Matric Theory
Professor. Chandra R. Murthy
Department of Electrical Communication Engineering
Indian Institute of Science, Bangalore
Basis, dimension

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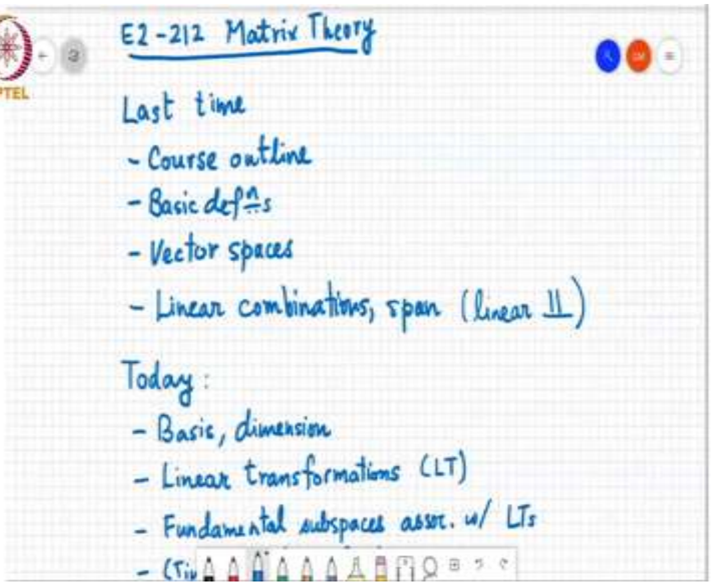

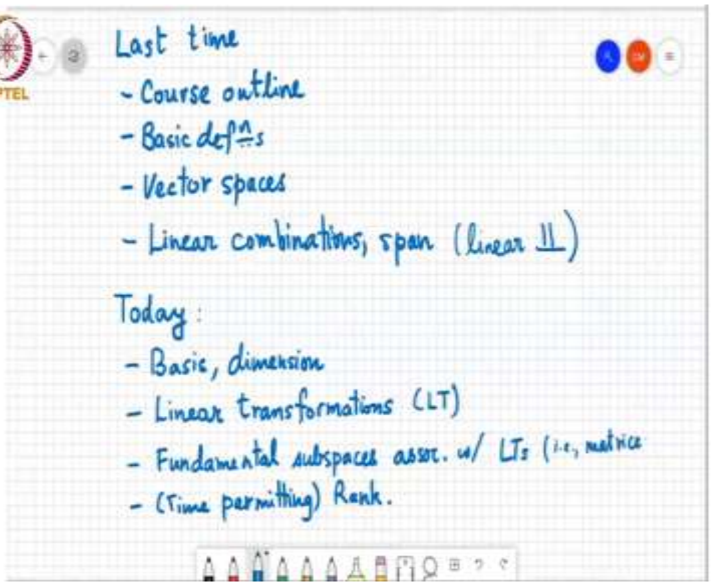

So, the last time we went to a course outline and we went through the basic definitions of matrices, matrix multiplication, matrix addition, so then we started discussing vector spaces, and in particular we discussed about, so we were discussing about linear combinations and in particular we discussed about linear independence.

So, linear independence is a very-very central concept to matrix theory and linear algebra and so I will just reiterate that a set of vectors are linearly independent if the only linear combination of the set of vectors that gives you the 0 vector is the all zero combination, that is all the coefficients must be equal to 0, that is the only way you can reach the all zero vector, then we say that that set of vectors are linearly independent otherwise they are linearly dependent that means there is a non-trivial linear combinations of those vectors.

Some of the coefficients are non-zero, but which when added together with that weighted combination gives you the 0 vector. So, today we will discuss many related concepts, specifically basis, dimension and last time somebody asked me about linear transformations and I was telling you that linear transformations are actually equivalent to matrices and matrices are equivalent to linear transformations.

So, the question was how do we define what is a linear transformation which then is related to a matrix, I am going to talk about that. Then I will talk about some fundamental subspaces associated with linear transformations or matrices and if time permits we will also discuss about the notion of a rank of a matrix.

