Matrix Theory

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Diagonalization

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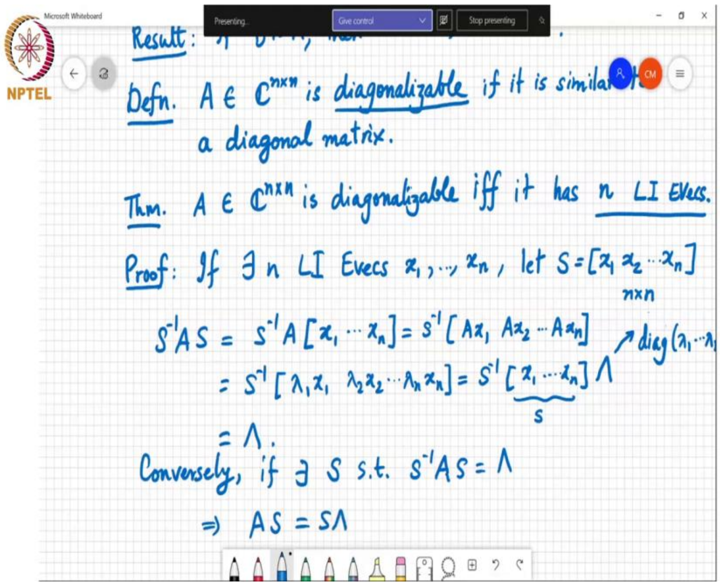

Diagonal matrices play, are actually very convenient. They are very easy to understand and for instance if you have a diagonal matrix its eigenvalues are the diagonal elements and so on. There is so many nice properties that diagonal matrices have and so, in general we want to know if a given matrix A is similar to a diagonal matrix, that is to say that the matrix A, it belongs to a certain equivalence class and does this equivalence class contain any diagonal matrix in it.

So, that is the following definition; is diagonalizable if it is similar to a diagonal matrix. So, when is the matrix diagonalizable? So, we have the following result, is diagonalizable if and only if it has n linearly independent eigenvectors. So, that is the requirement it must have n linearly independent eigenvectors.

And we have already seen that if a matrix has distinct eigenvalues, then the corresponding eigenvectors will be linearly independent and so, any matrix that has n distinct eigenvalues will necessarily be diagonalizable. Of course, a matrix could have repeated eigenvalues and still be diagonalizable and the identity matrix is an immediate example, all its eigenvalues are equal to 1, it has n repeated eigenvalues and it is already diagnosed. So, it is diagonalizable.

And in this case, you can also see that any nonsingular S is such that if I do S inverse times the identity times S gives me a diagonal matrix. So, the matrix S that transforms the identity matrix to a diagonal matrix can be any nonsingular matrix S. So, that also shows you that this matrix S that defines the similarity transform connecting A and B two matrices it need not be unique.

So, let us just see how this is proved. So, if there are n independent linearly independent eigenvectors x 1 through x n then let the matrix S just be a stacking of these vectors I will not write it with commas; x 1 stacked with x 2, x n. So, this is an n cross n matrix. We will show that this matrix works.

It will reduce A to a diagonal matrix. So, all we do is we consider S inverse A S which is equal to S inverse times A times this matrix x 1 through x n and this matrix so, if I expand this product I will get S inverse the first column of the product will be A x 1, the second column will be A x 2, A x n.

And because these are eigenvector, eigenvectors of this matrix A, this A x 1 is some lambda 1 x 1, S inverse times lambda 1, x 1, lambda 2, x 2 up to lambda n, x n. If I consider a diagonal matrix with lambda 1 through lambda n as its diagonal entries, then I can write this as S inverse times x 1, x n times this matrix lambda where lambda equals it is a diagonal matrix.

This is just rewriting this product here. But then this matrix is just S, and so I have S inverse S, which is the identity matrix, which is equal to lambda. So, we have shown that the matrix actually gets diagonalized. So, what this shows is that if there are n linearly independent eigenvectors, then the matrix A is diagonalizable.

By a similar transform, I have reduced A to a diagonal matrix. Since it is an if and only if condition I need to show the converse also. So, if what it, what we need to show is that if A is diagonalizable, then it must have n linearly independent eigenvectors. So, if there exists an S such that S inverse A S equals lambda, where lambda is some diagonal matrix, then it means that A S equals S lambda, I am just pre multiplying by S.

