Matrix Theory
Professor Chandra R. Murthy
Department of Electrical Communication Engineering
Indian Institute of Science, Bangalore
Errors in Solving System of
Linear Equation

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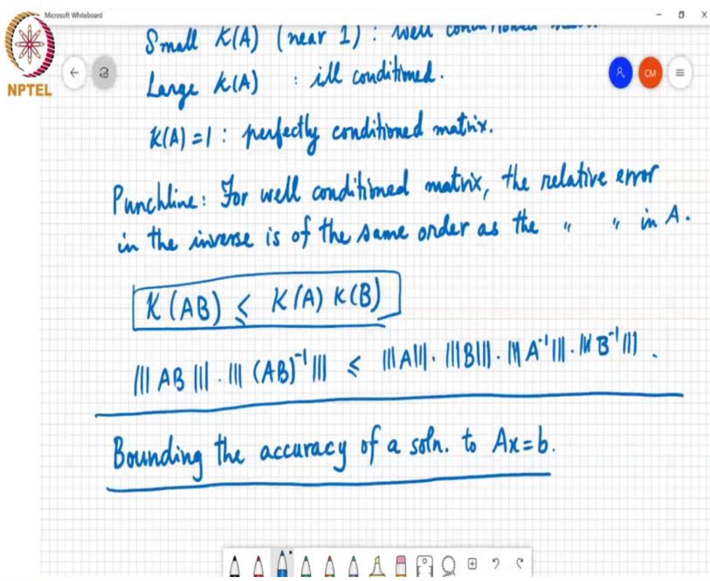

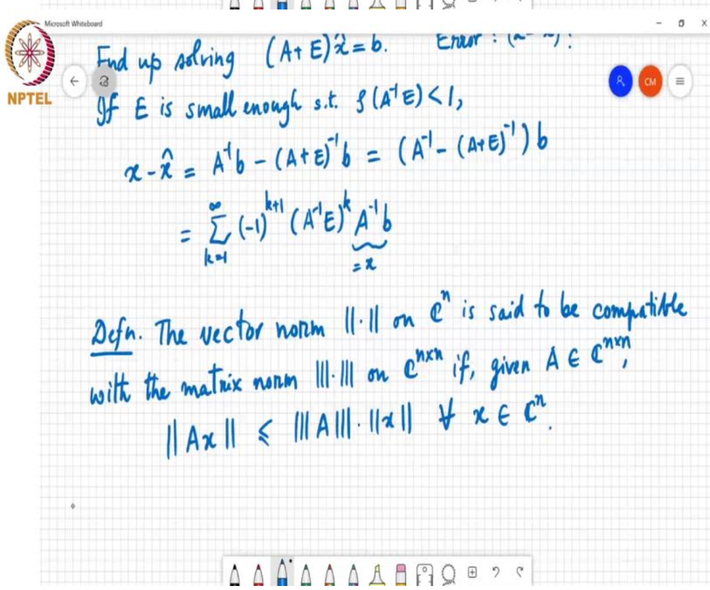

Another very, very important use of linear algebra is in solving systems of linear equations. And so, you are trying to solve the system of linear equations Ax equals b. So, let us consider a square system for now. There are more general things we will discuss later, but for now, we consider a square system. And we want to solve Ax equals b, but due to errors, we end up solving a nearby system of equations, say A plus E x equals b.

So, you can think of it as your knowledge of A is a little noisy. So, you had A plus E in your hand and then you solved A plus E times x equals b, and let us say you got a solution x hat. And what the question is, what can we say about the error is x minus x hat. So, what can we say about this? So, once again if E is, the entries of E, so, if the entries of E are small enough such that the spectral radius of A inverse E is less than 1, then we can do the following.

We can follow an approach quite similar to what we just did, and we can write x minus x hat is equal to, x is this b, and x hat is A plus E inverse b, which is then equal to A inverse minus A plus E inverse times b. And now, we see that this matrix is exactly the same form as what we saw in computing the errors in inverses. And so, we can use exactly the same approach and write this as sigma k equal to 1 to infinity minus 1 power k plus 1 A inverse E power k times A inverse b.

