Matrix Theory
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Matrix Norms

So, with this we will move on to the next topic which is Matrix Norms.

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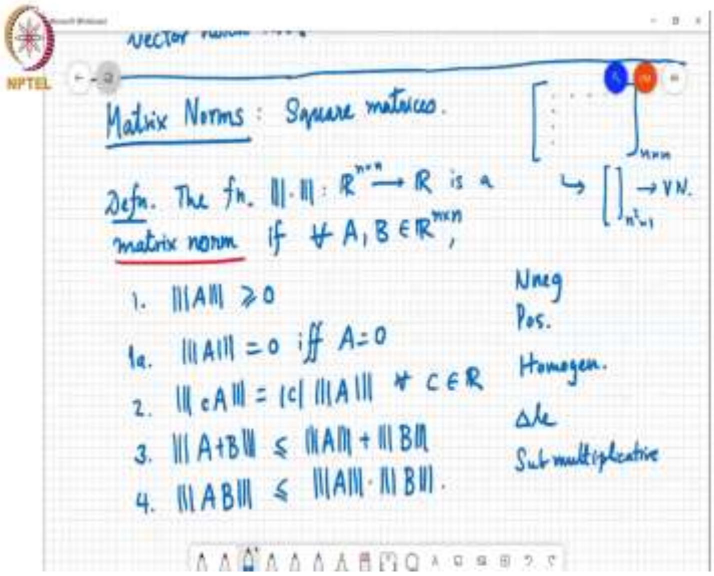

So, one, so now matrices are basically you know rectangular arrays of numbers and we want to let us say define the size of a matrix and compute it in the form of, call it a norm and compute the norm matrix. Now, one thing is that if it with respect to square matrices here.

Student: Excuse me. Sir?

Professor: Yeah.

Student: Sir, if we start with the pre-norm, can we get directly what could be the dual of dual of that norm.

Professor: Yeah, so the dual of the dual of pre-norm will be a norm, it may or may not coincide with the dual norm with the pre norm it will coincide with the pre-norm, if the pre-norm was actually a vector norm. If not it will, it may be a more restricted version or a different version of that pre-norm.

But, but once you find that dual of the dual norm that norm and this dual norm will alternate with each other going forward from there. Does that answer your question?

Student: Yes, sir. Sir, we cannot directly get it from pre-norm like by adding something to it anything, you cannot directly.

Professor: That is not, you will have to look at, so you are whether the dual norm of the dual norm of a pre-norm can be written in terms of the pre-norm itself by adding something or making some (())(1:58).

Student: Yeah, yeah.

Professor: I do not know that there is a simple modification to a pre-norm, that will make it satisfy the triangle inequality and therefore become a vector norm, which is, which is the same as the dual of the dual norm of that pre-norm, I do not know that there is such a thing. So, as you can see, the wonderful thing about matrix theory is that it is very easy to ask a question for which it is in fact very difficult to know, to find an answer.

So, we will discuss a lot of very, very deep results in this course. But always keep in mind that if you are lucky and in a later in life or in your research, if you are, if you are, if you are faced with a problem, which coincides with one of these deep results that we will be looking at, then you can use those results and you can say very interesting things.

But if it does not coincide with the results that we are discussing, it is oftentimes quite difficult to say what happens. So, that is the, that is one of the very interesting things about matrix theory. If you go slightly off the beaten path, you may be, you may quickly be off in some uncharted territory, where you will actually have to derive new results in order to understand what is going on.

So, what I was saying about matrix norms is that, if I take a matrix, which is of size n by n, what I can do is I have all these entries, I can simply vectorise them and I can construct an n squared length vector and then I can say that I will, now I can convert this into a big vector, a big long vector of size n squared by 1 and then I can say I use some vector norm.

Now, this is one way of measuring the length or size of a matrix, let me just say length rather than size, because size can also be taken to mean the dimension of the matrix. But there are, this is one immediate way by which you can measure the length of a matrix. However, matrices have multiplication defined on them and so we want to, we really look for matrix norms that will help us relate the length of A B to the length of A and B.

