Matrix Theory

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The Characteristic Polynomial

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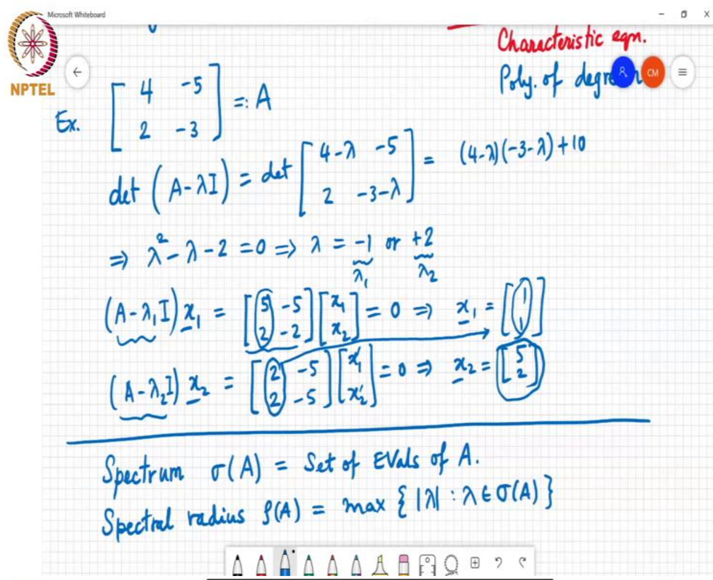

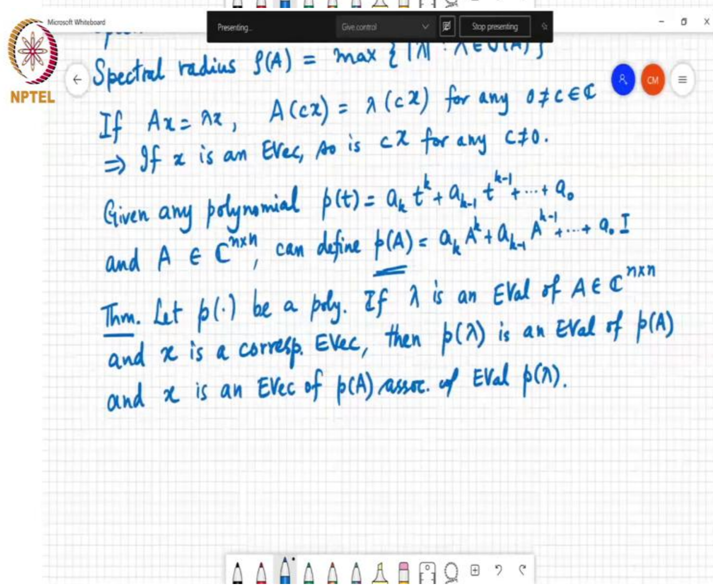

So, the spectrum of a matrix sigma of A, here is the set of Eigenvalues of A. And again the spectral radius rho of A is the largest magnitude Eigenvalue, which is equal to max of mod lambda, where lambda belongs to sigma of A. So, this is just writing the spectral radius in terms of sigma of A. Another small point to note is that, if Ax equals lambda x, then A times cx is equal to lambda times cx for any 0 not equal to c belonging to some complex space.

And so, basically if x is an Eigenvector, so is cx for any c not equal to 0. So, now, the other thing is, which will sort of lead us to the characteristic polynomial and studying it and, is that given any polynomial. So, let us say p of t ak t power k plus ak minus 1 t power k minus 1 plus a0, this is a polynomial in T. And A in C to the n cross n. So, we can define p of A, this is the polynomial evaluated at a matrix valued point A.

So, we will define it as ak times A power k plus a, ak minus 1 power k minus 1 plus a0 identity matrix otherwise you cannot add it to these terms. So, this is how we define the polynomial evaluated at a matrix. Now, this p of A it will always have the same Eigenvectors as the matrix A itself.