So, there is this extra b factor that is coming in here, so I do not need this bracket. So, other than this extra b factor, it is the same as what we had earlier. And A inverse b is just x. And so, this is equal to x. Now, in order to proceed further, see I want to bound for example, the norm of x minus x hat in terms of this norm of x. So, I need to find the way of connecting this norm of x minus x hat to norm of x out here.

But then there is a multiplication by this these, these matrices out here. So, for that you have to use another notion which is the notion of compatible norms. So, here is the definition, the vector norm C to the n is said to be compatible with the matrix norm on C to the n cross n is given A in C to the n cross n the norm of Ax, Ax is a vector is less than or equal to the matrix norm of A times the vector norm of x.

So, this is like a sub multiplicative property of matrix vector norms. So, earlier this was a matrix product and you had the matrix norm of A times the matrix norm of b. But now, this is a norm of a vector, which we are bounding by the matrix norm of A times the vector norm of b and this should hold for all x in C to the n. Then you have the following result. So, the question is, are there such things as compatible norms?

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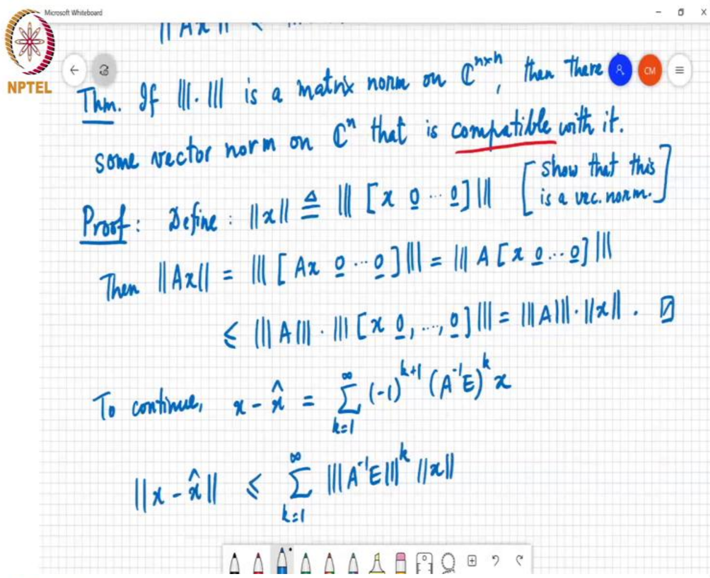

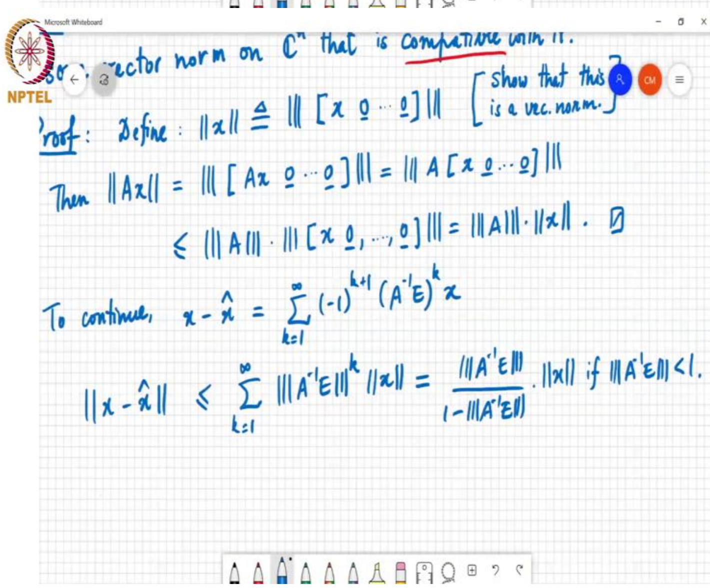

It turns out that any induced norm is compatible with the vector norm that induced the matrix norm and this is a small exercise that you can show. But we have the following theorem. If, is a matrix norm on C to the n cross n, then there is some vector norm on C that is compatible with it. So, there is always going to be one that is compatible with it. So, how do you show this? So, first of all we will define.

