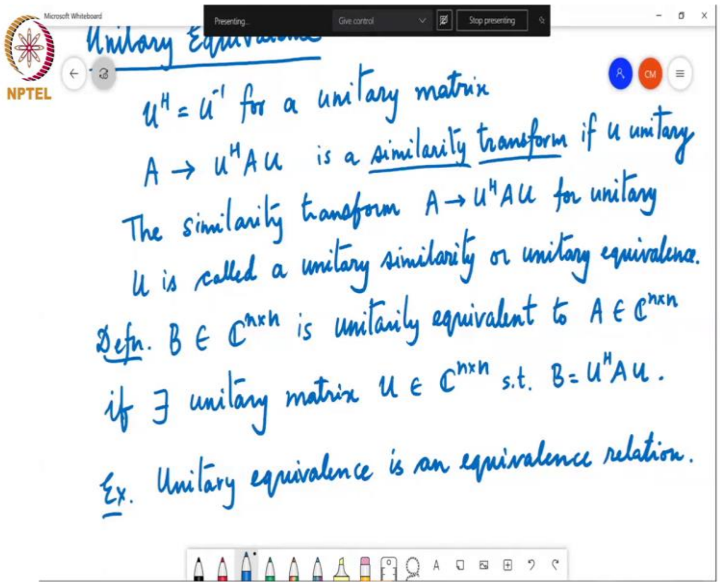

So, now let us continue on with unitary equivalence. So, I mean all this was kind of the prelude to get to unitary equivalence. So, basically for a unitary matrix we know now that u Hermitian equals u inverse for unitary matrix. So, that means that if I take the mapping from A to u Hermitian A u this is a similarity transform if u is unitary. So, by similarity transform what I mean is our definition S inverse A S where S is an invertible matrix so it satisfies that definition.

So, this mapping from A to u Hermitian A u is in fact a similarity transform. But it is the special similarity transform it is a transform where the matrix involved in the transform that the matrix S in our previous definition is a unitary matrix. And that is why we call it a unitary similarity or a unitary equivalence. So, the similarity A to u Hermitian A u for unitary u is or.

So, now that leads to the definition that B and C to the n cross n is unitarily equivalent to A in C to the n cross n. If there exists unitary so they going in circles orbit here. But matrix u in C to the n cross n such that B equals u Hermitian A u. So, I mean going by the fact that similarity transform is an equivalence relation. You can also show that unitary equivalence is also an equivalence relation.

So, it is not a misnomer it is called that for a reason. So, is that means that it is reflexive any A is unitarily equal to itself it is symmetric. A unitary equivalent to B implies B unitarily equivalent to A and it is transitive that is A unitary equivalent to B and B unitarily equivalent to C implies A is unitarily equivalent to C. So, you should show that just to convince yourself it is true.

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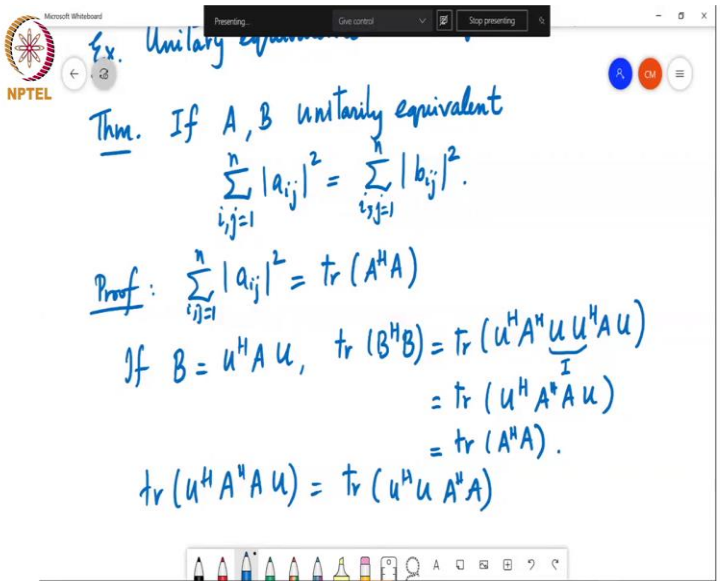

