Matrix Theory
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Lecture 23
Spectral Radius

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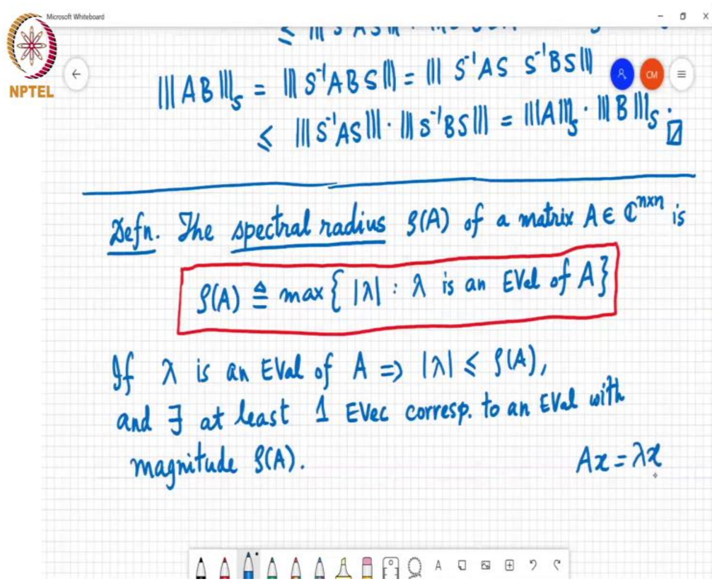

