Matrix Theory
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Similarity

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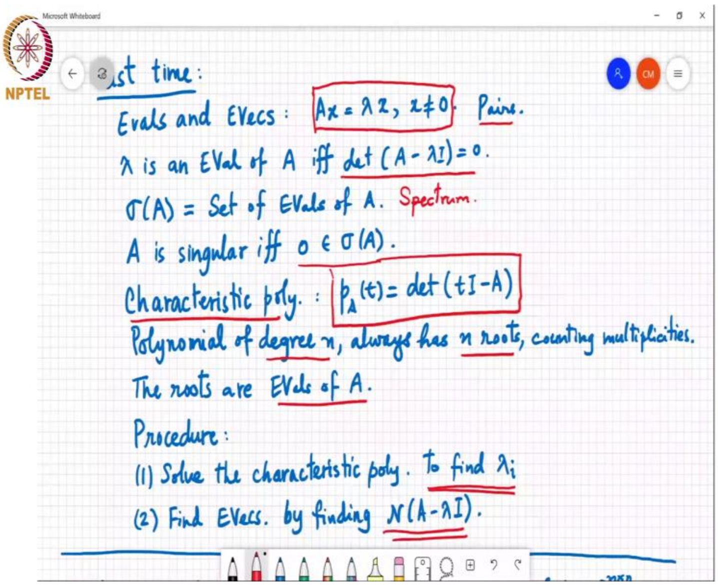

So, the last time we were looking at eigenvalues and Eigenvectors the basic equation is Ax equals lambda x and we need a solution such that x is not equal to 0. We also saw that this always occurs in I mean by definition it occurs in pairs, that is an eigenvalue has an associated eigenvector with it.

And further if Ax equals lambda x, A minus lambda times the identity matrix times x is equal to 0 which is a homogenous set of equations, which means that A minus lambda i is a singular matrix which in turn means that the determinant of the matrix must be equal to 0. So, from that we deduce that lambda is an eigenvalue of A if and only if determinant of A minus lambda i equals 0.

We also defined sigma of A to be the set of Eigenvalues of A and therefore A is singular if and only if 0 is in the spectrum of this matrix A. So, this is called a spectrum and we defined the characteristic polynomial as pAt of (())(1:34) polynomial of degree n and it always has n roots, counting multiplicities and these n roots are the Eigenvalues of A.

So, this so we have a procedure of how to find Eigenvalues, we first solve the characteristic polynomial or we find the roots of the characteristic polynomial and we find that gives us what lambda i's are. And then we find eigenvectors by finding the null space of A minus lambda I.

So, this is a procedure that will work reasonably well for small dimensional systems, but if you have very large matrices then you will have to use other methods to find Eigenvalues and eigenvectors. So, that is basically a short recap of what we did in the, what we saw in the previous class.

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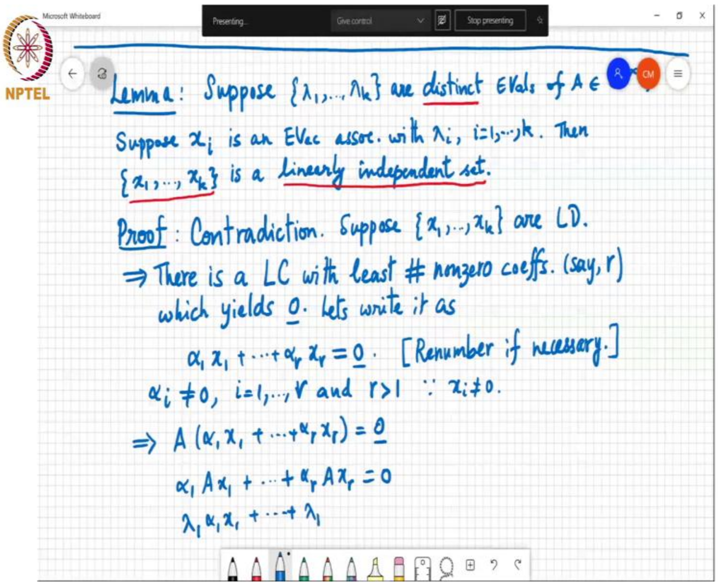