So, this is actually what I consider to be a kind of clever proof. So, the very first step is to define norm of x in terms of this matrix norm, and we are going to show that this particular definition of a vector norm is going to be compatible with this particular matrix norm. And so, without guessing this first step, it is a little hard to show this result. So, I will compute the matrix norm of the matrix where I append a set of n minus 1 0 vectors to this vector x to get an n cross n matrix.

And then I compute the matrix norm of that. So, I define this to be my, the vector norm that I am interested in. So, is this a vector norm or not is something that you should check, or show that this is a vector norm. So, it basically inherits all the properties of the matrix norm. And all you need to do is to show that because it is inheriting these properties of the matrix norm, it says, positivity, homogeneity and triangle inequality.

So, then the norm of Ax, this is an vector and I am computing this particular norm of it. This is equal to the matrix norm of a matrix whose first column is Ax, and all the other columns are 0, which is equal to. Now, because these columns are 0, I can actually pull out an A and write this to be A times x 0 0. Now of course, next steps are immediate.

So, you write, use the sub multiplicativity and write this as norm of A times the norm of this matrix whose first column is x and all the other columns are 0, and this by definition is the norm of x. So, that shows that norm of, so, there is always going to be a vector norm that is compatible with the matrix norm. But what I said earlier is that, if the norm, this norm is an induced norm, then the vector norm that induced it is a compatible, is compatible with this matrix norm.

So, that is something else, tried to show. So, based on this, we have, we were looking at the norm of. So, we, so to continue we wrote that x minus x hat is equal to sigma k equal to 1 to infinity minus 1 power k plus 1 A inverse E power k times x. We had A inverse E which is equal to x. And so, if we now take the norm on both sides x minus x hat is going to be less than or equal to, because the compatible matrix norm will satisfy this kind of sub multiplicativity type property.

But it is the sub multiplicativity of a matrix norm with a vector norm. And then I will further use multiplicativity to simplify A inverse E power k. So, k equal to 1 to infinity minus 1 magnitude is equal to 1 so, I can just drop that term norm of A inverse E power k times norm of x. And if the norm of A inverse E is less than 1 then this the right hand side. Let me do it this way, this is going to be equal to norm of A inverse E divided by 1 minus norm of A inverse E times the norm of x, if the norm of A inverse E is less than 1.

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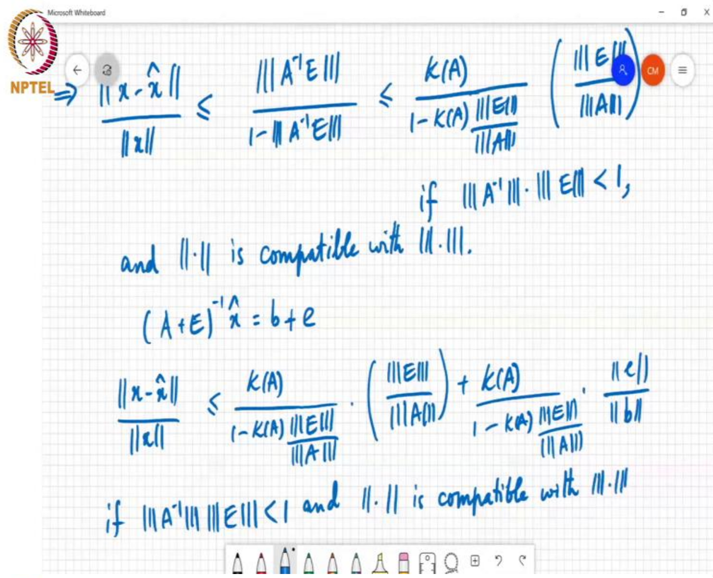