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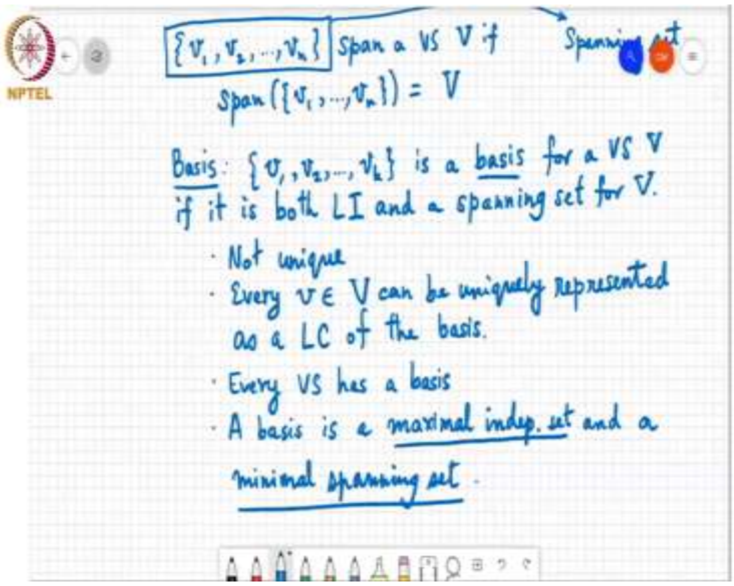

Now, to recall we say that, if you remember we just put down the last thing we discussed the last time in the previous class. So, a set of vectors V1, V2 up to Vn span a vector space V if V1 through V1 is equal to V. That means that every V, every vector in capital V can be written as a linear combination of V1 through Vn. So, basis, a set of vectors V1 through Vk is set of be a basis for a vector space V, which both linearly independent and spans the set V.

So, in this case we call this as spanning set, a vector space V if it is both linearly independent and a spanning set. So, some comments about the basis are in order. First of all basis is not unique, you can define many different basis for particular vector space and every V in this set spanning set V can be, every V in this vector space capital V can be uniquely written as a linear combination of the basis.

Obviously this is not true if you add or delete vectors from the basis. If you add vectors to the basis there are more than one way in which you can represent a vector that belongs to the vector space. If you delete vectors from the basis there are points in V which cannot be represented as a linear combination of the remaining set.

So, also every vector space has a basis and we say that, another way to say what I just said is in fact, the two popular phrases; basis is a maximal independent set and minimal spanning set. So, maximal independent set meaning that this is the maximum number of linearly independent vectors that you can pull out from this vector space V.

In other words, if you take a basis and you take any other vector from V and add it to that basis, that set of vectors now become linearly dependent and it is a minimal spanning set in the sense that if you take away any vector from the basis, then it can no longer span the vector space, there will be some points on the vector space, which cannot be represented as a linear combination of the remaining set.

So, another way to say this is that the set of, independent set of vectors in a vector space is a basis if and only if no proper superset of it is linearly independent. Also a set that spans V is a basis if and only if no proper subset of it still spans the vector space. These are things I have already said, I am just saying it in another way.

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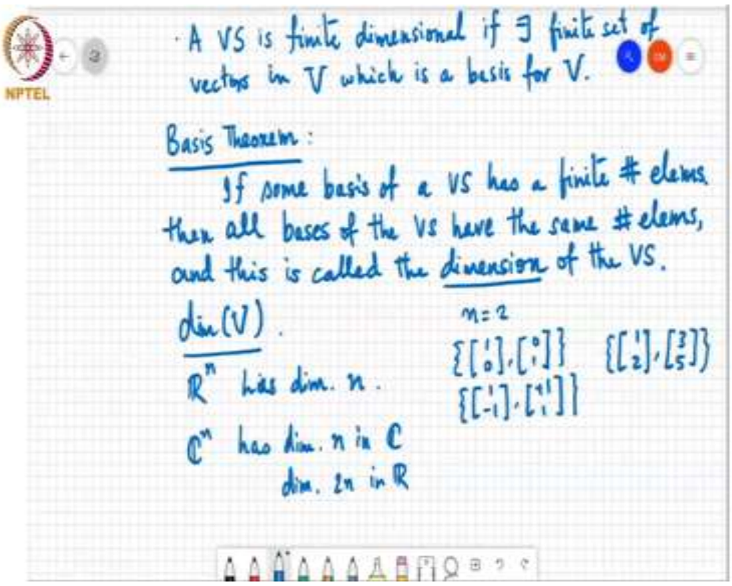

