Matrix Theory
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Determinant

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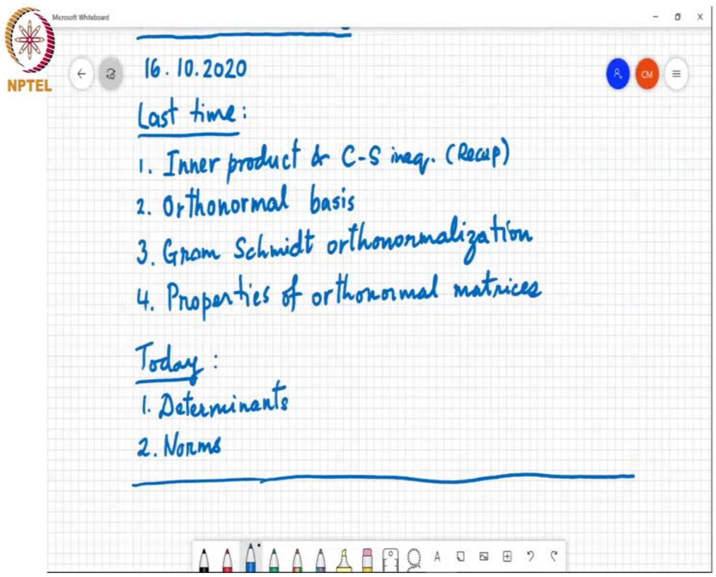

So, let us begin, the last time we looked at the inner product and the Cauchy–Schwarz inequality, this was just a recap. And then we discussed about orthonormal basis and some of their properties. And we went through the Gram Schmidt orthonormalization process, which can be used to produce an orthonormal basis for the vector space spanned by a given set of vectors. And then we looked at some properties of orthonormal matrices. So today, we will talk about two other topics. The first is determinants, and the other is norms. So, are there any questions about the previous class before we begin?

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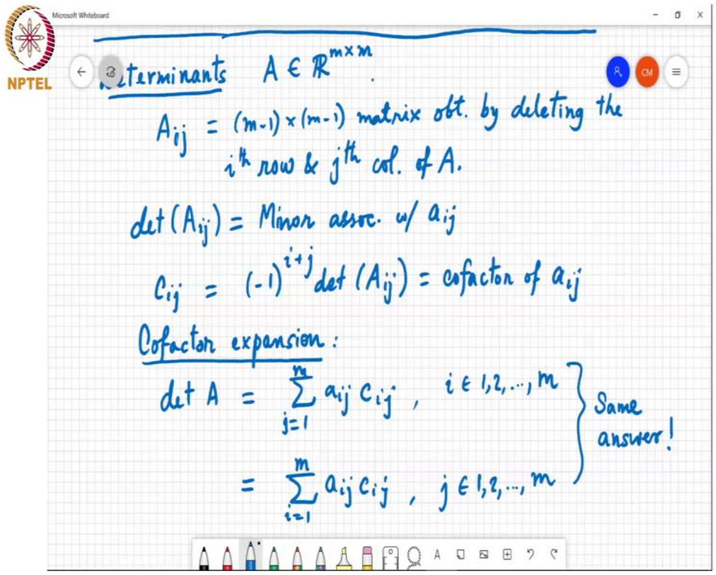

Okay, so we will start talking about determinants, so throughout we will consider square matrices A is in R to the m by m. Okay, so I think all of you certainly know what the determinant is, is there anybody in the class who has not heard of or seen what a determinant is? Okay, so I assume all of you know what it is, but we will just formally define it so that we are all on the same page on this. So if I consider the matrix A i j, to be the matrix obtained, it is an m minus 1 cross m minus 1 matrix, obtained by deleting the ith row and jth column.

Okay, then we define determinant of A i j, which is the determinant of a slightly smaller matrix to be the minor associated with the element, small a i j and we define c i g to be the cofactor of A i j, which is equal to minus 1 power i plus j times determinant of A i j, then, this is the definition of the determinant of A, this is called the cofactor expansion.

So, the way we define the determinant here, it is a recursive definition, the determinant of a matrix of size m by m is defined in terms of the determinant of a smaller size matrix of size m minus 1 by m minus 1. So, determinant of A is equal to the summation j equal to 1 to m A i j times c i j and note that there is a floating index (())(4:26) i belonging to 1 to up to m and I can also define it to be the summation over i going from 1 to m, a i j, c i j.