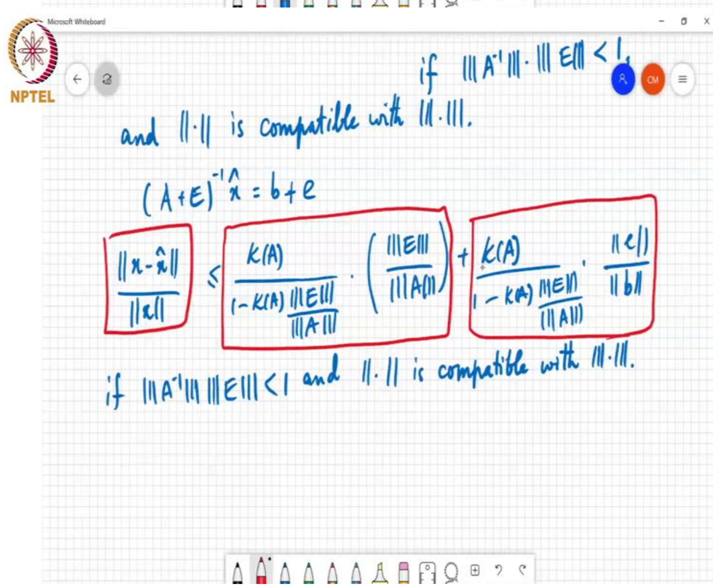

So, so, we taking this norm x to the other side, we have that norm of x minus x hat over norm of x is less than or equal to norm of A inverse E over 1 minus norm of A inverse E. And then we can use exactly the same arguments as the previous, A inverse computation to further bound this as K of A divided by 1 minus K of A times norm E over norm A times norm E over norm A. And this will be true if norm of A inverse times norm E is less than 1.

And, of course, there is the other reason, other, other point is that this norm is compatible with. So, you cannot use an arbitrary norm on the left and expect it to get bounded by the this quantity when you use some other norm over on the right, you have to use compatible norms on the left hand, right hand side. So, so we, we looked at basically solving, I mean, we have solved A plus E inverse x hat equal to b.

But, in fact, there could be errors on the, in the right hand side in measuring b also, this basically represents that you do not have exact knowledge of A. But if there is error in b also, I can write this as b plus some error vector e, small e. And then using the same procedure as earlier, we can write norm of x minus x hat over norm of x is less than or equal to K of A divided by 1 minus K of A times norm E over norm A times norm E over norm A plus K of A over 1 minus same thing, K of A norm E over norm A times norm e over norm b. So, this K of A appears in both the terms. So, this is true, if norm A inverse times norm E, we did not make any new assumptions here, is less than 1 and these two bars is compatible with.

So, the punchline here is that, the, see the left hand side here is the relative error in the solution. And that is equal to the sum of two terms, the first term is the relative error in E with the scaling factor kappa of A. And the second term, this is the relative error in the matrix A scale by the factor kappa of A. And this is the relative error in the right hand side, small e multiplied again by this K of A or kappa of A, which is the condition number of E.

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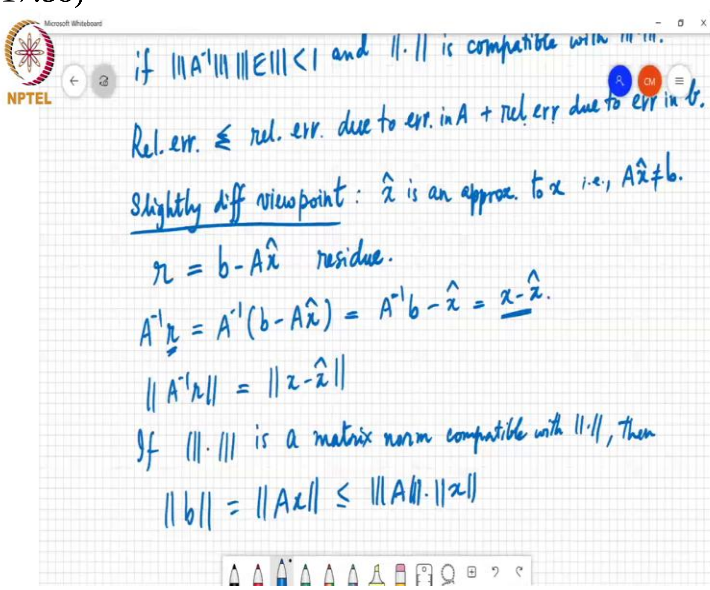

So, I will just write that for the sake of completeness, less than or equal to, the relative error due to error in A plus the relative error due to the error in b. And notice that these bounds, they do not directly involve exact the solution that you found. So, in some sense, they are giving you a bound on I mean, they depend on the actual error incurred, or at least the norm of the actual error incurred, but they do not have any direct dependence on A or on x hat.

