Matrix Theory
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Lecture 38
Properties of Unitary Matrices

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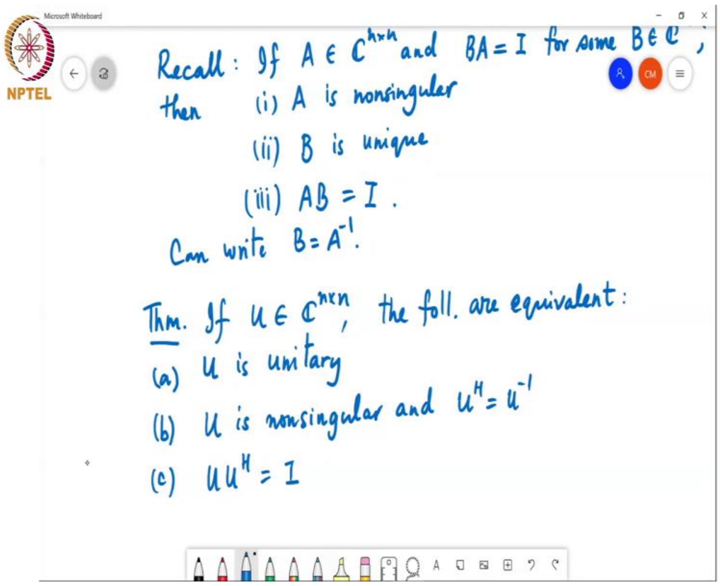

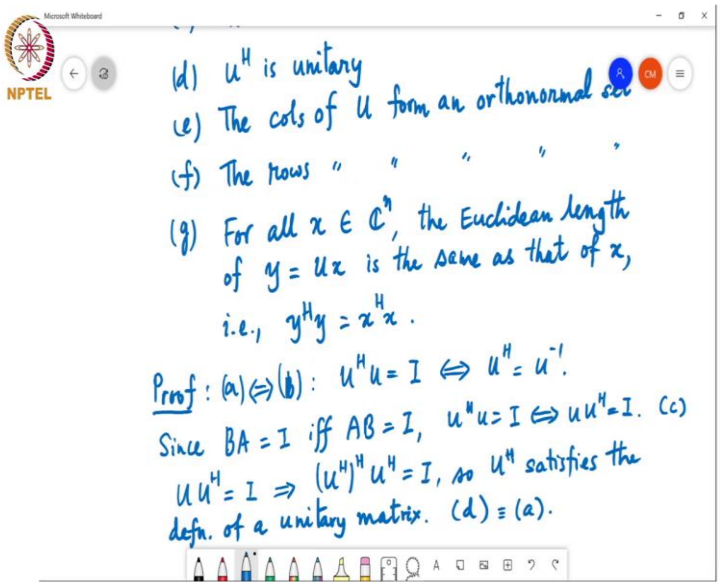

With this now we can state the next theorem. If u in c to the n cross n then the following are equivalent a u is unitary. So, basically, this is telling you various properties of unitary matrices and also various conditions which you can state which are equivalent to saying that u is unitary. So, u is non-singular and u Hermitian equals u inverse u u Hermitian equals the identity matrix.

u Hermitian is unitary. e the columns of u form an orthonormal set. The rows of u form an orthonormal set. And finally for all x the Euclidean length of y ux is the same as that of x. Which is y Hermitian y, x Hermitian x examination. So, it is making many statements about unitary matrices and equivalent ways of stating that a matrix is unitary. So, let us see how to show this.

The first few parts are very simple. Later we will just for the last part we will need to do a little bit more work. So, first of all, for equivalence we have to show that all these properties imply each other. Now, so if I take the first two properties if (())(3:40) then by definition u Hermitian u is I. And so u Hermitian is an inverse of u. So, u Hermitian u equals I, I will write it this way u Hermitian is equals u inverse it is a u Hermitian is an inverse of u.

And so these are reversible statements. And so, saying that A is unitary is the same as saying u Hermitian equals u inverse. So, this is actually establishing this both ways. And similarly, by the property I just mentioned BA equals I if and only if AB equals I. So, which means that uu Hermitian equals I and again this is a reversible statement. So, if I say u Hermitian u equals I. Let me write it a little more clearly.

Since BA equals I if and only if AB equals I u Hermitian u equals I is the same as saying u u Hermitian equals I. So, A and c are also equivalent statements. Now, u Hermitian is unitary that immediately follows from. So, if uu Hermitian equals I that is the same as saying u Hermitian whole Hermitian times u Hermitian equals I. So, you Hermitian satisfies the definition of a unitary matrix. So, essentially this means that d is true and in fact d is equivalent to a.

