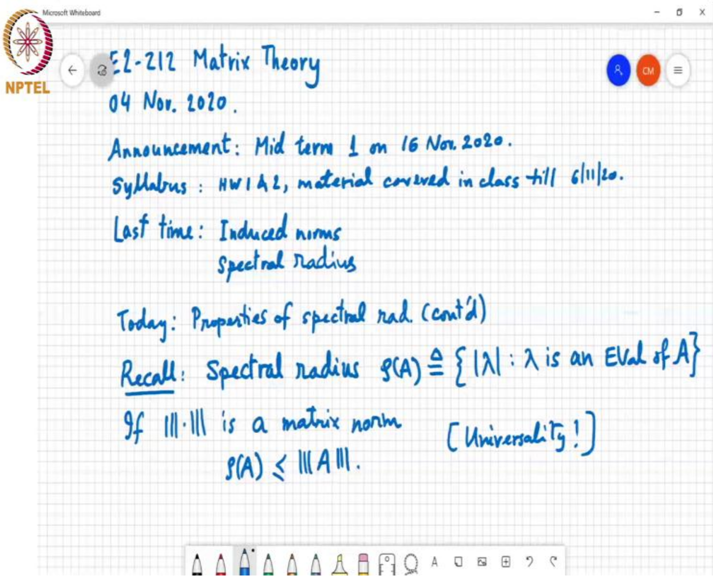
Matrix Theory
Professor Chandra R. Murthy
Department of Electrical Communication Engineering
Indian Institute of Science, Bangalore
Properties of Spectral Radius

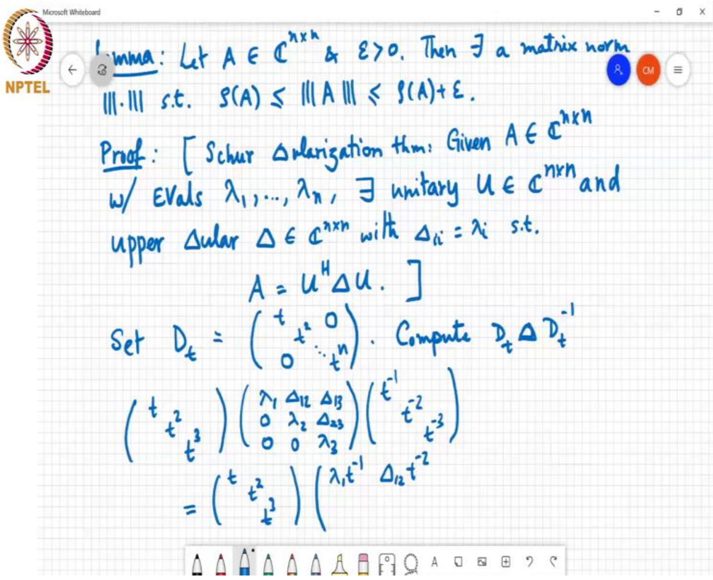
(Refer Slide Time: 00:15)



So, the last time we discussed a bit about the induced norms and also introduced the spectral radius not norm and today we will continue this discussion about the spectral radius. So, just to recall, so this (())(00:54) is defined to be the maximum magnitude Eigen value of the matrix A and we also saw that the spectral radius is a lower bound or any matrix norm.

So, if is a matrix norm rho of A is less than or equal to the norm of A, it is true for any matrix E and in fact, if you go back and look at the proof. So, in fact, so we saw that the spectral radius has this universality property that regardless of which matrix norm you choose to evaluate the norm matrix to the spectral radius of that matrix A. However, where spectral radius is not a norm. So, we were looking at some other properties of this spectral radius.

(Refer Slide Time: 02:53)



And so at the end of the previous class, we were trying to show the following lemma that, so, let A B and C to the n cross n and epsilon be greater than 0, then there exists a matrix norm such that in other words we can find a norm such that you can, the norm of this particular matrix is as close to its spectral radius as you wish.

Notice that if I, I mean this is for a specific matrix, so in other words, if I take a different matrix B, then it is not generally true that norm of B will also be between rho of A and rho of A plus epsilon. So, only for this 1 matrix A you can find a matrix norm such that rho of A, the norm of A is between rho of A and rho of A plus epsilon.

So, for the proof of this I said that the we use this show triangularization theorem, which we will be proving much later in the course. So, the first step is this, which says that any matrix A in C to the n cross n with Eigen values lambda 1 to lambda 2 then there exists a unitary U and so C to the n cross n and upper triangular delta with delta i i equal to lambda i such that A is equal to u Hermitian delta u.