So, there is a slightly (())(19:03) you can consider, which is that a slightly different, different viewpoint, which is that basically, we wanted to solve Ax equals b, but we computed some x hat where x hat is an approximation to x, i.e. Ax hat is not equal to b. So, we have solved something else, some nearby system or review some procedure, we got hold of an x hat and Ax hat is not equal to b.

So, we can define what is called a residual, which is equal b minus Ax hat. And this will be a nonzero quantity unless you exactly solve this equation Ax equals b, this is called the residue. And then we ask, what is the, what can we say about how close x hat is to x by looking at r. Of course, Ax hat is b minus r (())(20:34). So, if I compute A inverse r that gives me A inverse times, r is b minus Ax hat, which is equal to A inverse b minus A inverse Ax hat which is equal to x hat.

And A inverse b is x. So, this is x minus x hat. So, this is the error in x, when you. So, the error in x is related to the error in r through pre multiplication by the matrix A inverse. And so, we have that this norm of A inverse r is equal to the norm of x minus x hat. And so, if this three bars is a compatible matrix norm with two bars, vector norm then the norm of b is equal to the norm of Ax is less than or equal to norm of A times the two norm, two bar norm of x.

Which then means, this quantity, the norm of A times the norm of x over norm of b is greater than or equal to 1 if b is not equal to 0. So, as long as you are not solving a homogeneous set of linear equations, b is not equal to 0, then we can have such a bound like this. And then, because of this, if I look at norm of x minus x hat, this is less than or equal to the norm of A inverse r which is less than or equal to norm of A times norm of x divided by norm of b times the norm of A inverse times the norm of r, which in turn is equal to K of A times norm of r over norm of b times the norm of x.

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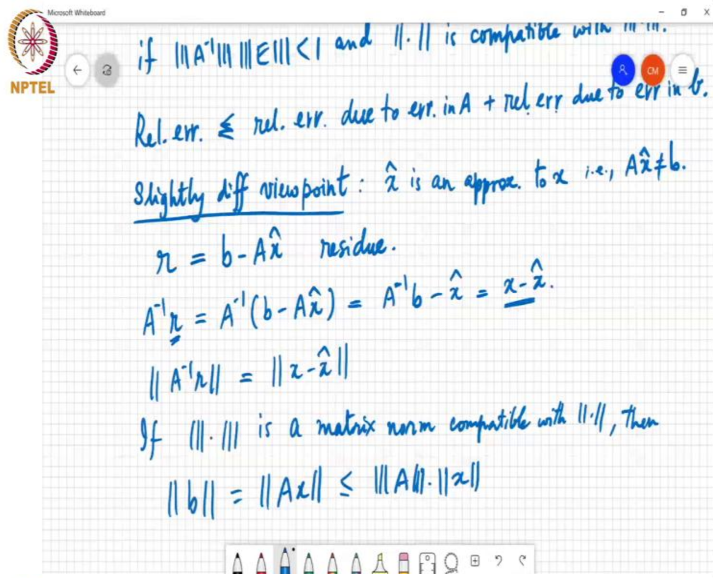