And now there is a floating coefficient j here and I can compute this for any j. Okay, so, they are basically m squared ways of compute, sorry, 2 m ways of computing the determinant, and they all give the same answer. Okay, so there are 2 m ways of computing the determinant. And all these give the same value. So this is another surprising fact about matrices is that you take a square matrix, and there are two m different ways, seemingly different ways of computing the determinant.

But whichever way you do it, you always get the same answer. How do we see this, in fact, there is a, it is not that difficult, all you have to do is to take the trouble to write out what the determinant expansion would be. And when you write it out, you will see that actually, whichever way you write this expression out, it is all the same terms that will finally show up in the determinant formula. And that is why they all actually should have the same value.

Now, in order to complete this definition, I need to know what how to compute the determinant of a 1 cross 1 matrix, because c i j itself is something that is minus 1 power i plus j times determinant of an m minus 1 by m minus 1 matrix.

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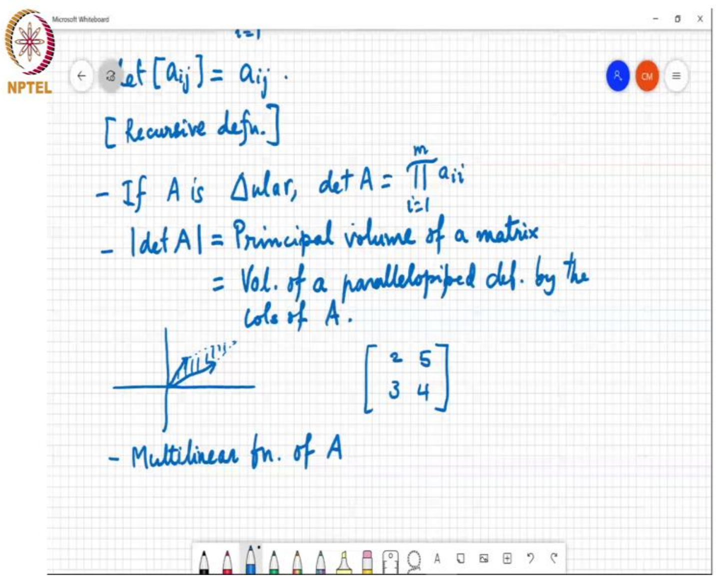

So, determinant of a single element or a scalar is that value itself. So this is a recursive definition. Okay, and yeah, so I think so this is the basic definition of a determinant. Now, we will discuss many properties of the determinant, but maybe just to keep the order here. One, so I will just make one or two small remarks, one is that if A is a triangular matrix, then the determinant is equal to the product of the diagonal entries.

Then okay, and then, the other thing is that if mod of the determinant of A is called the principal volume of a matrix. So this is the volume of a parallelopiped who is defined by the columns of.

Student: Sir.

Professor: Yeah.

Student: Can we define determinant for a non square matrix?

Professor: No, it is only for square matrices.

Student: Sir at the starting line, you define the matrix A which belongs to R power m cross n which is a non square matrix, right?

Professor: M by M. Maybe my handwriting is not clear, but this is m cross m.

Student: Okay Sir, thanks.

Professor: Okay, so this is the volume of a parallelopiped defined by the columns of A. So for example, if, I can only show you things on a two dimensional case, so if I define vectors associated with the two columns of A, so let us say it is 2 3 and 5 4 something like this, then I take 2 along the x axis, and then 3 along the y axis. This is a vector that looks something like this. And then I take 5 along the x axis, and I take 4 along the y axis, this is another vector like this.

So, I use these two and complete a parallelopiped and then I look at what is this area. So that is what I mean by the volume of a parallelopiped is defined by the columns of A, this is called the principal volume of a matrix. And this also happens to be equal to the magnitude of the determinant of A. The other small point I want to say is that this is a multi linear function of A. That is, it is actually linear in each element by itself. But it is a combination of various terms, so the same element could potentially. So it is a combination of various terms involving different elements of A.