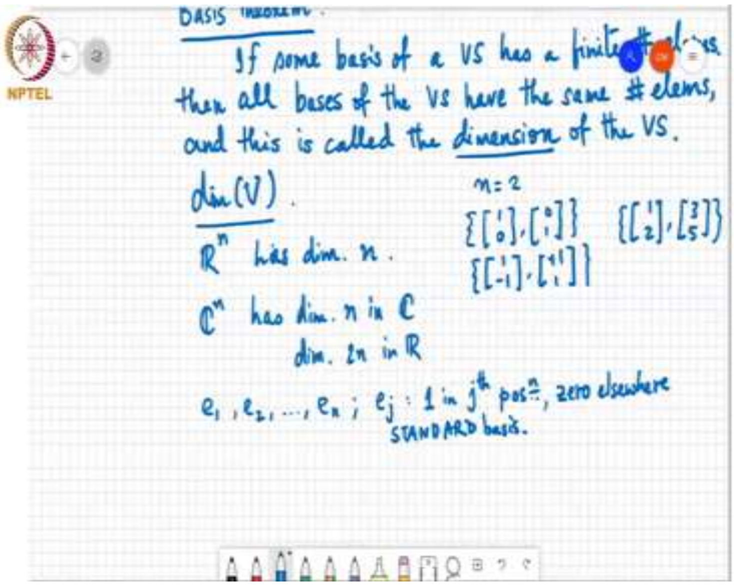

Now, very, another concept which is related to the basis is that a vector space is said to be finite dimensional if there exist a finite set of vectors in V which is a basis for V. So, in this course we will completely and exclusively look at finite dimensional vector spaces, if the basis for a vector space does not have a finite number of vectors in it, then we call the vector space infinite dimensional.

So, for example, if you take the set of all polynomials in one variable say x, then that is a infinite dimensional vector space because you can have x, x squared, x cube and so on going all the way up to infinity, if you take the set of all polynomials that you can define in x, then this is, this spans, this forms a vector space, which is infinite dimensional. But like I said in this course we will focus on the finite dimensional vector space.

Most of the results for finite dimensional vector space actually do extend to infinite dimensional vector spaces, but in some cases you will have to make some extension arguments which is beyond the scope of this course. So, here is one result related to basis. It is called the basis theorem. So, what do you think it says? Anybody wants to guess or just anybody know what the basis theorem says?

Student: Sir, from a number of the vectors present in the basis is actually the dimension of the matrices?

Professor: Yes, exactly, so that is the theorem. So, what it says is that if some basis of a vector space has a finite number of elements, then all basis of the vector space have the same number of elements and this is called the dimension of the vector space, it is denoted by dim of… So, essentially if you find a basis for a vector space and I find a basis for a vector space, the basis that you found could be different from the basis that I found.

But the number of vectors that you have used to form the basis is going to be exactly the same as the number of vectors I have used to form the basis. So, just to illustrate this idea, there could be different basis but they will have the same number of vectors, so first of all, if I take the space R to the n, this has dimension, how much?

Student: n.

Professor: n, so if I take for example, the case where n equals 2, then 10, 01, this is one basis and so if 1 minus 1, and minus 1, 1 and so is 12, 35. All these are basis, they will span R2, and you will see that they all have 2 vectors each.

Student: Sir, second example is not a basis, right, they are linearly dependent.

Professor: Now, it is a basis. Thank you. So, if I take C to the n, this also has dimension n in the field C and it has dimension 2n in the field of real numbers. Of course, if you take this n dimensional real space, this vectors e1, e2, up to en which are like this over here, where basically ej has 1 in the jth position and 0 everywhere else; this is called the standard basis.

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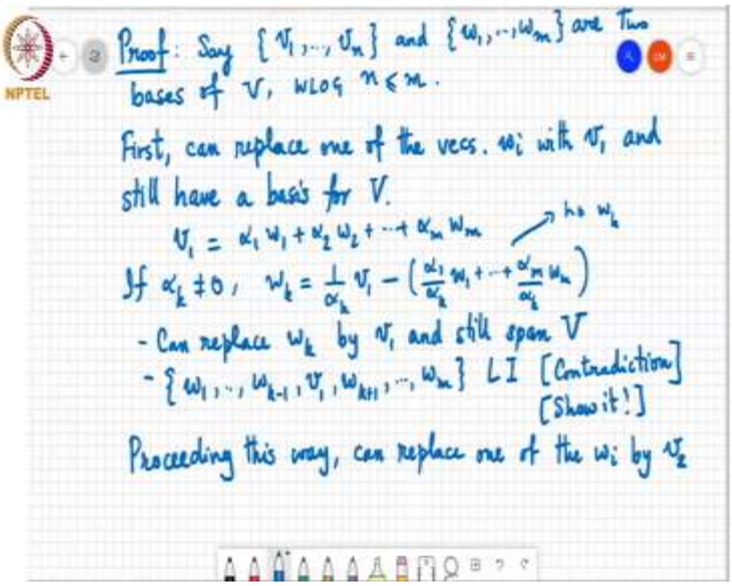