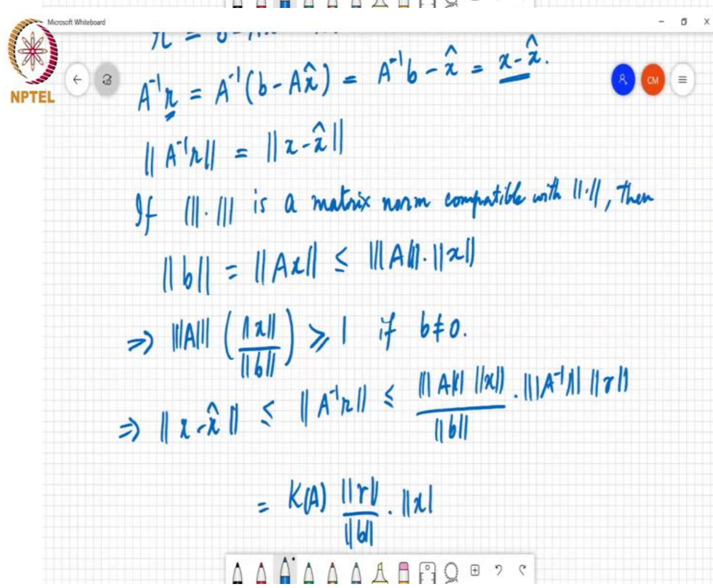

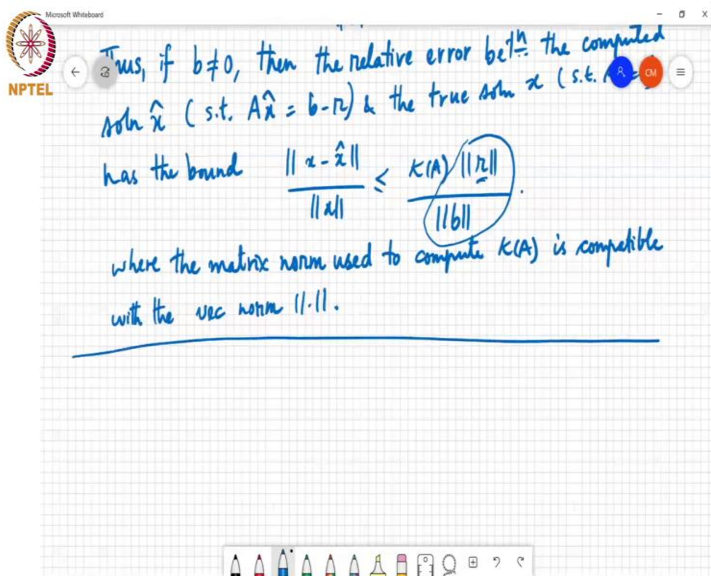

So, basically, yeah?

Student: Sir, should not norm of x minus x hat to be equal to the norm of A inverse r?

Professor: Correct. So, they are equal but just for the sake of writing and in inequality I write it, it is also less than or equal to. So, I wrote that here I think, so norm of A inverse r is equal to the norm of x minus x hat. But it is, this is also true, although I am loosening it a bit, but I can also write it this way. So, if b is not equal to 0, then the relative error between the computer solution x hat and such that Ax hat is equal to b minus r, r is the residual.

And the true solution x, which is such that Ax equals b satisfies norm of x minus x hat divided by norm of x is less than or equal to K of A times norm of r over norm of b. Where the only thing to keep in mind is that, the matrix norm gives the, this thing, yeah here we did not use any rule like the norm of A inverse E should be less than 1 or any such thing. This is a different formulation, there is no E matrix here.

So, the only thing we use was that, where matrix norm used to compute K of A is compatible with the vector norm. So, but anyway, the thing is that again it brings out the fact that, well condition are good matrices because the amplification to the relative error in b is going to be small, but if the matrix is ill conditioned, it is possible that the relative error, the, the error the relative error in x hat can be large, much larger than the error in, relative error in the residue.

So, you may think that you have solved it very accurately, because Ax hat is very close to b, but the error in x hat itself could be much larger than that, if the matrix is poorly conditioned. So, I think there is just one more way to analyze the sensitivity of linear systems. Let me see if I can cover that today.

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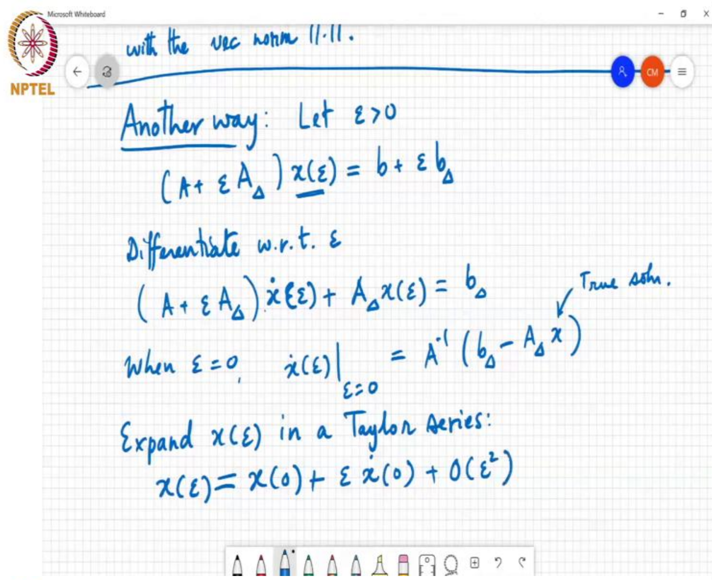