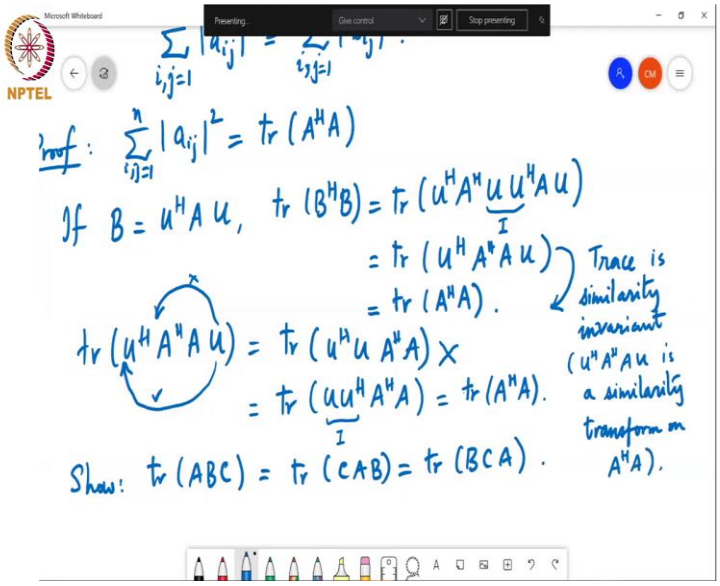

You have the following theorem. If A and B unitarily equivalent. Then the sum of the squares of all the entries of A is the same as the sum of all the squares of all the entries have B. So, intuitively this should be obvious. But just to see why this is true summation i j equal to 1 to n mod a i j square is the same as trace of A Hermitian A.

If you a look at the entries that arise when you are trying to conclude trace of A Hermitian A. It is exactly the same as this double summation i j equal to 1 to n mod of A i j square. Now, if B is u Hermitian A u then trace of B Hermitian B is equal to trace of I am just going to substitute for B. So, that becomes u Hermitian B Hermitian B B Hermitian. Sorry u Hermitian A Hermitian u u Hermitian A u.

So, I am just substituting for B as u Hermitian A u and u u Hermitian is the identity matrix. So, I am left with trace of u Hermitian trace of u Hermitian A Hermitian A u which is equal to trace of A Hermitian A. Why is this last step true.

Student: In the trace you can exchange the terms (())(12:14). So, it will be u Hermitian u A Hermitian A so u Hermitian A we want

Professor: So, you are saying that trace of u Hermitian, A Hermitian A u is the same as trace of u Hermitian u A Hermitian A? Is that what you are saying? This is not correct. So, one of the things about the trace is that you can you can cyclically permute the terms.

Student: A equal to trace of BA we can use that and…

Professor: Trace of A B equals trace of the B A which means that you are actually cyclically permuting. You cannot just pick two matrices and exchange the order. So, you can just shift this matrix out here and write it as u Hermitian u, A Hermitian A that is not correct. But you can shift this matrix all the way out here that is okay. So, I can write this says u u Hermitian A Hermitian A and that is equal to this is equal to the identity matrix.

So, this is equal to the trace of A Hermitian A. So, now you have one more exercise to show which is that if I have trace of ABC. Then it is equal to the trace of CAB, which is equal to trace of BCA. So, you need to show this. So, instead a simpler thing to do is to recognize that this u Hermitian A Hermitian A u is actually a similarity transform on A Hermitian A. Because u Hermitian equals u inverse.

So, since it is a similarity transform, we know that the eigenvalues of the matrix of a matrix remain invariant to its similarity transforms. And the trace is nothing but the sum of the Eigen values. So, basically it is a trace a similarity invariant. And so, this must be not true. So, that is a slightly simpler way to maybe think about it given what we know so far. So, trace is similarity invariant u Hermitian A Hermitian is Au is a similarity transform on A Hermitian A. So, that is about this I think we are almost out of time today.