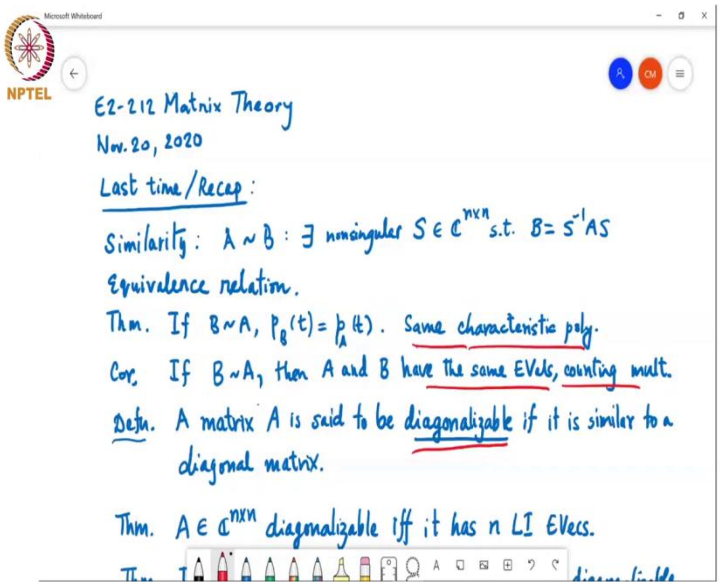
Matrix Theory
Professor Chandra R. Murthy
Department of Electrical Communication Engineering
Indian Institute of Science, Bangalore
Relationship Between Eigenvalues of BA and AB

(Refer Slide Time: 0:14)

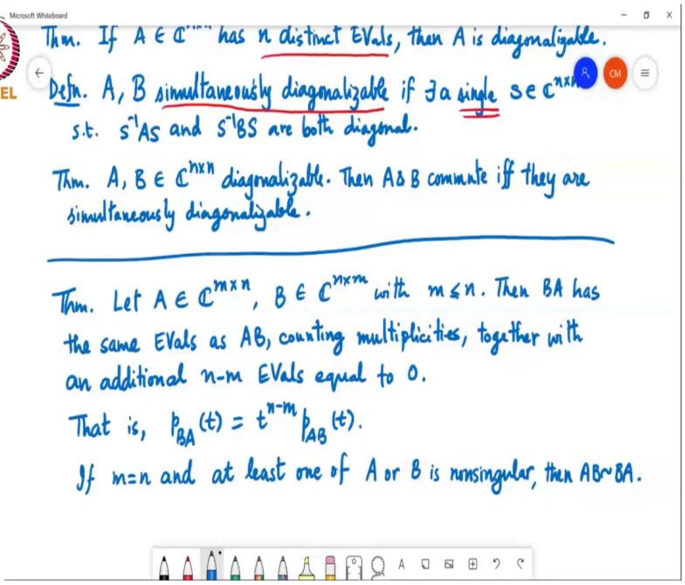


So, we will continue where we left off the previous time, but just to recap, we were discussing about similarity and we say that a matrix A is similar to matrix B if there exists a nonsingular matrix S, such that B can be written as S inverse AS. This is an equivalence relation. So, which in turn means that the relationship is reflexive, symmetric and transitive; reflexive meaning every matrix is similar to itself and if A similar and transitive means symmetric means if A similar to B then B similar to A.

And transitive means that if there is A triangular (())(00:58) that is if C similar to B and B is similar to A then C is similar to A. We saw that similar matrices have the same characteristic polynomial and the corollary to this is that, if two matrices are similar than they have the same eigenvalues counting multiplicities and we also defined the notion of diagonalizability and a matrix is diagonalizable if it is similar to a diagonal matrix.

And this similarity divides the space of all n cross n matrices into equivalence classes and if A similar to a diagonal matrix, then any other matrix belonging to the same equivalence classes diagonalizable into the same diagonal matrix and any matrix which is not in this same equivalence class will not be diagonalizable into this particular diagonal matrix.

(Refer Slide Time: 02:00)



And we saw that the condition for diagonalizibility is that it should have n linearly independent eigenvectors and, but we have not discussed when a matrix will have n linearly independent eigenvectors, one (portent) one such condition is that if A has n distinct eigenvalues, then A is diagonalizable.

When it has n distinct eigenvalues, it will have an n linearly independent eigenvectors and it will be diagonalizable. So, one thing is that if you perturb the matrix by a small amount, the eigenvalues also get perturbed by a small amount and we will see that result a little bit later. And so, even if A is not diagonalizable, you can always perturb it by a very small amount and obtain A matrix that is diagonalizable.