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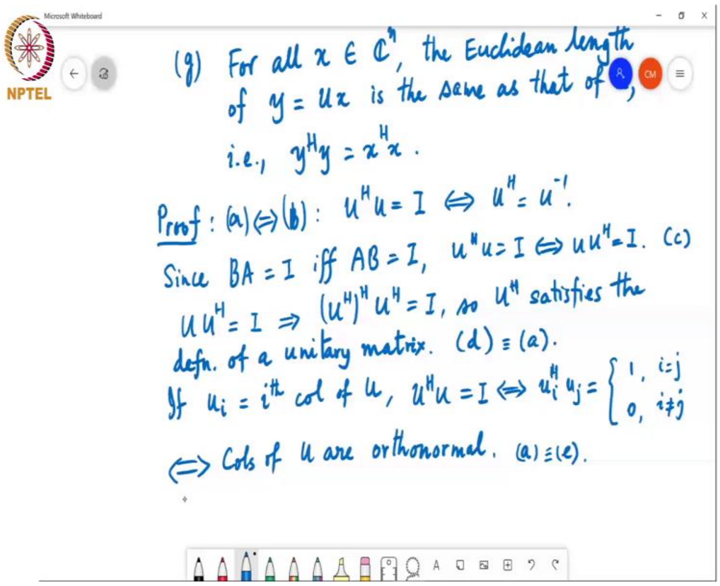

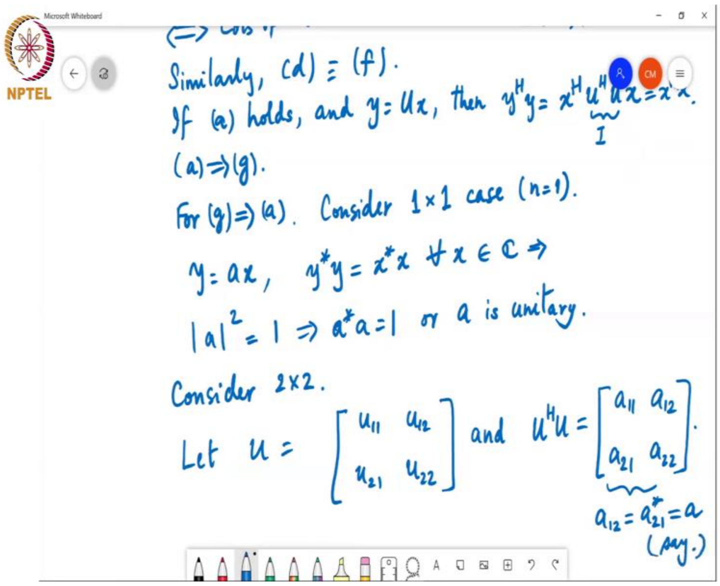

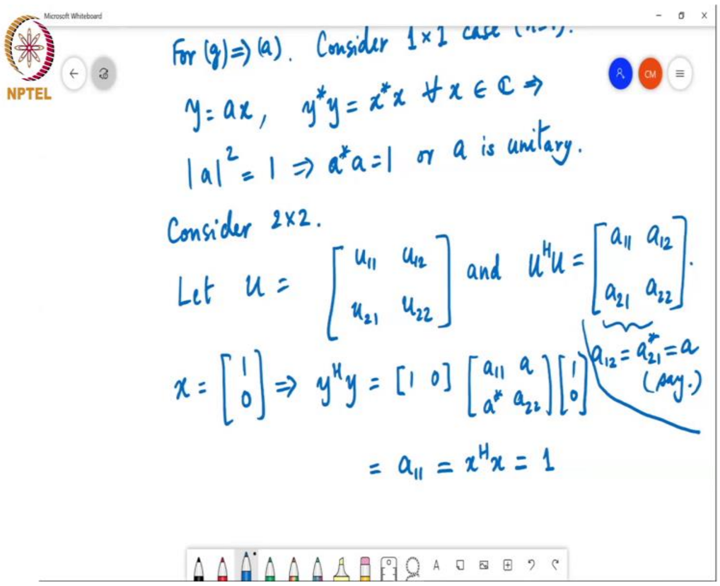

So, that establishes the equivalence of a, b, c, d. Now, if ui is the ith column of u. Then u Hermitian u equals the identity matrix is equivalent to saying u I Hermitian u j equals 1 if i equals j and 0 otherwise. And so that means that columns u are orthonormal. And so this is the same as saying a is equivalent to e. And so, he said that columns of u forms an orthonormal set by just doing the same thing with u Hermitian the rows of u form an orthonormal set.