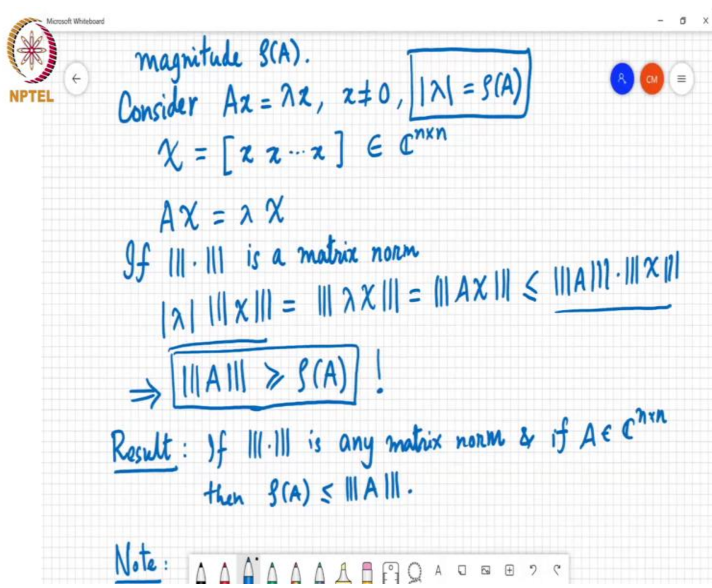

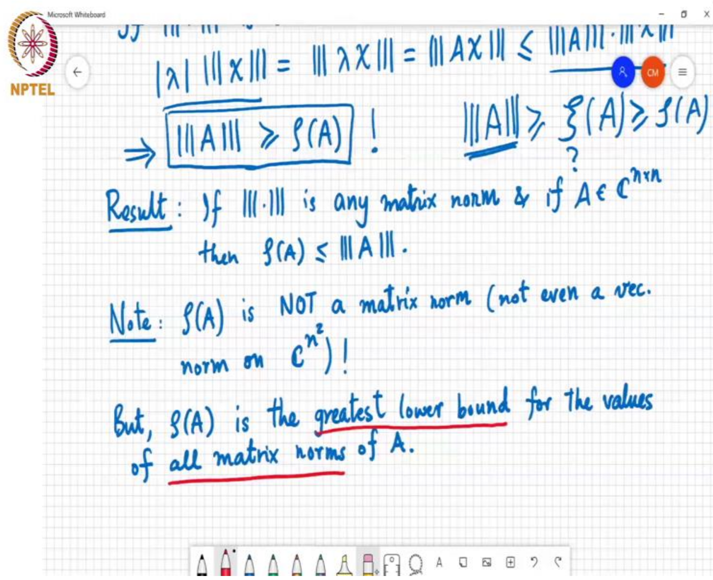

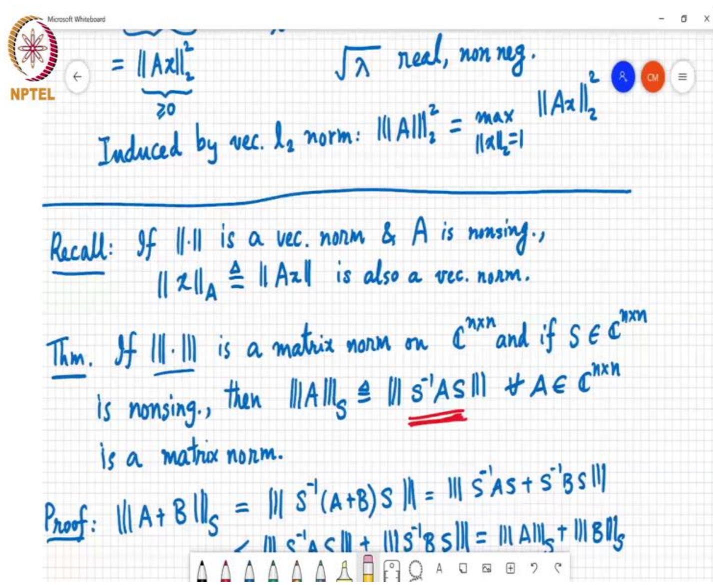

Now I want to discuss a new object

Student: (())(00:17)

Professor: Yeah, go ahead please.

Student: Sir, the triangle inequality for matrix norm how does it hold, it was not mentioned in the properties of…

Professor: Matrix norm satisfy triangle inequality, that is the definition of a matrix norm, a matrix now must satisfy non negativity, positivity, homogeneity, triangle inequality, and sub multiplicative.

Student: Oh okay.

Professor: So, basically matrix nom satisfies all the properties of the vector norm, in addition it needs to satisfy one extra property which is that of sub multiplicative. So, the spectral radius, rho of A of a matrix is rho of A is the maximum mod lambda such that lambda is an eigen value of A. So, this is the definition. So, given a matrix A you find all the eigenvalues of the matrix and then you look, you ask which eigen value has the largest magnitude, magnitude and that value is called the spectral radius of a matrix.

But you know this spectral radius is a fascinating object it, it has lots of very, very interesting properties. I will, I will tell you one to begin with. And so, it has a, it has the following universality property. So, suppose lambda is an eigen value of the matrix A. So, what that means is that mod of lambda is going to be less than or equal to rho of A because rho of A is the largest magnitude eigen value. And, and there is at least one eigenvector corresponding to lambda.

So suppose, so there is and there exists at least one eigen vector corresponding to an eigen value with magnitude rho of it. So now, of course, there is also an eigen vector corresponding to lambda, but I am interested in the eigen vector corresponding to the eigen value whose, whose…

Student: (())(04:19)

Professor: Whose magnitude is rho of A. Yeah, is there a question?

Student: Yes sir. Sir, what if lambda is imaginary?

Professor: Lambda can be complex value, no problem.

Student: Still there will be eigenvector?

Professor: Yes of course.

Student: Okay.

Professor: So, the point is that, given any n cross n matrix, it will always have n eigen values, some of these may be real valued, some of them may be complex values. So, since complex numbers are a generalization of real numbers, we can say that any n cross n matrix will have n complex valued eigen values. The, some of these eigen values may be repeated. Now corresponding to, we will see this later, but corresponding to every distinct eigen value there will always be at least one eigenvector.

When Eigen values are repeated, it is possible that you cannot find, by when I say distinct I mean linearly independent eigen vector. So, in other words, an n cross n matrix will always have n complex eigen values, but it may not have n linearly independent eigen vectors, but corresponding to every eigen, every distinct eigen value, there is always at least one eigen vector.

Because by definition, an eigenvalue, eigenvector pair satisfies the equation Ax equals lambda x. And so, if you can not find an x such that Ax equals lambda x then that cannot be an eigenvalue of the matrix. So, basically, since, since the rho of A is the maximum magnitude eigen value, if I consider that particular eigen value that gave me row of A there is a vector x such that, so if, so if I, if I consider, so let us say consider

Student: Hello sir.

Professor: Yeah.