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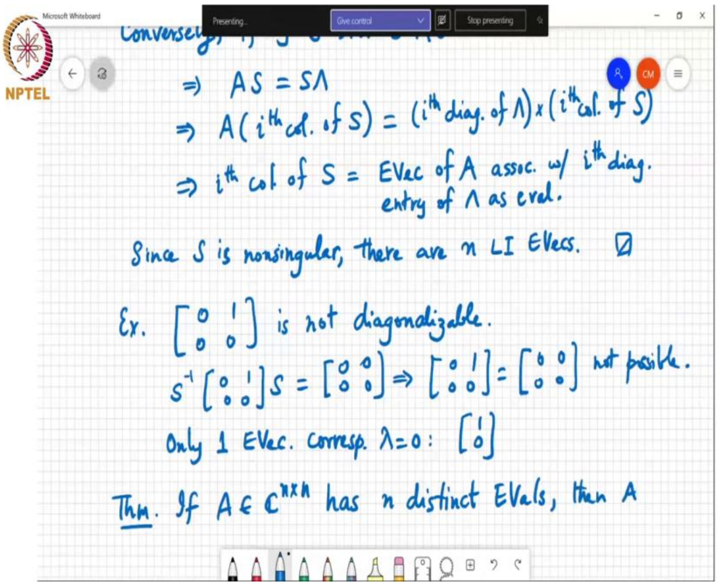

So, this just if I write out what this means, in words, this means that A times the ith column of S is equal to, but lambda is a diagonal matrix so that will be, so the ith column of, so this will be the ith column of A S. So, A times the ith column of S is equal to the ith diagonal entry of lambda times the ith column of S.

Which means that the ith column of S is the eigenvector of A associated with the ith diagonal entry of lambda as eigenvalue. So, basically what this means is that the columns of S are essentially eigenvectors of this matrix A and the diagonal entries of lambda are the eigenvalues of this matrix A.

And of course, since by definition, if A is similar to a diagonal matrix, this S is a nonsingular matrix. So, since S is nonsingular, they are or A has n linearly independent eigenvectors. So, that completes the proof. So, basically this result that is diagonalizable if and only if it has an n linearly independent eigenvector.

So, in principle this is a way to diagonalize a matrix, if it is indeed diagonalizable. So, all you need to do is to find eigenvalues and eigenvectors of the matrix A and then you check whether the eigenvectors linearly independent and if there are n linearly independent eigenvectors then you can just stack those eigenvectors together and that gives you the matrix S which is a diagonalizing similar, similarity matrix. So, this is one way to find how to diagonalize a matrix A.

So, again coming back to our previous example, this matrix 0 1, 0 0 is not diagonalizable. What it means is that it does not have two linearly independent eigenvectors. So, I mean if this was actually diagonalizable then there must be a matrix S such that S inverse times 0 1, 0 0 times S is equal to a diagonal matrix containing the eigenvalues which is the all zero matrix because both its eigenvalues are 0.

But then this implies that if I pre and post multiply by S and S inverse it means that 0 1, 0 0 must be equal to with 0 0, 0 0 which is not possible. So, this matrix is not diagonalizable and in fact, this matrix has only one eigenvector corresponding to lambda equal to 0 which is the vector 1, 0.

So, if I multiply this vector, this matrix by 1, 0 I will get 0, 0. So, that is so, 1, 0 times this matrix is equal to 0 times 1, 0 and so, 1, 0 is an eigenvector of this matrix corresponding to lambda equal to 0 and I cannot find any other linearly independent eigenvector of this matrix A.

So, we see that the number of linearly independent eigenvectors of a matrix that you can find corresponding to an eigenvalue can be less than the multiplicity of the eigenvalue. The multiplicity of the eigenvalue 0 in this matrix is two, it is an eigenvalue of multiplicity two, but the matrix has only one linearly independent eigenvector, it does not have two linearly independent eigenvectors corresponding to lambda equal to 0.

So, basically not every n cross n matrix will have a full set of n linearly independent eigenvectors. So, here is, here is another direct consequence of what we just saw if A in C to the n cross n has n distinct eigenvalues then A is diagonalizable.

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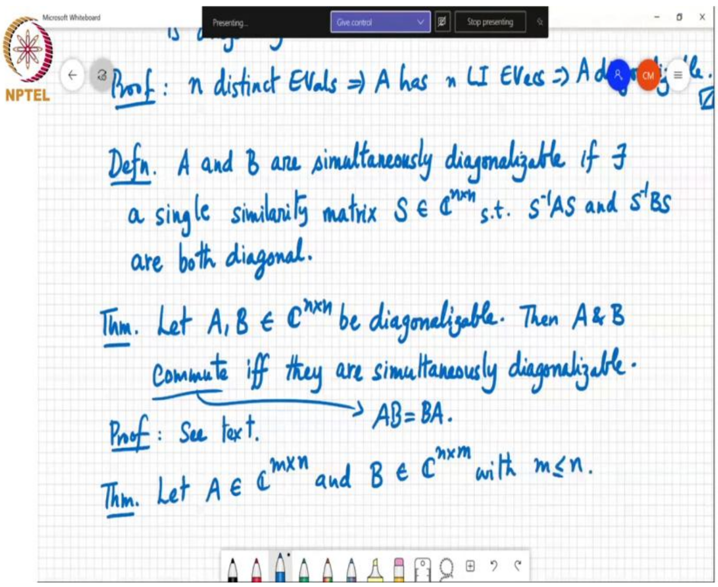

This is an immediate consequence because if it has n distinct eigenvalues then it has n linearly independent eigenvectors and so, we just saw that result A has n linearly independent eigenvectors which implies that A is diagonalizable. Obviously, the converse may not hold that if A is diagonalizable then it need not have distinct eigenvalues and I already gave you the identity matrix as a, as an example.