And the polynomials the Eigenvalues of p of A, p of A is an n cross n matrix. Because this is n cross n, this is n cross n, this is n cross n, when you add up all these terms you will get an n cross n matrix. And its Eigenvalues are closely related to the Eigenvalues of A itself. And so, that is the following theorem.

Student: Sir?

Professor: Yes?

Student: Sir, the screen is not updated sir.

Professor: Can somebody else confirm?

Student: It is getting updated for me sir, (())(04:06).

Professor: So, yeah maybe, if it just wait for a second if it does not update, then just drop off and join again, you should see the new screen, be a polynomial. And if lambda is an Eigenvalue A and x is a corresponding Eigenvector, then p of lambda is an Eigenvalue of p of A and x is an Eigenvector of p of A associated with the Eigenvalue p of lambda. So, how do we show this? This is very simple.

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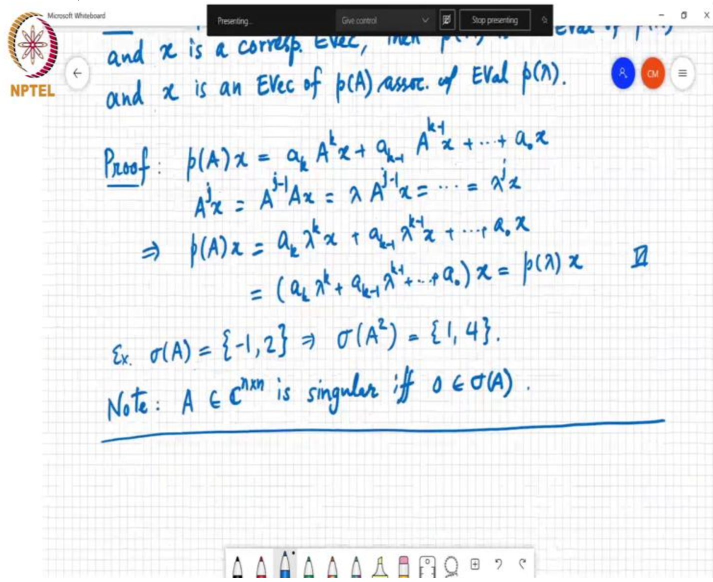

So, if I take p of A times x, this is going to be equal to ak A power k x plus ak minus 1 A power k minus 1 x plus etc plus a0 times the identity times x which is equal to x. Now, of course, A power j times x is equal to A power j minus 1 times Ax which is equal to lambda A power j minus 1 times x. And so, we can keep going like this, go to j minus 2 and so on. And so, this is going to be equal to lambda power j times x.

So, that means that p of A times x is equal to. So, it becomes lambda power k times x here and lambda power k minus 1 times x here and so on. So, it becomes ak lambda power k x plus ak minus 1 lambda power k minus 1 x plus etc, plus a0 times x, and x is multiplying all these terms so, I can pull it out, and write this as ak lambda power k plus ak minus 1 lambda power k minus 1 plus etc plus a0 times x and this is nothing but p of lambda.

This polynomial evaluated at the number lambda times x. So, we see that p of A times x equals p of lambda times x, this is an n cross n matrix, this times x equals, this is a scalar. So, p of lambda times x which means that, p of lambda is an Eigenvalue of A and the corresponding Eigenvector is x, p of lambda is an Eigenvalue of p of A. And the corresponding Eigenvector is x. So, based on this for example, if sigma of A were the two values say minus 1 and 2, then what is sigma of A square?

A is a two cross two matrix, it has two Eigenvalues, minus 1 and 2. So, sigma of A squared will be lambda 1 squared and lambda 2 squared, which is equal to 1 and 4. And as I already mentioned to you earlier, A is singular if and only if 0 is part of sigma of A. Because if A is singular if and only if ax equals 0 for some nonzero x, which means, which is true if and only if lambda equal to 0 is an Eigenvalue.

Student: Sir?

Professor: yes?

Student: Sir, in the above example, the polynomials bx equals x square, right?

Professor: Correct.