We also define the notion of simultaneous diagonalizeability, which is possible if there exists a single matrix. So, there is a common diagonalizing matrix S, such that both S inverse AS and S inverse BS and diagonalizable are diagonal matrices and one interesting property is that two matrices commute if and only if they are simultaneously diagonalizable. A and B commute meaning that A times B is the same as B times A.

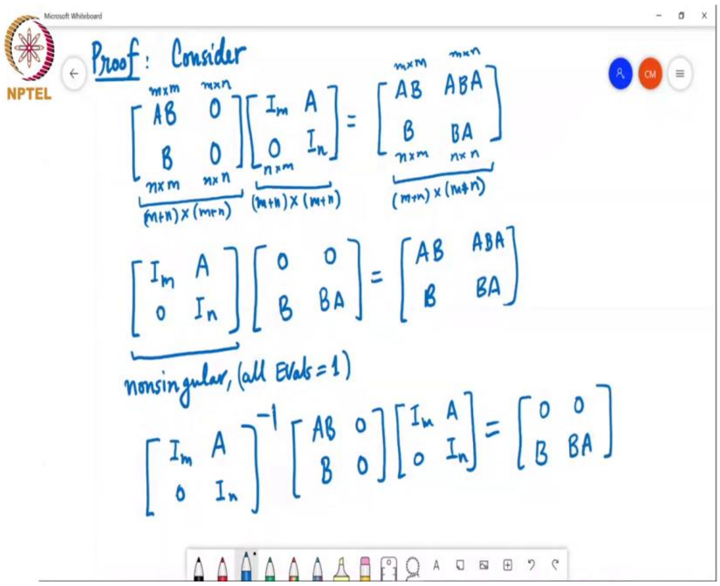
The last time and at the end of the previous class, we stated the following theorem, which says that, if you have two rectangular matrices A which is of size m by n and B which is of size n by m with m less than or equal to n, meaning that A is a fat matrix and B is a tall matrix.

But, because we have defined them as m by n and n by m, it is possible to consider what happens to BA and AB, both multiplications are kosher, they are both possible and these two matrices BA and AB have the same eigenvalues counting multiplicities but the matrix BA, which is of size n by n and it is a bigger matrix in size than AB which is of size m by m.

The extra eigenvalues BA has n minus m extra eigenvalues compared to AB and those extra n minus m eigenvalues will be equal to 0. So, in other words, the difference between the characteristic polynomial of BA and the character characteristic polynomial of AB is just this factor t to the n minus m.

You can see that this has n minus m 0es equal to 0, t equal to 0 and further if m equals n that is both these matrices are square and at least one of them is nonsingular then the two matrices AB and BA are similar. So, clearly if m equals n then they will have the same eigenvalues but remember having the same eigenvalues or having the same characteristic polynomial is not sufficient for two matrices to be similar to each other. But in this case, if one of them is nonsingular, then the two matrices are indeed similar to each other. So, let us see how to show this.

(Refer Slide Time: 05:21)



So, this is a slightly clever proof. There may be other ways to show it but this way is simple. But nonetheless, it is a clever proof. So, the way it goes is you consider the following. This times the matrix, so consider this product of these two matrices. Now, AB is of size m by m and B is of size n by m.

So, basically, they are this, this 0 here has m rows and n columns and this 0 here has n rows and n columns. So, overall, this matrix has a size m plus n by m plus n, it is a square matrix, and I am multiplying it by this matrix I m, A, 0, I n. So, basically, since this is I m, this has m columns.

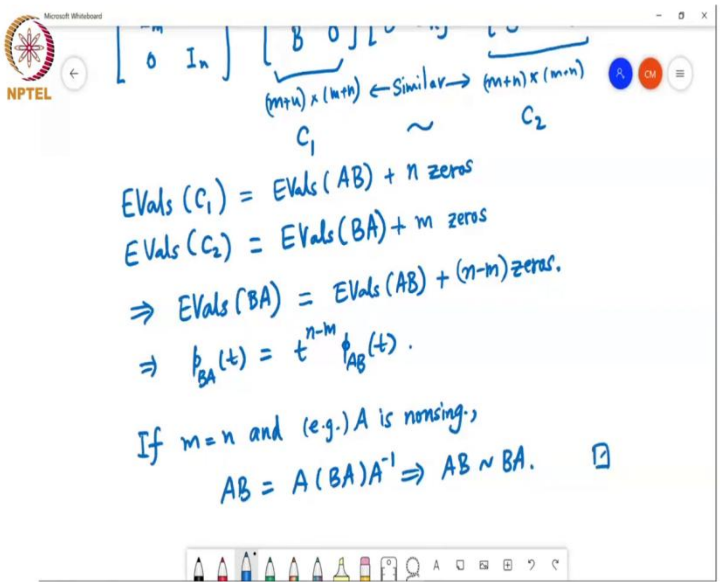
And since this is I n, this has n rows, and so this matrix is also of size m plus n by m plus n. So, this matrix here is of size m by n. So, it has the same rows as this and the same number of columns as this identity matrix. So, if I multiply these matrices equal to I m times AB, which is AB and so this times this gives me ABA, and this times this gives me B, this times this gives me BA.