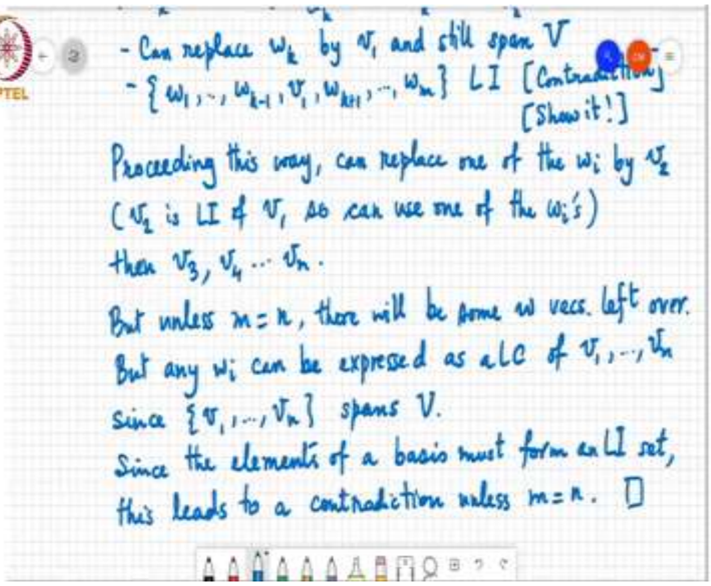

So, now let us prove this result. So, how do we prove result like this that, if some basis has a finite number of elements, then all basis of the vector space have the same number of elements and this is called the dimension of the vector space. So, the proof is by contradiction. So, suppose there are two, I found one basis and you found a different basis and they happen to have a different number of elements.

So, suppose V1 through Vn is one basis and W1 through Wm is a different bases of V and without loss of generality I can assume that n is less than or equal to m, otherwise I can simply switch what I call V and what I call W, so I can assume n is less than or equal to m whiteout loss of generality.

So, as a first step, so basically this has fewer vectors than this. What I am going to do is, I am going to take this bigger set here and replace one of these vectors with say V1 and then I will replace one other of these vectors with V2 and so on, that is what I am going to do. So, first we can replace one of the Wi with V1 and still have a basis for V.

So, why is that true? That is because suppose V1 was equal to alpha 1 W1 plus alpha 2 W2, alpha m Wm, I can always do this because W1 through Wm is a basis for this vector space V and V1 to Vn are also vectors that sit in V. So, if I take V1, it is a vector that is lying in V and so it can always be represented as a linear combination of W1 through Wm.

And obviously, not all these alphas are going to equal to 0, because if I make all these alphas equal to 0 then there the right hand side is 0 and the left hand side is V. So, this is true, so not all alphas are 0. So, if alpha k is one of the guys who is not 0, then Wk can be then written as 1 over alpha k times V1 minus all the other vectors.

So, in this series there is no Wk. I have skipped Wk and all the other terms are here, so I am just rewriting this first equation here. So, basically what this gives is that, notice that these are, this is just a linear combination of all the other Ws so I can replace Wk by V1 and still have a basis over this vector space V.

So, there are two things here, so we can replace Wk by V1 and still span vector space V and second is that this set of vectors that we get W1, Wk minus 1, V1, Wk plus 1 all the way up to Wm are still linearly independent. Why is that true? Simple, if you take a linear combination of these and you get 0, and suppose that is a non-trivial linear combination then all you have to do is to substitute for V1.

It sum linear combination of all these Ws and what that will end up showing you is that there is a non-trivial linear combination of W1 through Wm which gives you the 0 vector and therefore, this set of vectors W1 through Wm are not linearly independent. But that was one of our starting points, that this is the basis meaning that this is a linear independent set that spans V, so you can show by contradiction.

Student: Excuse me?

Professor: I would like you to try to show it, but what I was saying is the simple argument, what you do is you take a linear combination of these vectors W1 through Wk minus 1, V1, Wk plus 1 through Wm and you set it equal to 0. Suppose that there is a linear combination of these vectors which gives you the 0 vector.

And if that linear combination is a non-trivial linear combination, it means that these vectors are linearly dependent. So, you start by saying suppose it is true that I can take some beta 1, W1 plus et cetera to beta m, Wm where there is some beta k which is multiplying V1 and with not all beta is equal to 0, which gives you the 0 vector.