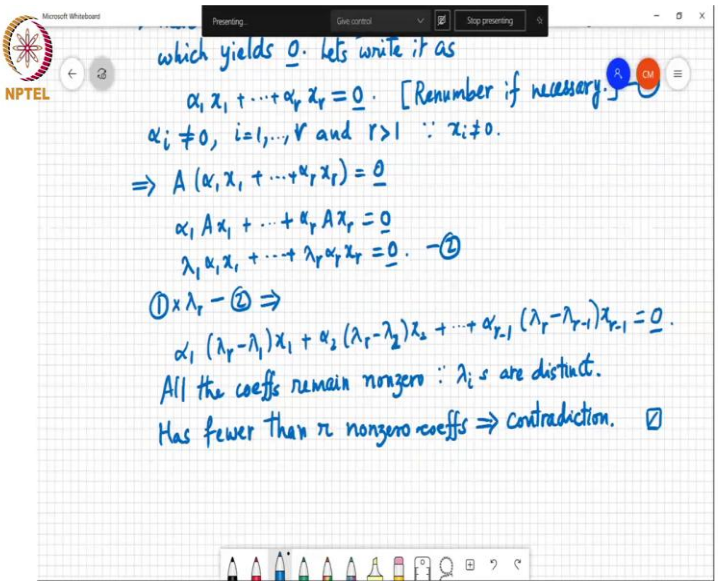

Today, we will continue this discussion and we will also discuss about one very important concept called similarity. So, in the last class I stated this result that if A matrix has distinct Eigenvalues, then the associated Eigenvectors will be linearly independent and so here is a lemma that essentially makes this point.

So, if lambda 1 to lambda k are distinct Eigenvalues that means that none, no two of these are equal of this matrix A, then corresponding to each distinct Eigenvalue there is at least one non-zero Eigenvector, non-zero vector which is an Eigenvector. And so, suppose x i is an Eigenvector associated with lambda i, i is 1 to k. So the k vectors, then these k vectors x1 to xk form a linearly independent set.

So, let us show this. This is a somewhat interesting proof. So, I thought I will just go through that with you. So, the proof is by contradiction. So, suppose it is not true and instead these k Eigenvectors are actually a linearly independent set. So, they are linearly dependent then what it means is that there is a non-trivial linear combination of these k vectors which will give us the 0 vector.

And in fact, one can find a minimal linear combination which will give us the 0 vector. So, that implies there is linear combination with the least number of non-zero coefficients, say r of them, which yields 0, 0 vector. So we will write that as so let us write that as say alpha1 x1 plus plus alpha r xr equals 0.

So, what I have done is I have assumed that it is the first r vectors here that gives you the least number of non-zero coefficients, which will give us the 0 vector, but that is okay, because I can always reorder or renumber these vectors if necessary. So, the point is that all these alpha i's are not equal to 0.

And further r is at least equal to 1, at least is greater than or equal to 2 or it is greater than 1. So, basically, what I want to say is alpha i is not equal to 0, i equal to 1 to r and r is greater than 1, because x i is not equal to 0. So you cannot just take one vector and find the non-trivial (()) (7:07) you will have to use at least two vectors and you are using some r vectors and all of these coefficients are non-zero.

So, all we do now is to multiply, pre-multiply this by A, so A times alpha 1 x1 plus etcetera plus alpha r, xr is equal to 0, because A times 0 vector is just 0. But the left hand side is equal to A alpha one times Ax1 plus etcetera plus alpha r Axr equal to 0, which means that Ax1 is equal to lambda 1 x1, lambda 1 x1 plus etcetera, plus lambda r alpha r xr equals 0.

So, now we can multiply, so I will call this equation 2, I call this equation 1. So I will multiply equation 1 by let us say lambda alpha sorry lambda r 1 times lambda r minus 2, keep the right hand side as the 0 vector. So, the right hand side remains 0, but if I multiply this by lambda r, and then I subtract this, then I will get alpha 1 lambda 1 sorry lambda r minus lambda 1 x1 plus alpha 2 lambda r minus lambda 2 x2 plus etcetera plus alpha r minus 1 lambda r minus lambda r minus 1 xr minus 1 and the last term is lambda r alpha r xr here and lambda r alpha r xr here, so they cancel so this is equal to 0.