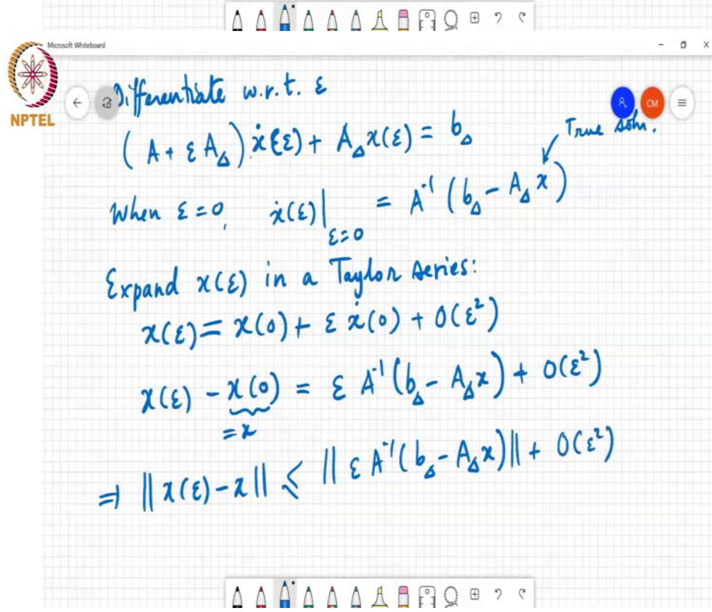

So, at least I can maybe start at giving you an indication of how, so another way. So, the way it works is like this. So, we consider a perturb system. And so, let epsilon be some small number greater than 0, and A plus epsilon times an error matrix A delta times x of epsilon is equal to b plus epsilon b delta. So, here A delta and b delta are some fixed matrices, but what we say is that, let us suppose that the A in our hand is a plus some small number epsilon times this A delta.

And the b in our hand is b plus some small number epsilon times this b delta. And corresponding to this linear system of equations or system of linear equations, we compute a solution to the linear system and we call it x epsilon. And now, we can try to look at how close x epsilon will be to x as epsilon starts becoming smaller and smaller. So, that means that, we are perturbing this matrix by a smaller coefficient times this A delta.

And we are perturbing the observations b by a smaller and smaller coefficient times this b Delta. And we want to know if very, very small perturbations can lead to large errors in x, that is x epsilon minus x is going to be a big number still. So, now there is a scalar epsilon throughout this thing. So, we can actually differentiate this with respect to epsilon. So, what I get then is, I have to use chain rule here.

So, A plus epsilon A delta times, there is some derivative x dot of epsilon, this is the derivative of the solution, which is a function of this epsilon. And this plus A delta times x of epsilon. So, I am differentiating. So, what I have on the left hand side here is a vector. And what I have on the right hand side is also a vector. So, I am differentiating a vector with respect to a scalar epsilon, which is the same as differentiating each entry of the vector with respect to the scalar epsilon.

And so, this time plus, so if you do that more carefully, you will see that what I am writing here is exactly correct. So, this is equal to b delta. So, for example, if I take the ith component of this, and differentiate it with respect to epsilon, the ith component is bi plus epsilon b delta i. And if I differentiate that with respect to epsilon I will get b delta i. And so that is why this whole vector on the right hand side is just b delta.

So, which means that when epsilon is equal to 0, we have x dot of epsilon, when epsilon equals 0 is actually equal to A inverse times b delta minus A delta times x at epsilon equals 0, which is equal to x. This is a true solution. So, we can use this derivative to expand x of epsilon in a Taylor series around 0. So, what we get is x epsilon minus, this is equal to x at 0 plus epsilon times x dot at 0 plus some term which is O of epsilon square.