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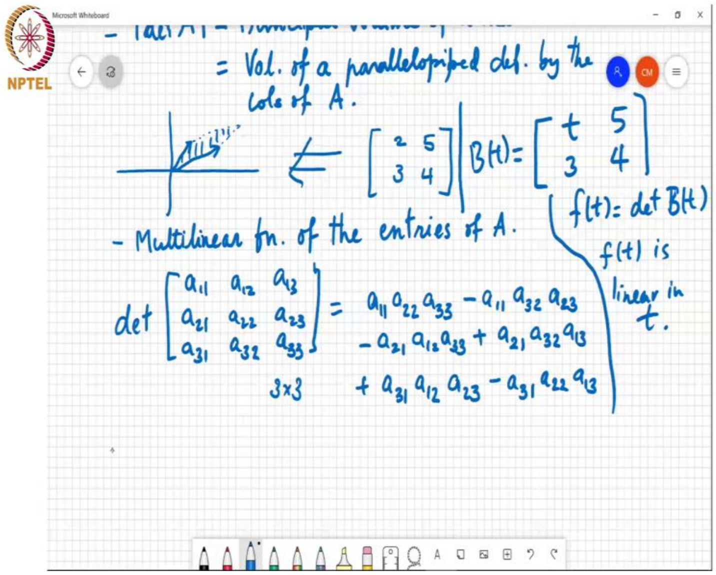

So just to, you know, make this point clear, I will actually take the trouble to write out the determinant of a 3 cross 3 matrix. And we also write it in this way by (two) put two bars around it, that can also denote the determinant. But for the moment, I would not do that, because you can see here I am using the two bars to denote the magnitude.

So, this determinant of this is equal to a 1 1 a 2 3. Sorry, a 2 2, a 3 3, minus a 1 1, a 3 2, a 2 3, minus a 2 1, a 1 2, a 3 3, plus a 2 1, a 3 2, a 1 3, plus a 3 1, a 1 2, a 2 3, minus a 3 1, a 2 2, a 1 3. So I should take any one element, like, for example, a 1 1, it appears in two terms, and it is a linear term that appears here.

So for example, if I unilaterally increase a 1 1, then the value of the determinant will also (uni) linearly change. It is a linear function of a 1 1 by itself. And if I take a 2 2, a 2 2 appears here, and here, it also appears in exactly two terms. If I take a 3 3, it appears here and here also appears in exactly two terms, and so on. And each term contains 3 terms from this, which is the size of the matrix, this is 3 cross 3. So each, each term that appears here contains exactly 3 terms. And then there are total of 6 terms. And each term appears linearly in it. So it is a multi linear function of the entries of a.

Student: Sir.

Professor: Yeah.

Student: Sir could you explain what is the meaning of multi linear.

Professor: It is linear but there are multiple terms, each of them being linear. So when you have a single variable, you know that the variable could, the function could be linear or nonlinear. But when you have multiple variables, then it could be linear in each component, but then it could be the sum of many such terms. So for example, if I had a function like a 1 1 a 1 2 a 1 3 a 2 1 a 2 2 a 2 3 a 3 1 a 3 2 a 3 3 this is also a multi linear function of a, it is linear in every one of the components. So the sum of multiple such terms.

Student: Sir.

Professor: Yeah, go ahead.

Student: But here, it is not sufficient to scale just one element, right? We will have to scale either one whole row or one whole column to scale the determinant.

Professor: No, no, no, I am just looking at what happens to the determinant. If you were to scale one particular entry of the matrix.

Student: It will become different. It would not, the determinant would not be scaled. Because the element a 1 1 appears only on two of this.

Professor: No, no, no, think of it this way. Suppose I consider a matrix t 5 3 and 4, okay. And I say, so suppose this is a matrix B. And then I call f of t equals determinant B. So this is actually a function of t. So if I define f of t, t to be determinant of B of t, then f of t is linear. This is what I am trying to say.

Student: Okay sir.

Professor: Okay, this is for this figure here.