Student: Okay, thank you.

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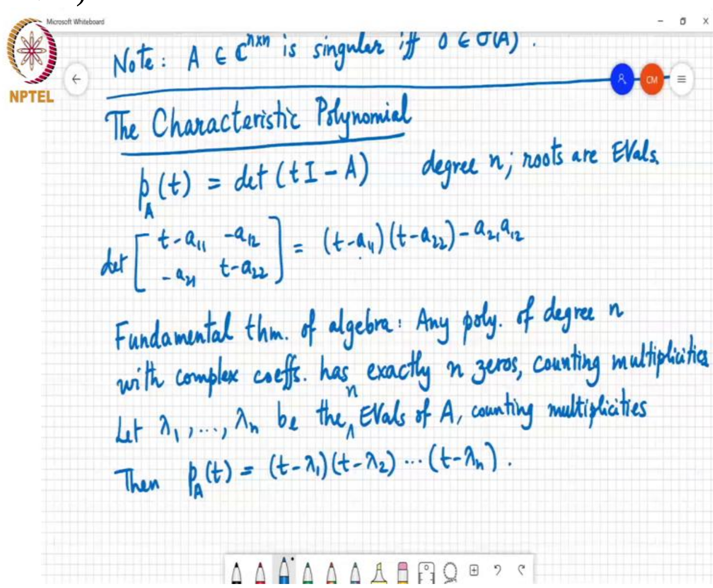

Professor: Okay, so, now let us discuss this characteristic polynomial in some more detail. So, this basically we answer questions about the Eigenvalues of A and for example, how many Eigenvalues does an n cross n matrix have, and when will it have an linearly independent Eigenvectors. So, those are the type of questions we want to answer. So, the characteristic polynomial will define it in a slightly different way from what I said earlier, it is just a change of sign.

We will define it to be pA of t which is equal to determinant of t I minus A. This is a polynomial of degree n and its roots are the Eigenvectors, are Eigenvalues. So, why is this a polynomial of degree n? It is, because if I think of it, I mean, I will just write it for the 2 cross 2 case, so, that you see what happens. This matrix will look like t minus a 1 1 minus a 1 2 minus a 2 1 and t minus a 2 2.

And if I take this determinant, it will be equal to t minus a 1 1, t minus a 2 2 minus a 2 1 a 1 2. And so, you can see that this is a quadratic polynomial. So, in the n cross n case by similarly, writing it out you will see that it has to be a polynomial of degree n and the coefficient of t is always 1. So, when I take the determinant of this I cannot suddenly get a polynomial whose degree is n minus 1.

It will always be of degree n. So, there is a, what is known as the fundamental theorem of algebra. We will not prove this here, we will take it on faith, but it says that any polynomial of degree n and complex coefficients, or with complex coefficients has exactly n zeros counting multiplicities.

So, multiplicity just simply means, how many times a particular lambda occurs as a 0 of pa, pa of lambda. So, if you count multiplicities then there are exactly n Eigenvalues. So, for example, the n cross n identity matrix has exactly n Eigenvalues, all are equal to 1. Now, because.

Student: Sir.

Professor: Yes?

Student: Sir commerce coefficients means real (())(14:00), no?

Professor: Yes.

Student: Okay.

Professor: Now, suppose we let lambda 1 through lambda n b the Eigenvalues of a, where repeated Eigenvalues are okay, they are just counted as lambda 1 lambda 2 to n. So, so, let lambda 1 lambda n be the Eigenvalues of A counting multiplicities, the n Eigenvalues of A counting multiplicities. Then, these are the zeros of the characteristic polynomial, so we can write pA of t equals t minus lambda 1, t minus lambda 2, t minus lambda n.

So, this is a factorized form of the characteristic polynomial, and this polynomial is identical to this determinant of t i minus a. So, in particular, if I look at the coefficient of t to the n minus 1 in this polynomial, that is just the sum of the Eigenvalues. And similarly, if I look at the coefficient of t power 1 here, you can see that it is just going to be t power 1 will be minus a 1 1 minus a 2 2. And the coefficient of t power n minus 1 here will be minus of lambda 1 plus lambda 2 plus etc up to lambda n.