Student: Sir based upon the definition you gave, will it be the interpretation like this that the spectral radius defines the capability of matrix A to the maximum amplification it can give to a vector x.

Professor: That is correct.

Student: Okay sir.

Professor: But it is something that you have to proof. So, and when you say amplification, you have to also keep in mind, when you say amplification, in what sense? What, what property of the vector x are you thinking of getting amplified. So, for example,

Student: (())(07:22)

Professor: Hm?

Student: Direction of x will not be changed, in the same direction amplification should be considered.

Professor: That is true.

Student: (())(07:31)

Professor: That is true, that is it for eigen vector of A. That is true for any eigen vectors of A, not necessarily, not only for the eigen vector corresponding to the eigenvalue whose magnitude is rho of A.

Student: Yeah.

Professor: So, when you are thinking about, so you made a statement, you said that the rho of A is the maximum amplification that a vector can experience when it is multiplied by A. So, in other words, in your mind, what you are thinking of is that the vector x has a certain size or length, you measure it using some, typically you want to measure it with the notion of a norm.

So, for some particular norm, the vector x has some value. And then you look at the norm of Ax, and you see how much bigger the norm of Ax is compared to the norm of x. In other words, you are looking at the ratio, norm of Ax divided by norm of x. And you are saying that the maximum value this can take, norm of Ax divided by norm of x, the maximum value it can take is roh of A.

So that is correct. But it is important that to keep in mind that this is measured in terms of the Euclidean norm. It is only when you measure it in the Euclidean norm that the largest possible value of norm of Ax divided by norm of x is going to be rho of A. So now, let me just continue with this argument here.

So, suppose I consider this, the particular Eigen vector corresponding to rho of A. So, there are some lambda corresponding to rho of A. And corresponding to that Eigen value, there is an Eigen vector x, and that satisfies Ax equals lambda x. Now, I will consider a matrix x, which is of size n cross n, and whose columns are all equal to x. So, I just repeat this.

Then what I will do is, and note that Ax is equal to the same scalar lambda times the matrix X. That is because each of these columns are multiplying A, and then when you multiply this column with A it will become equal to lambda times that column and then every column gets multiplied by lambda you can pull that right out of the matrix and you will have lambda times X.

Now, so if, if you are given any matrix norm is a matrix norm, then if I consider mod lamda times matrix norm of this X, this is equal to the matrix norm of lambda X, this is just by homogeneity, but lambda X is equal to Ax and then by sub multiplicative this is less than or equal to the norm of A times the norm of X.

Now, X is nonzero and therefore, the matrix norm of this matrix capital X is always going to be nonzero. So, if I compare this last step with this first step, I can simply cancel out the matrix norm of X and I have the conclusion that norm of A is greater than or equal to mod of lambda which is rho of A. So, this is a fascinating property. So, we defined rho of A to be the largest magnitude eigen vector of, eigen value of the matrix A and it turns out that rho of A is a lower bound on any matrix norm you can define on A.

So, however you define the matrix norm, it can never give you a value which is less than this spectral radius of A. So, I will write that out because it is a very, very interesting and fascinating result. If is a, is any matrix norm and if is A n C to the n cross n then rho of A is less than or equal to the norm of A. So, a natural question you can ask is, rho of A is the maximum magnitude of all the eigen values. So, basically it maps an n cross n matrix to a non negative number and so, is rho of A a matrix norm.

Student: Excuse me Sir?

Professor: Hm?

Student: I have doubt sir.

Professor: Yes?

Student: Sir in the above expression you have written lambda, mod of lambda is less than or equal to mod of A into like matrix norm of A into matrix norm of X, then, then how did you convert from here to next step? Is it for all lambda above expression?

Professor: No, no, this is for, so I wrote that here, mod lambda is rho of A.

Student: Only for that particular lambda. Thanks.

Professor: Yeah, so I am considering the eigen vector

Student: (())(13:49)

Professor: corresponding to the eigen value whose magnitude equals rho of A.

Student: Thank you.

Professor: Of course, the other eigen values in the matrix A have a magnitude less than rho of, less than or equal to rho of A. So, the norm of A is actually going to be greater than or equal to the magnitude of any eigen vector, eigenvalue of the matrix A. And in particular, it will be greater than or, I mean norm of A is greater than or equal to the maximum magnitude eigen value of A which is rho of A.