Then, what you do is you substitute for V1 from here, V1 is some non-trivial linear combinations of these Ws you substitute for V1 into that equation involving beta 1 W1 plus etcetera up to beta m Wm equal to 0. And you manipulate that equation a little bit and you end up showing that there is a non-trivial linear combination of W1 through Wm that is also giving the 0 vector, which means W1 through Wm are not linearly independent.

But that is a contradiction because we started by assuming that V1 through Vn and W1 through Wm are bases of V. So, that is the argument. So, you can, so now that you have replaced one of the vectors in this set by V1, think of this as your new basis and you do the same argument and replace one of these vectors with V2. Now, clearly V2 is linearly independent of V1, so you can use one of the other Wi's to replace it.

You do not need to use V1 to replace it because clearly V2, when I write V2 as a linear combination of these vectors here, the coefficient of V1 may or may not be 0, but one of the other coefficients will certainly be non-zero and that coefficient that you use to rewrite it like this and then say you can replace Wk by some Wk prime by V2 is linearly independent of V1 so we can use 1 of the Wi's.

So, then we can do then V3, V4 and so on. But unless m equals n, what we will have then is we will have a set which has V1 to Vn and some of the Wi's and since V1 to Vn span this vector space V, it means these Wi's can be written as a linear combination of these V1 to Vn and which means that this new basis that we found is not linear independent any more. So, there will be some Ws left over.

But any Wi can be expressed as a linear combination of V1 through Vn since V1 through Vn spans V. So, that means that this new set that we have is no longer a linearly independent set and this leaves to a contradiction. So, that is the proof. Any questions?

Student: Yes, sir. While proceeding with, replacing every Vi by Wi, you said that the coefficient of certain Wi is 0 to take another Wi, but while going from V3 to V4 to Vn, if we ran out of the non-zero coefficients, I mean, there is no particular Wi left for which is coefficient is non-zero and I can replace it by V.

Professor: Yes, that is not possible. Because for example, suppose you have replaced V1, V2 and V3 with some of the Wi's and now you have a set which has Wi's and it has V1, V2, V3 in it. Now, you are looking at replacing V4. V4 is linearly independent of V1, V2 and V3, so you cannot express V4 as a linearly combination of this set which contains V1, V2 and V3, and the rest of the Wi's but with the non-zero coefficients being only in V1, V2 and V3.

Because this V4 is linearly independent of V1, V2 and V3, so the coefficients of one of the other Wi's has to be non-zero. Is that clear? And that Wi can be used to replace.

Student: Okay.

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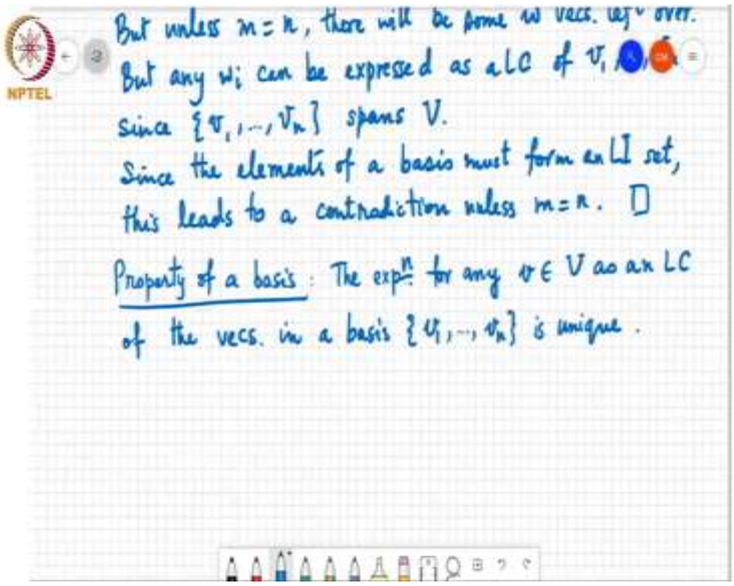

Professor: So, the next thing I want to talk about, just maybe one remark. I think I have already mentioned this, but a property of a basis is that if you take any vector in the vector space V and express it as a linear combination of the vectors and the basis, that linear combination is unique.

So, the next thing I want to talk about, so this is also something you can show by the way. What I am doing right now is really just reviewing some basic concepts from matrix theory that you must have seen in your undergraduate. So, I am not proving all of these results, once we started discussing the new material or the, once we get past all these background material, I will generally try to prove every result that I put down.