So, similarly, b is equivalent to f. So, now the last part is this g which says that for all x the Euclidean length of y equal to u x is the same as that of x that is y Hermitian y equals x Hermitian x one way is very easy. If a holds that is u is unitary and y is equal to u x then y Hermitian and y is equal to x Hermitian u Hermitian u x just substitute, in y equals u x which is equal to u Hermitian u is the identity matrix. So, this is equal to x Hermitian x.

So, this goes one way so a implies g and so what remains is to show the other way that is g implies a. So, for g implies a so let us first consider the 1 cross 1 case. So, n is equal to 1 then what happens here is that if I take y equal to a x this is a scalar thing. So, everything here is a scalar. Then if I look at y Hermitian which is the same as the conjugate y this is equal to the condition says that this is equal to x Hermitian x for every x.

Now, we need to show that this implies the matrix a the 1 cross 1 matrix a is unitary. So, if this is true this implies that mod a is square equals so y Hermitian y is equal to mod a square times x star x. So, if I take x star x to the other side, I get mod a squared equal to 1 which implies a conjugate a equals 1 which is the same as saying a is unitary. This is the definition of a unitary matrix. So, the 1 cross 1 case it is easy.

Now, for the 2 cross 2 case. Now, let us consider the 2 cross 2 case and then we will generalize it to the n cross n case. So, consider then here let us let u equal to the matrix u11, u12, u21, u22 and u Hermitian u. So, let us say this is a matrix a11, a12, a21, a22 is just some notation I am defining. So, now here by definition this is a Hermitian symmetric matrix. So, here we have that a12 equals a21 star. And let us call this both equal to a say.

Now, we will take some specific choices for this vector. So, if x equal to this vector 1 0 then so once again just to recall what we are trying to show here is that if y Hermitian y equals x Hermitian x for all x. Then u must be a unitary matrix. So, if I take y Hermitian y that is equal to 1 0 times u Hermitian u which is this matrix a11, aa star a22 this times 1 0.

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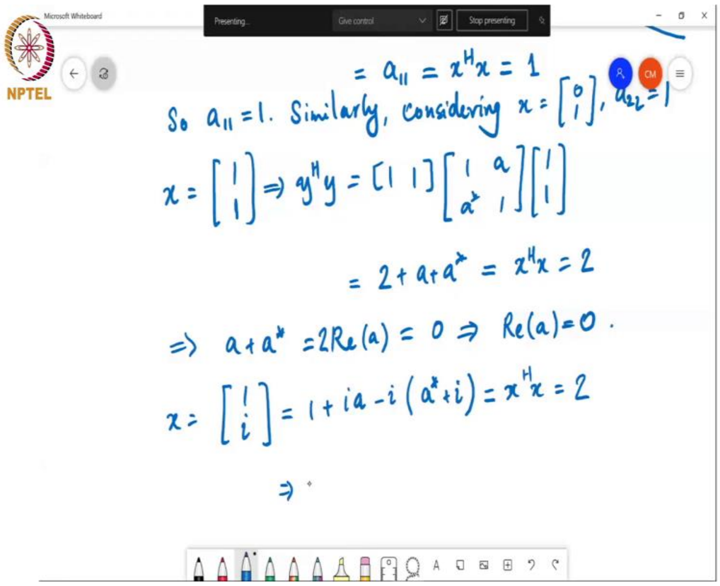