Student: It is like tighter bound, this is tighter bound then, tighter lower bound

Professor: It is lower bound on, so rho of A is basically a lower bound on any matrix norm.

Student: Yes, yes. Yeah, got it.

Professor: So, rho of A is not a matrix norm. In fact, it is not even a vector norm on C to the n square. You can show this, it is not difficult, but it is a lower bound on any, for any norm of A. Of course, you know I can always say, if I want a lower bound on norm, I can always say a lower bound on the norm of A is norm of A greater than or equal to zero. It is a non negative, it is the non negativity property of the matrix norm, but that is a trivial lower bound, it is not very interesting.

But norm of A greater than or equal to rho of A is a very non trivial lower bound. And so, so, that is the property of this rho of A. Now, it turns out that although rho of A cannot be a matrix norm it is, it is actually the, it is the biggest lower bound you can find, but rho of A is the greatest lower bound for the values of all matrix norms. What do I mean by this? So, we just discussed that norm of A is greater than or equal to rho of A.

So, pick any norm, the value of that norm, that norm of A is going to be at least equal to rho A. So, I found one lower bound on norm of A, you can ask can I come up with a different, a bigger lower bound on norm of A, that is instead of rho of A think of maybe you can find some, some other function, so and write an inequality that this is greater than or equal to some other function zeta of A, then what is this zeta, and this zeta of A is greater than or equal to rho of A.

Can you find such a function, as something that is an even better lower bound on any matrix norm you can define on A. The key thing is it has to hold for every possible matrix norm you may choose to define on the matrix A. Even once that we have not yet discussed and we have, that maybe people have not yet discovered. Can you find a, a bigger lower bound? The answer turns out to be no. So, rho of A is in fact, the greatest lower bound you can find that will hold for all matrix norms of A. That is what we will show next.

Student: Sir?

Professor: Yeah?

Student: Sir I did not get the concept of how it is amplification that a matrix can give.

Professor: So, that is a side point, we will come back to that later. Essentially, we were discussing about if you look at norm of Ax and then you divide that by norm of , this is, this is in one way, you can you can define this to be what is the amplification that x is experiencing when it is multiplied by A and the spectral radius is the largest such amplification you can get.

Student: Sir is this same as spectral norm only?

Professor: Spectral radius and spectral norm are two completely different things. So, if I go up here,

Student: Sir, actually, my apprehension was it requires it to be in same direction What if there is another vector that is rotated but has still greater amplification by A?

Professor: So, I think you will need to spend a little time thinking about it weather this is possible or not. So first of all, let me maybe make one small point that the, I mean think, there is many more things we have to discuss before I can properly answer this question and if I start discussing those, we will just completely leave the point here.

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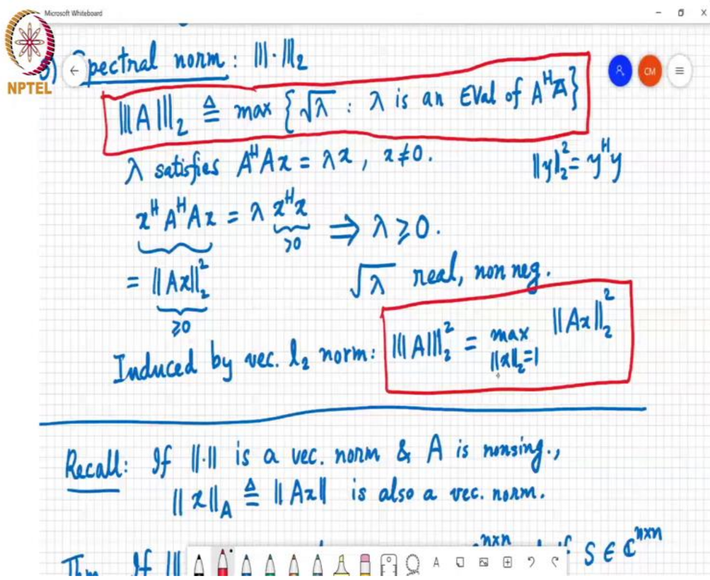

So, but the only point I want to make right now in terms of clarifying this is that, the spectral, the spectral norm, is defined like this, it is the maximum root lambda where lambda is an eigen value of A Hermitian A.