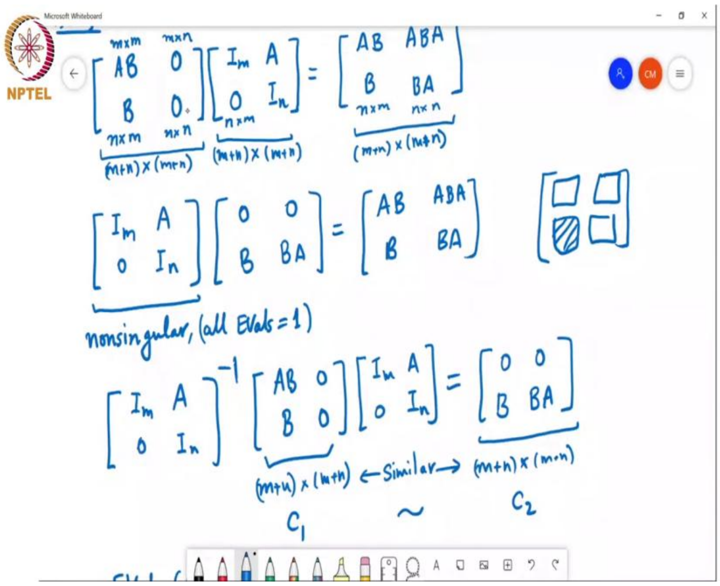
So, once again this AB is of size m by m ABA is of size m by n, this B is of size n by m and this BA is of size n by n and so, overall this matrix is also of size m plus n by m plus n as it should be the same size as this. Similarly, if I consider I n, A, 0, I n times the matrix 0, 0, B, BA, what do I get?

I get AB, I get B, I get BA. So, this is the same as this, so these two matrices are the same and so, and further this block matrix here, I m 0, this is a block upper triangular matrix. In fact, these two blocks are both diagonal and so this is actually an upper triangular matrix with ones on the diagonal.

All its eigenvalues are equal to one and so, this matrix is nonsingular, all its eigenvalues, this is actually not required for the proof, but this is just a side observation. Since this matrix is nonsingular and these two products are coming out to be the same matrix what we have is that if I consider it, if I pre multiply this by the inverse of this matrix then I will get this matrix here. So, that what I mean is I m A 0 I n inverse times AB 0, B 0 times I m A, 0 I n will be equal to this matrix here 0 0, B BA.

(Refer Slide Time: 11:20)





So, basically what this means is that this matrix and this matrix are similar to each other and this matrix. Yes?

Student: Sir, what do you mean by block when you say a block upper triangular matrix?

Professor: So, an upper triangular matrix is like this, all the elements above the main diagonal are nonzero and everything below the diagonal are 0 and a block upper triangular matrix consists of blocks which go down like this and all these, all the blocks have. So, let me let me erase here.

So, a matrix which you can divide into blocks so, there is a block here, a block here, a block here and a block here and if this block is always 0 you call such a matrix a block upper triangular matrix. Okay? It is arbitrary you can divide it in whichever way you like, but if you can divide it into blocks where everything below those blocks is 0 then you call it a block upper triangular matrix.

Student: Sir, individual blocks they could be non-upper triangular right?

Professor: Correct, correct, but in this case, you see that the individual blocks on the diagonal are identity matrix and therefore, they are actually upper triangular, in fact, they have ones on the diagonal. So, this matrix contains only ones on the diagonal and so, it is nonsingular, okay.

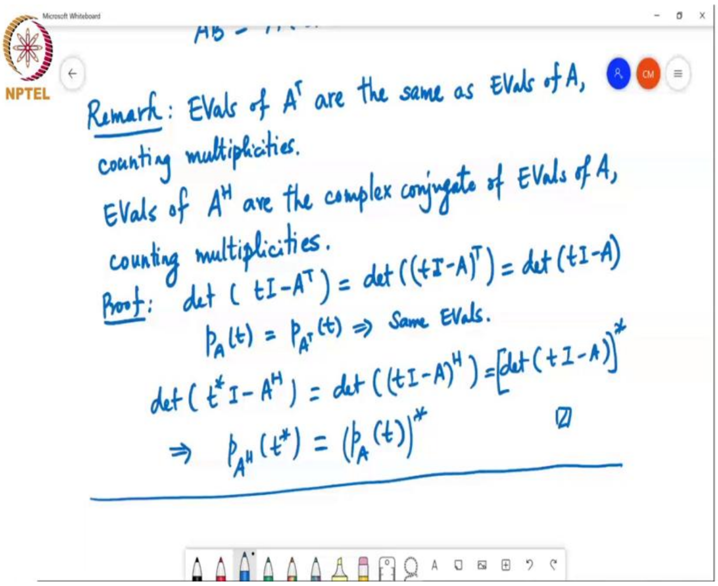
So, these two matrices are similar, they are both m by n sorry m plus 1 n cross m plus n and they are similar. So, basically if I call this matrix, say C 1 and this matrix C 2 they are similar to each other and the eigenvalues of C 1 is now, this is a block lower triangular matrix. So, and the size of this 0 here was n cross n.