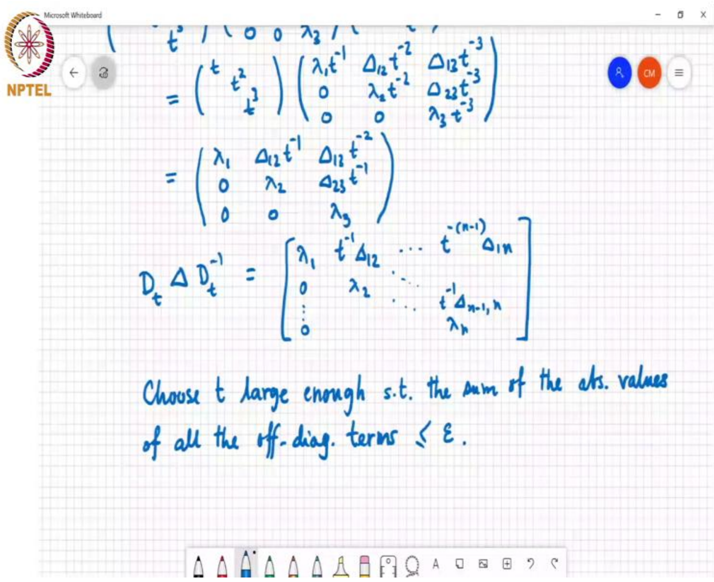
This is such a decomposition is possible for any n cross n matrix A, you can always find a unitary U and that upper triangular delta satisfying these properties. Now, we will set D t to be the matrix with t t squared up to t power n on the diagonal and zeros everywhere else and then for such a matrix, if I compute for example, I just since I went through the little bit quickly in the previous class, let me just.

So, here I will just say compute D t delta d t inverse, where delta is the upper triangular matrix you get by doing the (())(7:04) decomposition of this matrix A. So, since I went through it a little bit quickly the previous class let me just take 3 cross 3 example and just show you how this works out.

So, D t is D t squared D t cube times this delta matrix will have lambda 1, lambda 2 and lambda 3 on the diagonal it is upper triangular, so it will have zeros below the diagonal and let us say it has delta 1 delta 1 3 and delta 2 3 here and then D for a diagonal matrix when you invert it, all the diagonal entries get inverted. So, it is t inverse t to the minus 2 and t to the minus 3 and zeros everywhere else.

So, if I do this, this is equal to I will multiply these two first, so it is t squared t cube times which you multiply this you will get lambda 1 t inverse and this will be delta 1 to t to the minus 2 and delta 1 3 t to the minus 3 and this entry will be 0 and lambda to t to the (())(8:44), and that t to the minus 3 0 0 lambda 3 t to the minus 3. Now, if I complete this multiplication, I will get this will become lambda 1, then delta 1 to the inverse, this times.

(Refer Slide Time: 09:20)

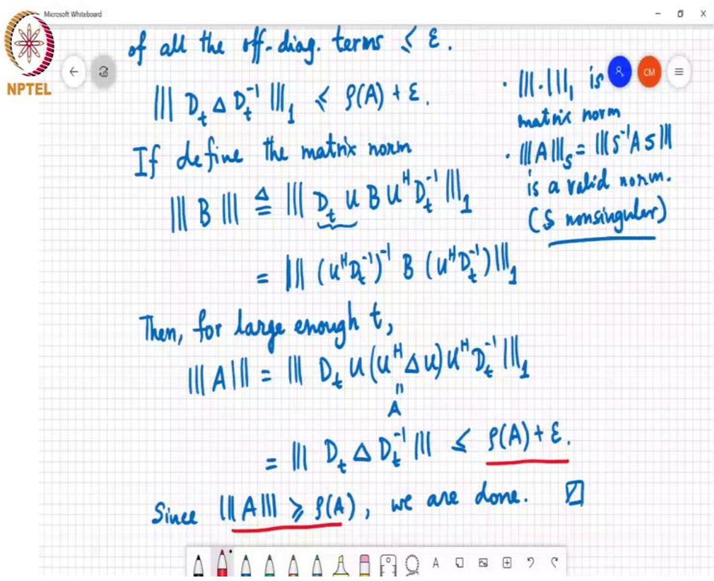


Then times the first row will become delta 1 3 t to the minus 2 then 0 lambda 2 delta 2 3 t to the minus 1 0 0 and lambda 3. So, you can see that when you do such an operation D t delta D t inverse, it will retain lambda 1 lambda 2 lambda 3 on the diagonal, the first of diagonal, the first that is the super diagonal, the entries will all get multiplied by t inverse. The second of diagonal will get multiplied by t to the minus 2 and so on.