So, those two coefficients must match if the polynomials are, if we say two polynomials are equal, it means that the coefficients of t power n, t power n minus 1 all the way down to t power 0 must match, t power n obviously matches because the coefficient of t squared is 1. And here also when I take the coefficient of t squared in the 2 (())(16:19) in the lambda 1 lambda 2 case, that will also be equal to 1.

But basically, if I look at what happens to the coefficient of t to the n minus 1 here, it is going to be equal to a 1 1 plus a 2 2, which is a coefficient, which is equal to, which must equal lambda 1 plus lambda 2, which is the coefficient of t to the n minus 1 here. So, basically, we and similarly, if I take the constant term here, that is the product of these Eigenvalues and the constant here, here is going to be, all I have to do is that the constant term will be obtained by setting t equal to 0 in this pA of t.

And so the constant term is just nothing but the determinant of the matrix A. So, the product of the Eigenvalues equals (())(17:10). And the sum of the Eigenvalues equals the trace of the matrix.

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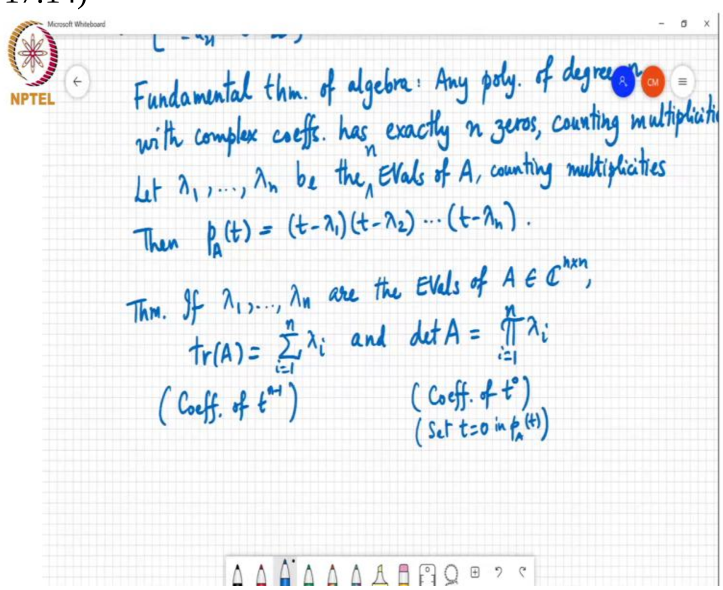

So, that is the following theorem. If lambda 1 through lambda n are the Eigenvalues of A. So, the determinant of A is the product of the Eigenvalues, and the trace of A is the sum of the Eigenvalues.

So, just to maybe make it clear, if you look at the if I set t equals 0 here, what I get is minus lambda 1 into minus lambda 2 up to minus lambda n, which is minus 1 power n times the product of the Eigenvalues. If I set t equals 0 here, I will get determinant of minus A, which is again going to be equal to minus 1 power n times determinant of A. Those two must be equal, and therefore, minus 1 power n cancels with minus 1 power n.

And what you are left with is determinant of A is the product of the Eigenvalues. And similarly, if I look at the coefficient of t power n minus 1 here, it is going to be minus lambda 1 minus lambda 2 up to minus lambda n. And here if I look at the coefficient of t power n t power 1, I have a 1 1 plus minus of a 1 1 plus a 2 2, which is nothing but the negative trace of the matrix. So, negative trace of the matrix is the same as the negative sum of the Eigenvalues.

And so, we have this part here. So, for the sake of completeness, I will just say here coefficient of t to the n minus 1 and this is obtained by looking at the coefficient of t power 0 or the constant term, or the other way is set t equal 0 in. So, there couple of others results. Before that, maybe I will just write this one thing out, which is what is the procedure to find Eigenvalues and Eigenvectors?