It does not have the distinct eigenvalues, all its eigenvalues are equal to 1 but it is of course diagonalizable. So now, you have seen that this similarity matrix that say diagonalizes a matrix may not be unique. So, a related question is, is it possible that there is a single matrix, similarity matrix that will diagonalize both two different matrices. So, we say that A and B are simultaneously diagonalizable if there exists a single matrix S such that S inverse A S and S inverse B S are both diagonal.

What does in words means is that there is a basis in which the representations of both these linear transforms are diagonal. So, both these are diagonal, they need not be the same matrix the same diagonal matrix, all you need is that S inverse A S and S inverse B S are both diagonal.

In other words A and B need not be similar to each other, but they can be simultaneously diagonalizable. So, here is one result, which I will not prove, but nonetheless true is let A and B be n cross n matrices and suppose these two matrices are diagonalizable, then A and B commute that means A B equals B A if and only if they are simultaneously diagonalizable.

The proof is not difficult, it is just somewhat long and so, I do not want to do that in class. It (())(18:21) an induction argument on the matrix size to show that they commute if and only if they are simultaneously diagonalizable. There is one other important result which we will use quite a lot in this course and it is also very useful result.

So, that is this result; so let A in C to the m by n, now no longer square matrices and

Student: Hello Sir.

Professor: Yes?

Student: Sir, what do you mean by A and B will commute even there?

Professor: I just said it in words but.

Student: Okay, okay sir.

Professor: Okay?

Student: Yes sir.

Professor: With less m than or equal to n.

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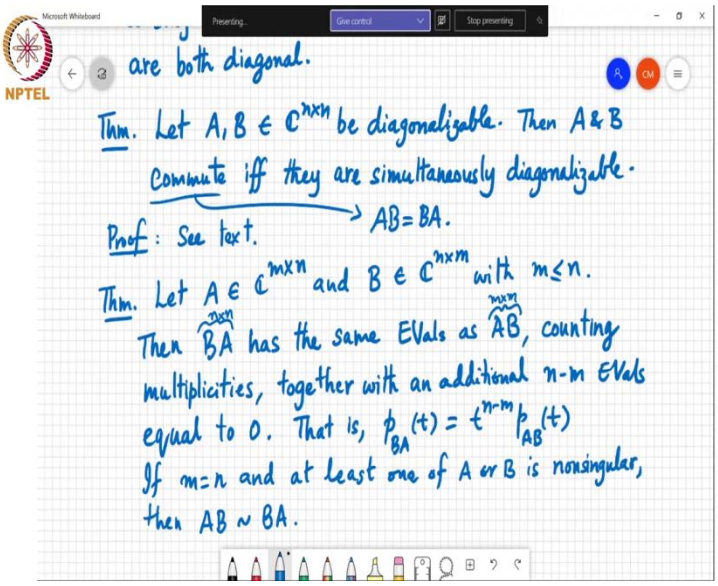

Then B A has the same eigenvalues as A B counting multiplicities, together with an additional n minus m eigenvalues equal to 0. So, another way to say it is that, that is if I look at the characteristic polynomial P BA of t that is equal to t power n minus m P AB of t. So, this n minus m extra zeros because t equal to 0 is a repeated eigenvalue 0 of this polynomial, with repetition n minus m times.

So, there are additional n minus n zeros. Of course, the matrix B A is of size (n by m) n by n and this is a matrix of size m by m and m is smaller than n, less than or equal to n and so, this has more number of eigenvalues it has exactly n minus m additional eigenvalues more on top of whatever are the eigenvalues of AB, but the, there are the m eigenvalues of AB will also appear as m eigenvalues of BA and in addition BA will have n minus m extra eigenvalues which are all going to be equal to 0.

If m equals n and at least one of A or B is non-singular, so they are both square matrices of the same size then A B is similar to B A. So, this is one example where from this result we see that if I apply this result to the case where m equals n, then A B and B A will have the same eigenvalues, but we have already seen that two matrices could have the same eigenvalues but not be similar to each other.

But what this is saying is that if at least one of these two matrices is nonsingular, then these two matrices will be similar to each other. So, this is one example, where having the same eigenvalues is actually sufficient in for the matrices to be similar but provided two other conditions are satisfied; namely that m equals n and at least one have A or B is nonsingular.

Of course, the thing is we are only comparing, checking whether AB and BA are similar. So there is no time to see the proof of this in this class. So we will see the proof in the next class, which will be Wednesday of next week. That is all for today.