So, the eigenvalues of C 1 are the eigenvalues of AB plus n zeros. Similarly, the eigenvalues of C 2 is the eigenvalues of BA plus this matrix and this matrix is of size. So, this matrix B is of size m by m. So, this matrix is of size m by m. So, gives the eigenvalues of BA plus m 0es.

Thank you. So, the eigenvalues of C 1 are the eigenvalues of AB plus n zeros. Eigenvalues of C 2 are the eigenvalues of BA plus m 0es and so, this means that just comparing these two the eigenvalues of BA, so basically this n is greater than or equal to m. So, the eigen, the eigenvalues of BA are the same as the eigenvalue, they have same eigenvalues, they are similar matrices.

So, we have that this is the eigenvalues of AB plus n minus m 0es and as a consequence the characteristic polynomial p BA of t is going to be t to the n minus m times p AB of t. Now, to close out the proof if m equals n and say A is nonsingular then AB equals A times BA times A inverse which means that AB is similar to BA.

(Refer Slide Time: 17:09)



Now, so, one small remark before I start discussing the next thing is that how are the eigenvalues of A are related to the eigenvalues of A transpose and the eigenvalues of A Hermitian. So, eigenvalues of A transpose are the same as eigenvalues of A counting multiplicities. Also eigenvalues of A Hermitian are complex conjugate of the eigenvalues of A counting multiplicities.

So, basically A and A transpose have the same eigenvalues A and A Hermitian have the same eigenvalues except for the complex conjugate operation. So, we will just see this very quickly. So, if I consider determinant of t I minus A transpose. The solutions or the zeros of this equation are the eigenvalues of A transpose.

And this is equal to determinant of t I minus A transpose which is equal to the transpose operation does not change the determinant t I minus A and so, the basically t I minus A and sorry A and A transpose have the same characteristic polynomials and so, they have the same eigenvalues.

Similarly determinant of if I take t complex conjugate I minus A Hermitian that is equal to determinant of t I minus A whole Hermitian and when I take the Hermitian every element gets the conjugate operation. So, the determinant of this you can think of it as this transpose and then taking the complex conjugate and determinant of this matrix transpose is same as the determinant of this matrix and the complex conjugate simply complex conjugates every element of the matrix and so, this is equal to determinant of t I minus A whole complex conjugate.

So, this means that P A Hermitian of t complex conjugate is equal to p A of t complex conjugate and so, this means that if you find zeros of this and then you take its complex conjugate, you will get the zeros of this matrix and so, the eigenvalues of A and the eigenvalues of A Hermitian are related through the complex conjugate operation.

Now, we know that if two matrices are similar, then they will have the same characteristic polynomial and the same eigenvalues counting multiplicities but what we have shown here is that A transpose and A have the same eigenvalues and the same characteristic polynomial counting multiplicities.

Obviously, we cannot conclude from this that A is similar to A transpose. This is not I mean, so, if two matrices are similar then they have this same eigenvalues, but if they if two matrices have the same eigenvalues, it does not mean that they are similar. So, do you think A and A transpose are similar to each other? In other words, if I give you any matrix A will you be able to find an invertible S such that A transpose equals S inverse A S? What is your guess?

Student: Sir, A similar to itself so, if we take a transpose of that.

Professor: A transpose will be similar to A transpose.

Student: Okay.

Professor: From that you cannot conclude that A similar to A transpose.

Student: Sir, if we take transpose in the definition of similarity.

Professor: So, but the definition of similarity is that it says that they are similar if there exists an invertible S. So, it goes to mean if you take the transpose in that. It simply gives you the reflexive relation that is if it is true that A is similar to A transpose it will just say that then it means that A transpose is similar to A.

But it will not really give you a proof that A is similar to A transpose. So, this is one of those results which is you know in matrix theory which makes this this subject very intriguing, it turns out that A and A transpose are actually similar to each other. You will be given whatever A you will you can find an invertible matrix such that A transpose equals S inverse AS.

So, they are actually similar to each other. But to show that we have to use this something called the Jordan canonical form, which we will cover later in the course and, but using that form we can show that A is actually similar to A transpose that we will come later.