And since these lambdas are distinct, all these are, these coefficients will remain non-zero. But then what we have done now is we found a linear combination involving only r minus 1 of these vectors. But we started with the assumption that the this alpha 1 x1 plus etcetera up to alpha r xr is the least number of non-zero coefficients required to get the 0 vector. So it contradicts the, it is that sort of this has fewer than r non-zero coefficients which is a contradiction. So, now we can move on to another topic, which is that of similarity.

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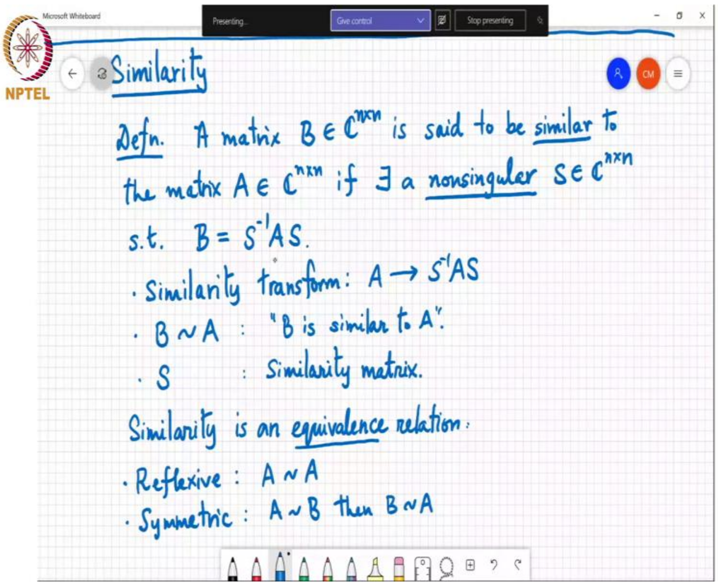

So, we will start by defining what this is, so B, a matrix B is said to be similar to the matrix A in C to the n cross n if there exists a nonsingular S in C to the n cross n such that B equals S inverse AS. So, this transformation S inverse AS applied on A is called a similarity transform and what it really represents is a change of basis of a linear transform.

So, if S represents a change of basis matrix, so given a linear, set of linear equations, say y is equal to Ax, if I can write in a new, if I represent x in a new basis as S times z, where z is in the new, is the coordinates of x according to the new basis, then, if I can compute y equal to Ax as y equal to A times S times z and this y is now again in the old coordinate system and so, if I want to transform it back to the new coordinate system, I have to multiply by y by S inverse.

So, S inverse y becomes S inverse As times z. So, S inverse y is like w which is the coordinates of y in the new coordinate system. So, S inverse AS represents the same linear transform as A but in a different basis or a different coordinate system. That is one way to think about this similarity transform. So, this similarity transform is a mapping from say A to S inverse AS.

So, given a (matrix), given a non-singular matrix S, if you can map A to some other matrix S inverse AS and this kind of a transformation is called the similarity transform. And we will also use the notation B tilde A to say that B is similar to A and this matrix S is called the similarity matrix.

So, this similarity is actually what is called an equivalence relation. What we mean by that is that it is reflexive, which means that A is similar to A, of course, I can write A as identity matrix inverse times A times the identity matrix. So, A is similar to A and it is symmetric meaning that if A is similar to B, then B is similar to A. So, if B equals S inverse AS I can write A as S BS inverse. So, there is another matrix such that, so you can call that matrix t which is equal to S inverse then A will be equal to t inverse Bt and so, B is, A is also similar to B.

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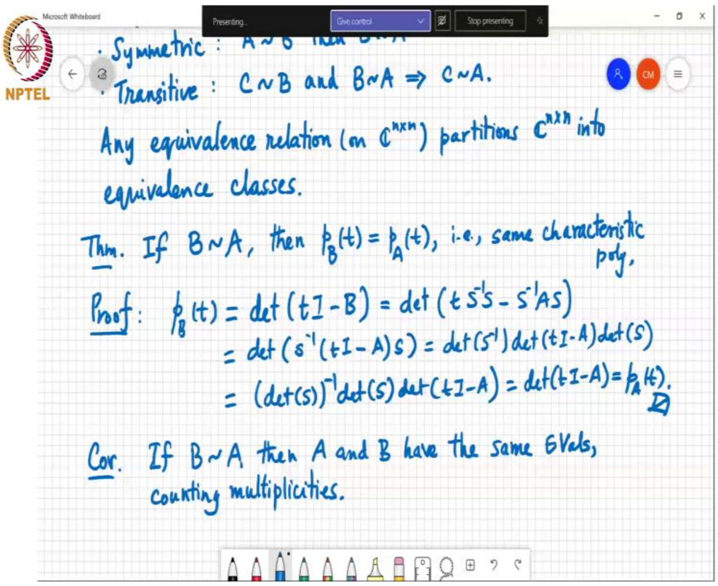