The individual lengths of a and b and that is an extra property that is desirable when you are looking for matrix laws and so the, therefore the definition of a matrix norm, in fact involves the condition involving products of matrices. So, here we go. So, the function and as I told you in a previous class, I am going to use three bars to denote a matrix norm and this is a mapping from R to the n cross n, to R is matrix norm.

If for every A and B belonging to R to the n cross n, we have 1 the norm of A is greater than or equal to 0, 2 or actually 1a as we called it earlier norm of A equals 0 if and only if A is the all 0 matrix, 2 is the homogeneity property for every C in R. And 3 is the triangle inequality. So, so far we have not deviated from the definition of the vector norm.

In fact, if I vectorise a matrix like this and then I compute any vector norm on this vector that will satisfy all these four properties and so the last property is the one that distinguishes vector norm from a matrix norm which is called a sub multiplicativity property. So, why do we need this, we will be clear when we discuss matrix norms going forward where this property turns out to be very useful showing that a matrix norm defined like this, does turn out to be quite useful.

And so again just for the sake of completeness, this is what we call the non-negative, this is what we call the positivity property, this is what we call the homogeneity property, this the triangle inequality and this is called the sub-multiplicativity. If only these properties are satisfied, then this is called a generalized matrix norm.

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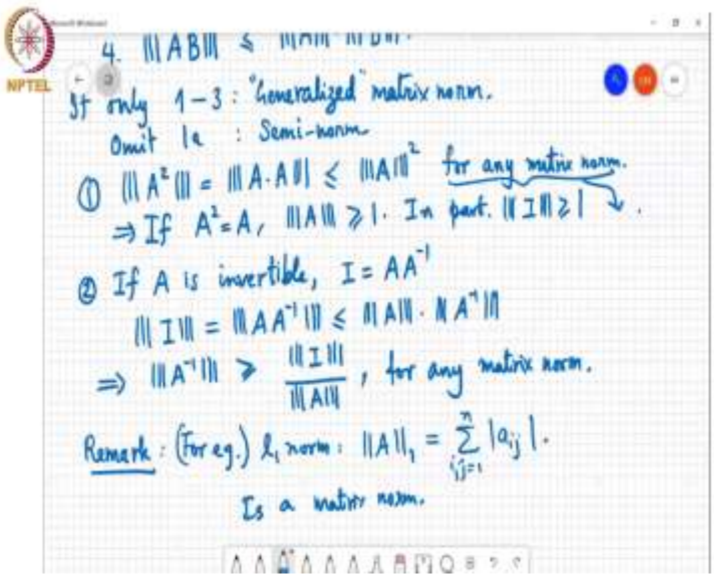

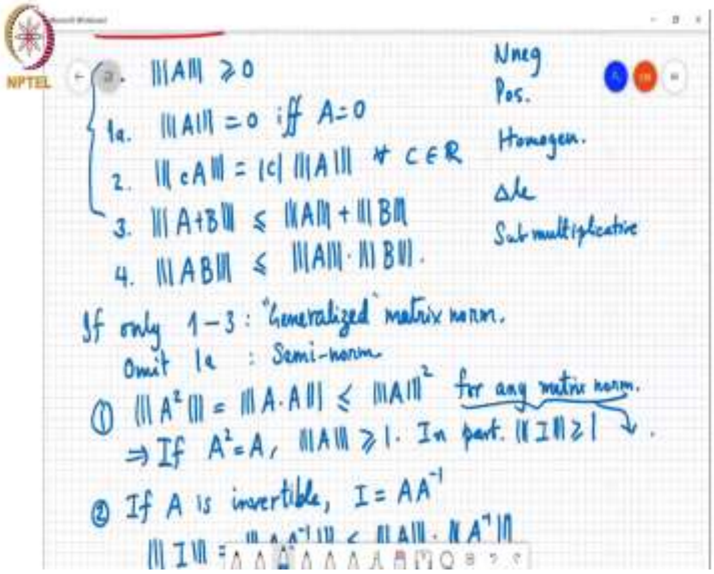

And if 1a is not satisfied, then as usual, we call it a semi-norm. So, this is the definition of a matrix norm it has one extra property besides what we use for vector norms. So, I can make a few immediate observations. The first is that if I take the norm of A square, A squared is A times A and by the sub multiplicatively property, this is less than or equal to norm of A times the norm of A, so I can write it as norm of A squared and this is true for any matrix norm.