Whereas the spectral radius is the maximum mod lambda, where lambda is an eigen value of A itself. It turns out I mean, it is, it is very, it is a good point to make, because it is, it is, what is interesting here is that these two are very different objects. The spectral norm is a matrix norm, whereas the spectral radius is not a matrix norm.

So now just, let us just compare these two. So, keep this in mind, norm A2 is the max root lambda, where lambda is an eigen value of A Hermitian A. And in fact, we can show something like this.

Student: Yeah.

Professor: The square of this is equal to, I have to look at the largest value of Ax squared subject to x2 equal to 1.

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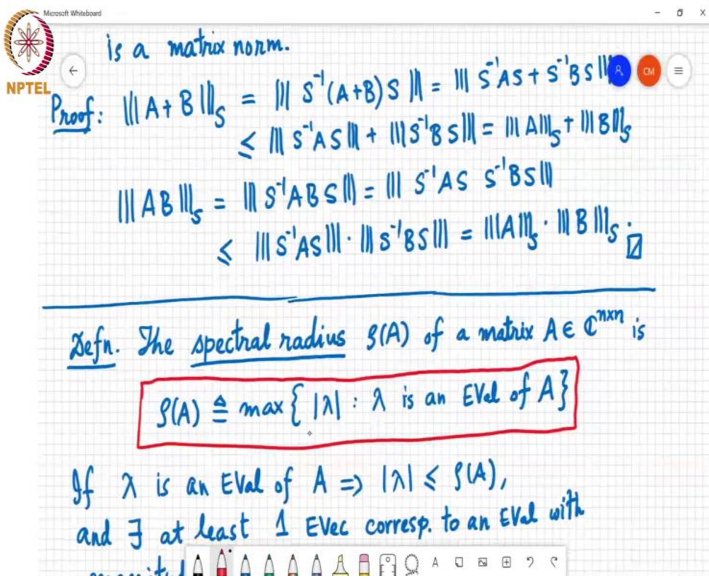
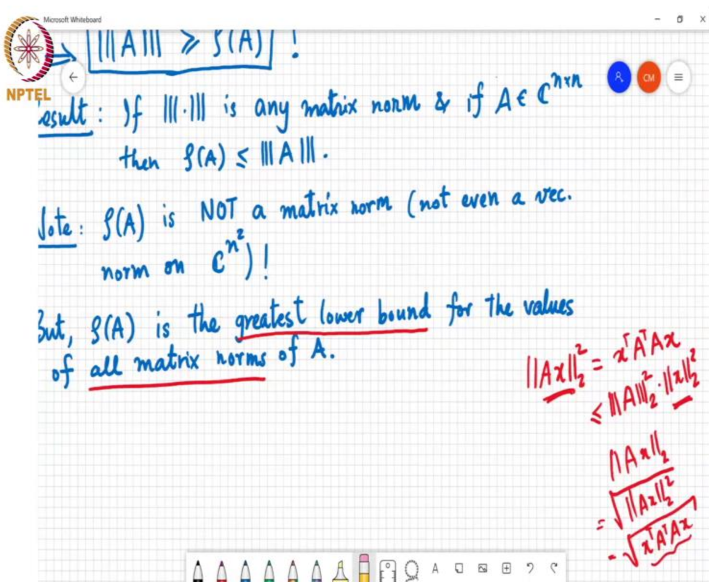




If $|||\cdot|||$ is a matrix no…

$$|\lambda|\,|||x||| = |||\lambda x||| = |||Ax||| \le |||A|||\cdot|||x|||$$

$$\Rightarrow \boxed{|||A||| \ge \rho(A)} \;! \quad \forall A,\ \forall |||\cdot|||$$

Result: If $|||\cdot|||$ is any matrix norm & if $A \in \mathbb{C}^{n\times n}$ then $\rho(A) \le |||A|||$.

Note: $\rho(A)$ is NOT a matrix norm (not even a vec. norm on $\mathbb{C}^{n^2}$)!

But, $\rho(A)$ is the greatest lower bound for the values of all matrix norms of A.