So, we have that x of epsilon minus x of 0, which is actually equal to x is equal to epsilon x dot of 0, which is A inverse b delta minus A delta x. Then plus the second order term O of epsilon square. So, then we have that norm of x epsilon minus x is equal to the norm of this quantity, which I will use, again triangle inequality, or actually this O of epsilon square, when I write it like this, this is a vector whose entries are all scaling with, scaling down as epsilon square.

And so when I take this triangle inequality on it, I will get the norm of O of epsilon square and the norm of a vector whose entries are all of order epsilon square is also of order epsilon square. And so, this actually is less than or equal to the norm of epsilon A inverse b delta minus A delta x plus a term that is still O of epsilon square.

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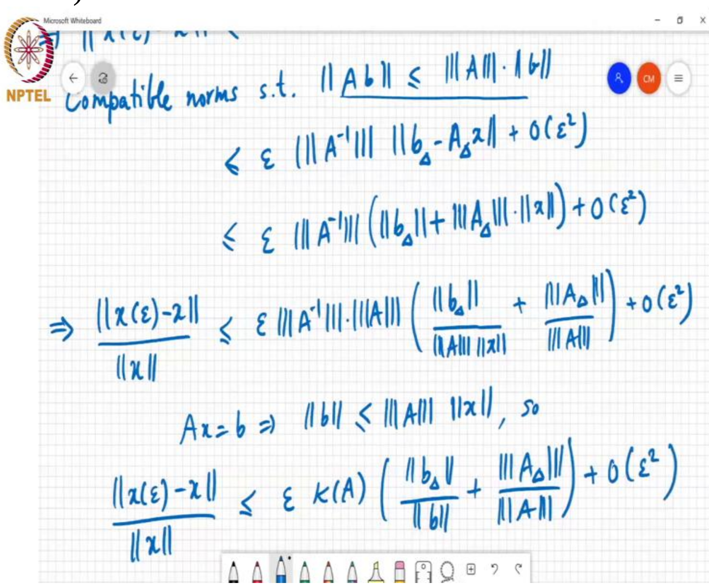

Now again, we will consider compatible norms, such that Ab is less than or equal to the matrix norm of A times the vector norm of b. So then, we can write this right hand side as being less than or equal to epsilon norm of A inverse norm of, vector norm of b delta minus A delta times x plus till this O of epsilon square. In turn, I can bound as less than or equal to epsilon times norm of A inverse.

And I will use triangle inequality and then sub multiplicativity here and write this as norm of b delta. And I am using triangle inequality, but then I have to use plus because I am taking norm inside and the mod of minus 1 is plus 1. So, plus and then since it is additive term, I can use the, use this property one more time to write this as norm of A delta times the norm of x plus O of epsilon square.

(())(36:14) is less than or equal to and as usual, I can insert multiply and divide by norm of A epsilon times norm of A inverse times norm of A times b delta. So, I have to divide by norm of

A. So, norm of b delta divided by norm of A times, there is a norm x from here. So, divided by norm of x plus the norm of A delta, the norm of x cancels, but I will be left with the norm of A sitting here. Again, plus O of epsilon square, dividing by norm of A and norm of x does not change the epsilon dependency here. And so this in turn, so basically, again, use this compatible norm thing.

So, so Ax equals b implies norm of b is less than or equal to the matrix norm of A times the vector norm of x. So, this is a bigger number. So, if I replace the denominator here with norm of b, that will only increase this quantity. So, the, my bound is not affected. So, the, this x of epsilon minus x over norm of x is less than or equal to epsilon, this quantity here is K of A times norm of b delta over norm of b plus norm of A delta over norm of A plus O of epsilon square.

So, this is nice, you can see that it is showing how, oops, it is showing how the relative error in x of epsilon is related to the oops, something happened. I will fix that. And what happened here? Haan, here we go. So, yeah, yeah this quantity is the relative error in b. This quantity is the relative error in A. And this is the condition number. And so it is showing how the relative error in the solution depends on the condition number times the relative error in b, plus the relative error in A times this epsilon itself. So, we will stop here for today and continue on Wednesday.