So, if you take the length of a matrix, the length of the square of the matrix is at most the square of the length of the matrix and that is true for every norm and in fact, if A is such that A squared, if A is such that, A squared equals A, what do we call such matrices?

Student: (())(10:00).

Professor: (())(10:00) exactly, if A squared equals A on the left side also I get norm of A and so I can cancel this norm of A on both sides and I can then say that norm of A should be greater than or equal to 1. So, for any a such that A squared equals A, then norm of A is greater than or equal to 1 and in particular, the identity matrix is one such matrix for which A squared equals A and therefore, norm of the identity matrix is greater than or equal to, greater than or equal to 1, for any matrix norm.

And the second point is that if A is invertible, then we have that identity matrix equals A times A inverse and once again, if I use my sub multiplicative property, the norm of the identity matrix, which is equal to the norm of AA inverse is less than or equal to the norm of A times the norm of A inverse, which means that if I asked what is this the length of A inverse, it is at least equal to the norm of the identity matrix divided by the norm of A.

Now, there are many more properties, we will discuss them as we go along. Basically, as I mentioned, we can always vectorise a matrix and then compute a norm on it and some of the vector norms that we have looked at are in fact, matrix norms when you apply it to R to the n cross n, but others are not.

So, I just make a remark here. So, if I take for example, if I take the l1 norm and if I define, so I will define it with two bars here and this is just a function and I will define it to be sigma i j equal to 1 to n, mod aij and I asked, is it a matrix now and the answer is, this is a matrix norm.

Now, clearly, this because it is an l1 norm, it already satisfies the first, these three properties, norm A greater than or equal to 0, this this and this triangle inequality, the only thing you need to show is that it will satisfy the sub-multiplicativity property also, because the l1 norm is in fact a vector norm, it satisfies all these three properties. So, we will show that the next time and continue discussing other norms.

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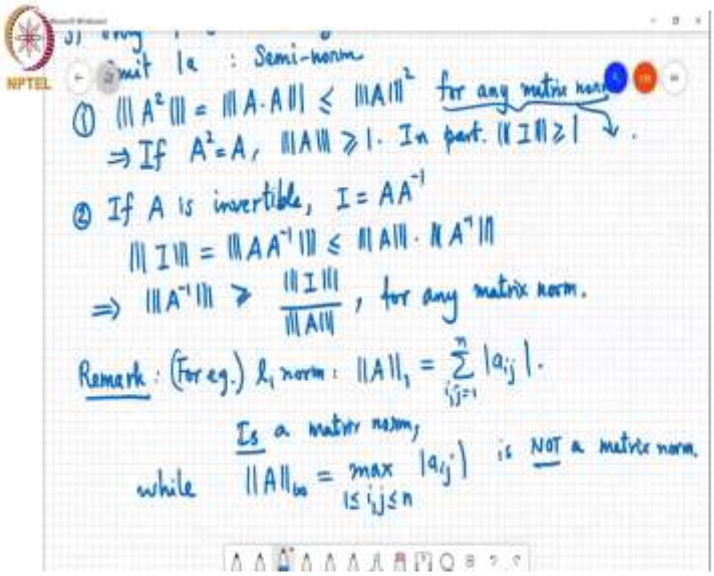

And so norm and I would say while the, if I define A infinity to be equal to max 1 less than or equal to ij, less than or equal to n, more aij. So, instead of taking the max, I am taking the max of the vectorised version, this quantity is not a matrix norm. So, we will stop here for today and continue the next class.

Student: Sir, I had one question that I mean these metrics norm translate to the tensors as well. So, we have let us say 3D matrices, we then.

Professor: A lot of has, just like we are discussing here. Some parts translate, some parts do not. But the analysis of tensor norms is beyond the scope of this course. So, I would not be discussing that here. I will discuss about matrix norms, but if you are interested in tensor norms, I can point you to some references. Actually the one of the, one of the reference textbooks does cover quite a lot about tensor norms. You can look at the textbook and learn for yourself about tensor norms.