So, in general, what we have is that D t Delta DT inverse is equal to lambda 1 0 t inverse delta 1 2, all the way up to t to the minus n minus 1 delta 1 n lambda 2 up to lambda n on the diagonal and this will be t inverse delta n minus 1 and n. So, you can fill in the rest of the entries. So, this is how it will look.

So, all the first off diagonal, the first super diagonal entries will get multiplied by t inverse, then t to the minus 2 all the way up to t to the minus of n minus 1. So, basically if I choose t very large, all these off diagonal terms can be made small enough. So, specifically what we will do is we will choose t large enough, such that the sum of the absolute values of all the off diagonal terms here is at most epsilon, is less than or equal to epsilon.

(Refer Slide Time: 12:20)



So, what this means is that, if we look at d t delta d t inverse and we consider its l 1 norm, what is the l 1 norm, it is the maximum column sum norm. So, I will have to take the sum of the magnitudes of the entries in every column and see which column is giving me the biggest number and that is this maximum column sum norm. So, when I take the magnitude of the entries here, I get mod of lambda 1, here I will get mod lambda 2 plus mod of t inverse delta 1 2.

But the sum of all these entries is at most epsilon and so, this one term in magnitude cannot, the magnitude of this one term it cannot exceed epsilon. So, that is also going to be smaller than epsilon. So, the this column this the l 1 norm of this column can be at most mod of lambda 2 plus epsilon, the l 1 norm of this column also can be at most mod of lambda 3 plus epsilon because this is the sum of the magnitudes of just two terms, whereas we have already made sure that the sum of the magnitudes of all these terms is at most epsilon.

So, this quantity here will be at most, just a sec and the maximum of all these lambdas is rho of A by definition and so, this is going to be at most rho of A plus epsilon. In other words, 1 of these columns is going to have the largest Eigen value in magnitude that largest Eigen value plus the sum of all these terms together in magnitude is an upper bound on the maximum column sum norm of D t delta D t inverse and so, this is true. So, that was the kind of key step in the proof and so now, the rest of it is kind of connecting the dots here.

So, if we define the norm. So, they will consider D t U D U Hermitian d T inverse and maximum column sum norm. So, we have already seen that if I take any invertible matrix and I consider S inverse A S where the norm of S inverse A S that is also a valid norm. So, this is a valid matrix norm. So, l 1 this maximum column sum norm is a matrix norm and this matrix D t or U Hermitian D t inverse this matrix is an invertible matrix. So, this is in fact a valid norm. So, this is what I will define. Then.

Student: Sir, can you please repeat.

Professor: Hold on 1 second.

Student: Okay, sir.

Professor: The volume is too low on my machine, yeah please repeat your question.

Student: Sir, can you please repeat why it is valid norm.

Professor: So, we have already seen that okay first of all is a norm and the second point is that given any norm A S S inverse A S is a valid norm the only requirement is that S should be norm singular, but that is true here because this I can always, U Hermitian D t inverse inverse B U Hermitian D t inverse.

This is the theorem we stated and proved in the previous class. So, this is a valid norm and then if I compute under this norm what the norm of A is so, then for large enough t we have norm of A is equal to the l 1 norm of D t U, now A itself I can write as U Hermitian delta U, this equals A, you Hermitian D t inverse the maximum column sum norm of this, but U U Hermitian and U U Hermitian are both the identity matrix.

So, I have D t delta d t inverse which is less than or equal to rho of A plus epsilon as we just showed here. And of course, for any norm and we have already seen that, for any norm this rho of A is a lower bound on the norm, we are done, I mean that is all we wanted to show, which is that the norm of A is between rho of A and rho of A plus epsilon. These are the two things we wanted to show. So, that is this result, this completes the proof we started the previous class. So, now we will continue.

So, just one remark is that what this result shows is that rho of A is essentially the greatest lower bound for the values of all matrix norms of, so the value of all matrix norms of A. In other words, I can find a norm such that the norm of A is as close to rho as I wish and so, there is no way anybody else can find some other, zeta of A or something which is bigger than rho of A and such that no matter which norm of A I choose to evaluate that will still be bigger than zeta to A. The rho of A is the biggest lower bound I can find on any possible matrix norm of A.