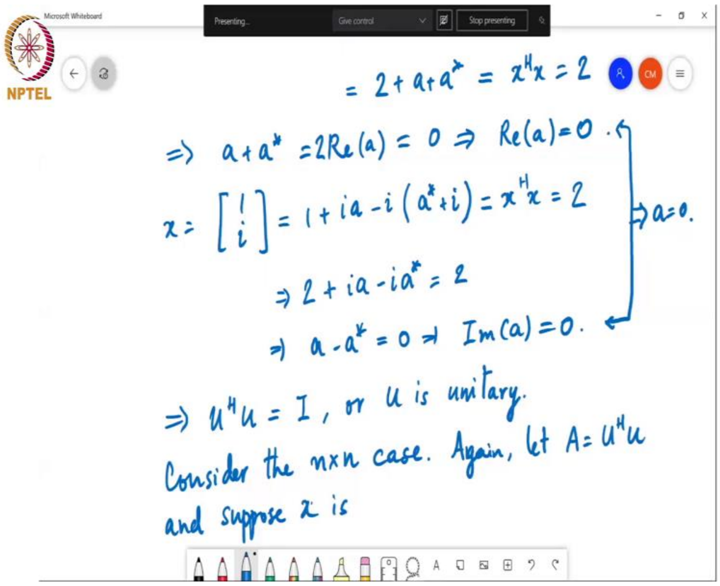

Now, this product here it just pulls out the entry a 11. And what we are given is that this is equal to x Hermitian x is equal to 1. So, a 11 is equal to 1. So, a 11 equals 1. Similarly, so considering x equal to 0 1 a 22 equals 1. So, this matrix is of the form 1 a a star and 1. So, we just need to show that a equals 0 and u Hermitian u will be the identity matrix or u will be unitary.

Now, if I take x equal to the vector 11 then y Hermitian y is equal to 11 times the matrix 1 a a star 1 times the vector 11. which is in turn equal to see if I expand this out this is 1 1 times 1 plus a a star plus one. So, it becomes 1 plus a plus a star plus 1. Which is the same as 2 plus a plus a star. And this is what we are given is that this is equal to x Hermitian x which is equal to 2.

So, this means that a plus a star which is equal to the real part of a 2 times the real part of a is equal to 0. So, which in turn implies that the real part of a equals 0. Similarly, if you choose x equal to 1 i then and you do the same thing you will have 1 i. This matrix times 1 i. So, then this becomes 1 minus i a a star minus i and then you do this times 1. So, that gives me 1 plus i a plus i times this made a star plus i sorry here it will become minus i. Because I am taking the conjugate transpose.

So, it is minus i times a star plus i. And so that becomes equal to… So, minus i squared is minus 1 which cancels with this. And so, I will be left with i a minus i a star. And that is supposed to be equal to the norm of this vector. Which becomes 1 minus 1 which is 0 norm squared of this vector. So, this is equal to 0 just for the sake of clarity and write it as x Hermitian x which is equal to 0. Did I say that correctly? Or have I made a mistake?

Student: Sir, how is (())(17:44)

Professor: Thanks for asking that. So, the 1's are not cancelling minus i squared is actually plus 1. So, this is actually 2 plus i a minus i a star. And similarly, explanation x is not 0. If I take x Hermitian x it becomes the inner product between 1 minus i and 1 i. So, that becomes 1 minus i square and so this is also equal to 2. So, as before I get 2 plus i a minus i a star is equal to 2.

So, that means i a plus i a star equals 0 and I can take out i n take it to the other side. So, I will be left with a minus a star which is equal to 0. Which implies that the imaginary part of a is also equal to 0. So, we have shown that the real part of a is 0. And we have shown that the imaginary part of a is 0. So, that implies a equals 0. So, that means that u Hermitian u is equal to the identity matrix for u is unitary.

So, what is left is to show that this holds in the n cross n case also. So, here they will use exactly this idea that we just discussed but what we will do is so once again let a equal to u Hermitian u. And suppose x is such that I will just write it like this.

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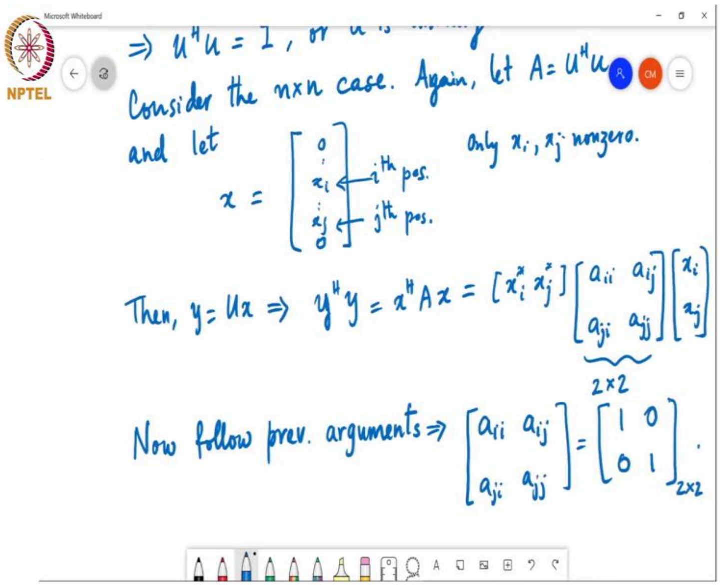