And finally, it is transitive meaning that if C is similar to B and B is similar to A, then C is similar to A. Now, what an equivalence class does is it splits the space of all n cross n matrices into equivalence classes. So, within an equivalence class any pair of matrices are similar to each other.

And if you take one matrix from a given equivalence class and another matrix from a different equivalence class, they will not be similar to each other, you cannot find an S such that B equals S inverse AS. So, equivalence relations that is one property of equivalence relations, so equivalence, so let me put it this way, it is in fact true of any equivalence relation not this particular one, but not only this one, but any equivalence relation on C to the n cross n partitions C to the n cross n into equivalence classes.

So, any pair of matrices in the same equivalence class are similar to each other and any pair of matrices coming from different equivalence classes are not similar to each other. So, basically you can ask what properties two matrices in a given equivalence class share and in fact, they share many, many properties and this is what we are going to study in some detail.

So, the first thing that, the first thing result about what they share is that they share the characteristic polynomial. If B is similar to A then pB of t equals pA of t, they have the same characteristic polynomial. This is very easy to show it is essentially a couple of lines proof.

So, pB of t by definition is the determinant of tI minus B which we can write as determinant of t S inverse S identity matrix is S inverse S and S here is this similarity matrix that will take A to B. So, minus B is S inverse AS and what I can do now is I can pull out S inverse from the left and pull out S on the right. So, this is equal to determinant of S inverse tI minus A times S.

But we know the determinant of A B equals determinant for A times determinant of B. So, this is equal to determinant of S inverse determinant of tI minus A determinant of S. But determinant of S inverse is 1 over the determinant of S. So, that is equal to determinant of S inverse determinant of S determinant of tI minus A, then these two obviously cancel which is equal to determinant of tI minus A which is equal to pA of t.

So, a corollary to this is that if B is similar to A or A, B are similar matrices, then A and B have the same Eigenvalues counting multiplicities. So, they not only have the same distinct Eigenvalues, but also the number of times the Eigenvalue appears as an Eigenvalue of A is the same as the number of times it appears as an Eigenvalue of B. So, now question, is the converse true if two matrices have the same Eigenvalues will they be similar?

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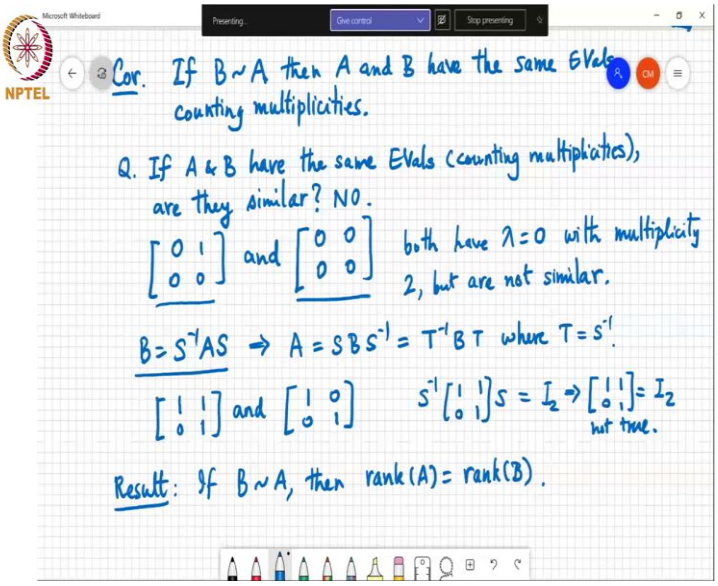

Student: Yes, yes sir.

Professor: So, who said yes?

Student: Sir, Dhruv.

Professor: Dhruv, okay. If A and B anybody else have an opinion on this? Counting multiplicities, are they similar?