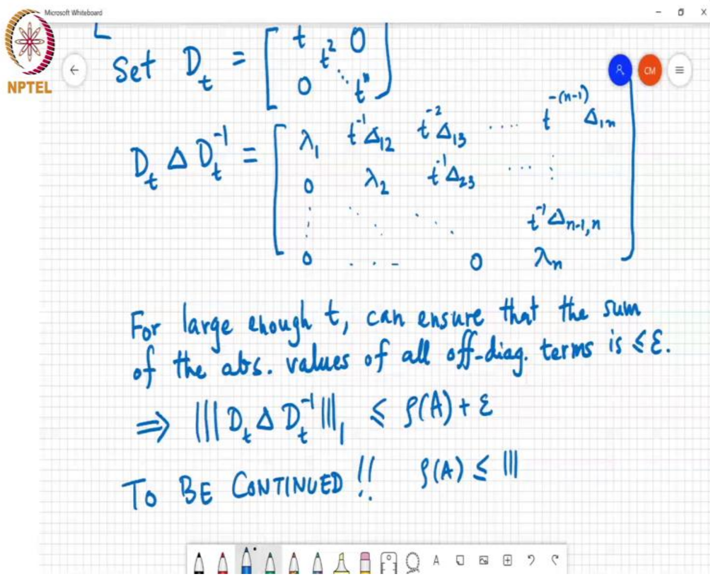
Lemma: Let $A \in \mathbb{C}^{n\times n}$ & $\varepsilon > 0$. There is a …

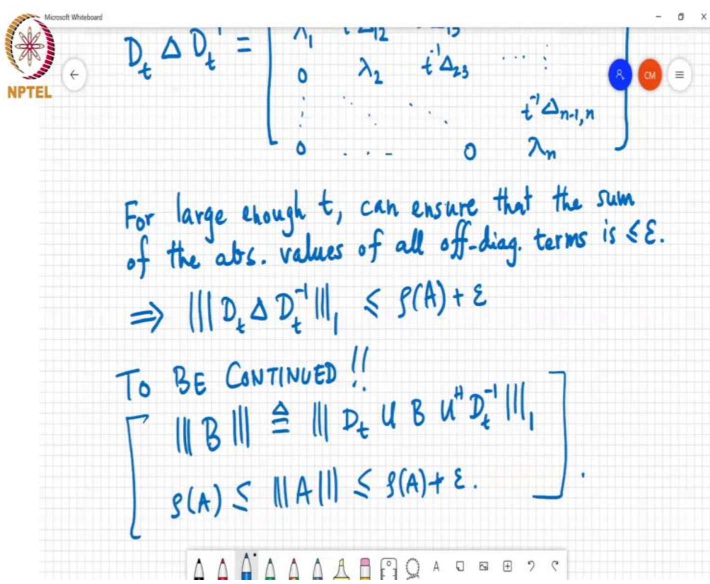
But, $\rho(A)$ is the greatest lower bound for the values of all matrix norms of A.

Lemma: Let $A \in \mathbb{C}^{n\times n}$ & $\varepsilon > 0$. There is a matrix norm $|||\cdot|||$ s.t. $\rho(A) \le |||A||| \le \rho(A) + \varepsilon$.

Proof:

Schur triangularization thm:
Given $A \in \mathbb{C}^{n\times n}$ w/ EVals $\lambda_1, \dots, \lambda_n$, $\exists$ a unitary matrix $U \in \mathbb{C}^{n\times n}$ s.t. $A = U^H \Delta U$, where Δ is upper Δular w/ diag entries $\Delta_{ii} = \lambda_i$.





Whereas, the spectral radius is rho of A is the maximum mod lambda, where lambda is an eigen value of A. So, these are two different objects.

Student: Sir isn't that definition the induced, induced by l2 norm that we have for spectral norm? Isn't that thing that this is the maximum amplification that we can get in terms of Euclidean norm?

Professor: Yes. So, they are both related. And if you are looking at a quantity like, what you can show is that this quantity is actually equal to x transpose A transpose Ax. And this you can show is less than or equal to this norm of, not four bars.

So, I think your point is basically that, when you are talking about amplification to this norm of x, you should really be comparing this, you should be looking at when you are looking at Ax l2 this is actually equal to square root of Ax l2 squared, which is equal to square root of x transpose A transpose Ax. And so, it is actually more related to the spectral norm than to the spectral radius.