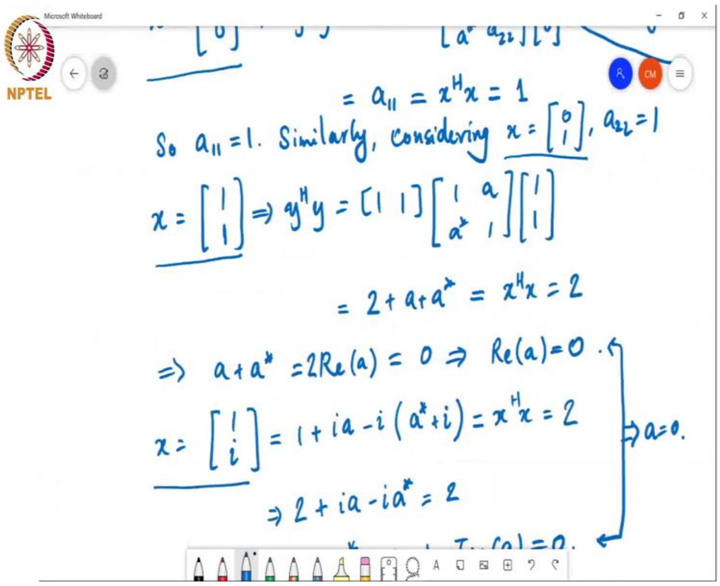

X is this vector with 0’s there is going to be two 1’s and 0’s everywhere else. Where these two are in the ith and (())(20:38) position. So, then if I take if I define y equal to u x implies that if I take y Hermitian y this will pull out this is equal to x Hermitian A x which can then be written as I will just do one thing because so let us call this x i.

It is a vector which has 0’s everywhere except x i in the ith position. And x j in the jth position because I want to use x i equals 1 xj equal to 0. Then x i equals 1 x i equals 0 x j equals 1 and then x i equals 1 x j equals 1. And finally, x i equals 1 and x j equal to i, that is what we did. We considered four cases we considered x equal to 1 0 and 0 1 x equal to 1 1 and x equal to 1 i.

So, those are the four x i x j values that I am considering later. But for the purpose of writing this I will just say x i x is a vector where only the x i and x j are nonzero. So, then if I do x i x Hermitian A x. I will get none of the other entries matter it becomes x i star x j star times a ii a ij a ji a jj times x i x j and so now this is exactly the 2 cross 2 case we considered earlier. And so now we can follow the previous arguments which means I will choose xi xj equal to 1 0 0 1 1 1 and 1 I to show that this matrix aii, aij, aji, ajj is nothing but the identity matrix 2 cross 2. And so, but i and j are arbitrary here.

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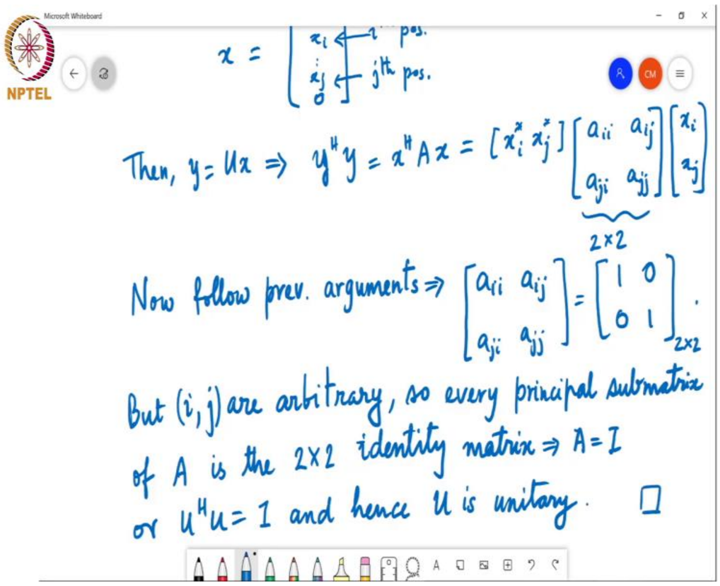

So, every principle 2 cross 2 sub matrix of A is the identity is the 2 cross 2 identity matrix. Which implies A is equal to I or u Hermitian u equals I and hence u is unitary. So, those six statements are equivalent seven statements.