Student: Sir, I do not think it is compulsory.

Professor: Why?

Student: Sir, if they have same Eigenvalues, then they will have same characteristic polynomial but that is determinant of tI minus A is equal to determinant of tI minus b. But, even if the equivalence relation does not hold for different values of t and B not equal to (())(24:50), it can be, they can be similar. It is not.

Professor: So, here is a very simple argument, if I consider this is 0 0 0 and 1 here, these matrices are not equal. But what are the eigenvalues of this matrix, what are the eigenvalues of the all 0 matrix?

Student: Both of them (())(25:40) 0 only sir.

Professor: There are these two matrices, so one thing you can keep in mind is that if a matrix is triangular, then the diagonal entries of the matrix are the Eigenvalues of the matrix that is easy to show and you should also try that out for yourself and convince yourself this is true that if the matrix is upper triangular, the diagonal entries are the Eigenvalues.

So, for matrices like this, which are upper triangular, you can just read off the diagonal entries those are the Eigenvalues. So, both these matrices have 0 comma 0 as their two Eigenvalues, but they are not similar, why are they not similar? Because if there was an such that S inverse, this matrix times S was equal to the all 0 matrix, where S is a non-singular matrix, you can simply pre multiply and post multiply by S and S inverse.

Then you will get an absurdity that 0 1 0 0 equals the all 0 matrix. So it is not possible that these two matrices are similar. So, they are not similar although they have the same Eigenvalues. So, the answer to this question is no.

Student: Sir.

Professor: Yes?

Student: Sir you said, I was looking into that that if B is similar to A then you write B equal to S inverse AS right? So, if we multiply S and post multiply S inverse then we get A equal to S BS inverse?

Professor: Yes.

Student: But we also know that if A similar to B then B is similar to A, so from there, we can write A equal to S inverse BS?

Professor: No, not throw the same similarity matrix that is important. So, this is what you said is correct. If B is S inverse AS, I can also write this as A equals SB S inverse, S and S inverse are not the same matrix. So I can write this as T inverse B T, where T equals S inverse.

So, actually the, when we say two matrices are similar, depending on the direction in which I want to execute the similarity transform, the matrix S, means the matrix S depends on the direction in which I want to execute the similarity transform. So I can write, so for example, so that is why when we say B is similar to A, what we mean is that there exists an invertible S such that B equals S inverse AS you write it like this.

But of course, it also means that A is similar to B which means that there is a different matrix T such that A is equal to T inverse B T, but that T is not the same as S. T is actually equal to

S inverse. And in fact, this matrix S, this may not be unique, we will see that later. There are possibly many different S's such that B equals S inverse AS.

Student: Okay, sir. Thanks.

Student: Sir?

Professor: Yes.

Student: If two matrices have distinct I mean, same Eigenvalues which are not 0. Then will they be similar always?

Professor: So that is something to think about, we will see many more results coming up and then the answer will become obvious. Just hold on.

Student: Okay, sir. Thank you.

Professor: So a direct consequence of this, but maybe I can just answer this question in this way. So, suppose I consider the matrix 1 1 0 0, sorry, now clearly these two matrices still have the same Eigenvalues both are equal to 1. But clearly, it is also not possible that there is a matrix S such that this identity matrix equals S inverse times this matrix times S.

So, if there exist such a matrix, now this is the identity matrix 2 cross 2 implies that if I now pre multiply and post multiply by S and S inverse, I should have 1 1 0 1 equals the identity matrix, which is not true. So, it is possible that the matrix has non-zero Eigenvalues and the Eigenvalues are the same, but the matrices are not similar to each other. But what if the Eigenvalues were non-zero and distinct, that we will see.

So, since similar matrices have the same characteristic polynomial, they have the same number of non-zero Eigenvalues counting multiplicities and the number of non-zero Eigenvalues equals the rank of the matrix. And so, we have the result, that is one way to think about it, but I will tell you another way. The other way to think about it is if B equals S inverse AS, multiplying a matrix by a non-singular matrix does not change its rank.

And so as a consequence, left or right multiply, left and right multiplying by the non-singular matrix retains the rank of the matrix. So, if B is similar to A, then A and B may be I just write it, so similar matrices have the same Eigenvalues counting multiplicities and similar matrices also have the same rank.