However, we will see the connection between these two more closely in a few classes. Let me get rid of this. I do not know if that answers your question.

Student: Yes, sir. It is more related to spectral norm, but in general, that is true for spectral radius.

Professor: Yeah, yeah.

Student: With respect to any norm.

Professor: Correct. So, so later, we will actually explicitly discuss this kind of amplification concepts, when we discuss this thing known as the Rayleigh quotient. And in fact, that can be used to derive an algorithm to find the eigen values of a matrix, in which is different algorithm compared to finding the characteristic polynomial and trying to solve for the zeros of the characteristic polynomial.

So, we will discuss that more later. But right now, I want to say that, rho of A is a lower bound on the, on any norm on the matrix, you can you may choose to define on the matrix A. And in fact, you cannot find a better lower bound, this is the greatest lower bound and that is because of the following lemma.

Let A be a matrix of size n cross n and epsilon be some number, small number which is strictly greater than zero, then there is a matrix norm such that rho of A is less than or equal to this matrix norm of A. So, this matrix norm I am going to denote by these three bars is less than or equal to rho of A plus epsilon. So, what is this saying, it is saying that no matter how small an epsilon you choose, I will be able to find a matrix norm such that the matrix norm of this particular matrix A is between rho of A and rho of A plus epsilon.

In other words, you cannot find a different lower bound, which will work for all A's and all norms. So maybe I will write that here. So, it works for, for all A, and for all norms. So, you would not be able to find another norm, another lower bound, which is actually bigger than this rho of A. I can get, I can define a norm such that the norm of A is as close to this rho of A as I wish, that is what this lemma is saying.

So, how do we show this. The proof actually uses one result that we are going to again show later, which I will state here, but you have to take it on faith for now, but we will prove it later on. It uses a result called the Schur triangularization theorem. What this says is that given A of size n cross n with eigen values lambda 1 through lambda n, then there exists a unitary matrix u which is of size n cross n such that A is equal to u Hermitian delta u, where delta is an upper triangular matrix with diagonal entries delta ii equal to lambda i.

So, this is the result that we will use. Any matrix, any n cross n matrix can be decomposed as u Hermitian delta u, where u is a unitary matrix and delta is upper triangular with the eigen values along its diagonal. So now let us set Dt to be the matrix, the diagonal matrix with t t squared up to t to the n along the diagonal and zeros everywhere else. And then if you compute Dt delta Dt inverse, you can show, you can try a simple 2 cross 2 example to convince yourself that this is true, but you can show that this is equal to lambda 1 then t inverse delta 12 t to the minus 2 delta 13 up to t to the minus of n minus 1 delta 1m 0 lambda 2 t inverse delta 23 etc. And so, all the diagonal entries will remain the same and get lambda n down here.

And I will have t inverse delta n minus 1 comma n down here. And all these zeros and sub diagonal, you have zeros everywhere. If this is what happens if you take Dt delta Dt inverse, where Dt is this diagonal matrix and delta is an upper triangular matrix. So, basically, if I choose t to be a very large number, I can make all the off diagonal entries as small as I wish. So, so for large enough t, we can ensure that the sum, sum of the absolute values of all off diagonal terms is less than or equal to epsilon.

So that, that means that this, if I look at Dt delta Dt inverse, this, the one norm is going to be less than or equal to rho of A plus epsilon. I think I am out of time, there is actually a couple of more steps and I do not want to rush this last part of the proof. And so, it will take me maybe three or four minutes and I do not want you guys to feel like I rushed through this proof, so I will just say to be continued.

So, a fun exercise for you to, for you could be to see if you can actually show that norm of A is less than or rather. So, ultimately what we want to show is that rho of A is less than or equal to for this particular. So, let me, let me do this. I just say one thing here. We will define the norm. So that we want to define a norm such that the norm of A is between rho A and rho of A plus epsilon, and we are going to define it to B the l1 norm of Dt u B u Hermitian Dt inverse.

So, this is using that S inverse A S form. So, if I define it like this, it turns out that for this particular norm, rho of A is less than or equal to A is less than or equal to rho of A plus epsilon. So, this is what we want to show, but we will continue this and show it a little more slowly in the next class.