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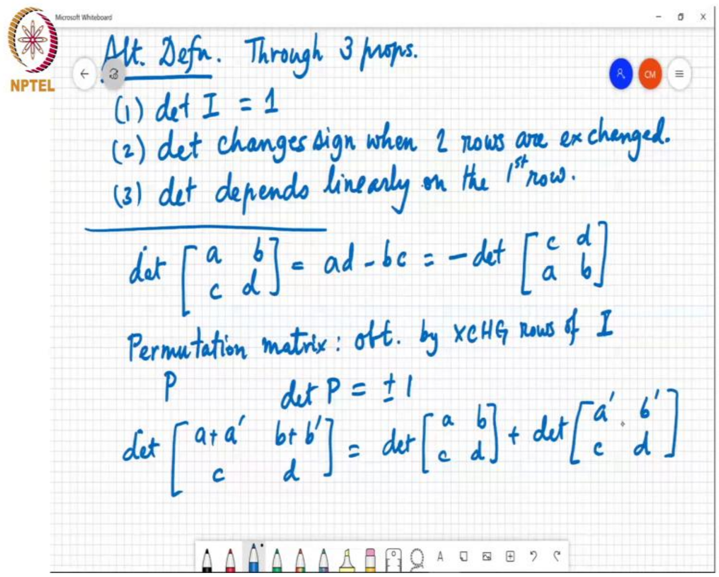

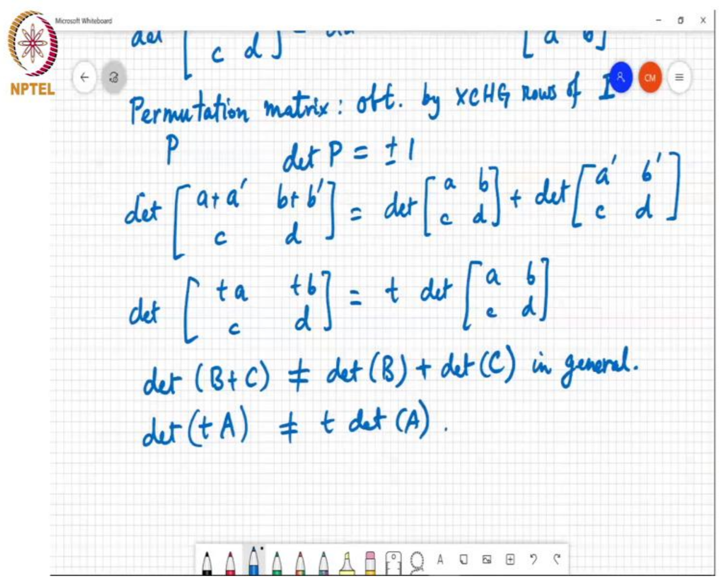

Now, there is an alternative definition which is sometimes useful by itself. Okay, so, it is through three properties. So, the first is that the determinant of the identity matrix is always equal to 1 no matter what the size of the identity matrix is, two, the determinant changes sign when 2 rows are exchanged. And three, the determinant depends linearly on the first row. So, any function set which satisfies these three properties is a valid definition of a determinant.

So, basically what we are doing here is we are mapping a matrix to a number. And any mapping of a matrix to a number which satisfies these three properties is, in fact, a valid definition for the determinant. And it is the same definition as what we discussed earlier. Okay, so, this is another definition. So, just for you, and we will come back to this later when it is useful. Okay, so now, several properties.

Well, before I discuss these properties, maybe I will just explain what I mean by this. So, if I take a matrix say determinant of a b c d, this is equal to a d minus b c, which is equal to the negative (())(17:59) the data of the matrix c d a b. And also if I take a permutation matrix, so, for the moment, just keep in mind or I will just say that a permutation matrix is a matrix obtained by exchanging, so this is my shorthand notation for exchanging rows of the identity matrix, this is one way to think of a permutation matrix.

A permutation matrix if I denoted by P, then determinant of P is always equal to plus or minus 1. So, for any permutation matrix since the it is obtained by exchanging rows of the identity matrix and in exchange rows, the sign of the determinant changes the determinant of t is plus or minus the determinant of the identity matrix which is equal to 1 and by linear in the first row, what I mean is that, if I take the matrix so, determinant of a plus a dash b plus b dash c d this is equal to determinant of a b c d plus the determinant of a dash b dash c d.

So, I can take linear combinations of the first row, or any row. And the determinant can be split as the determinant of not linear combinations, I can consider the first row to be linear combination of two vectors a b and a dash b dash, then the determinant splits as the sum of determinants, the first where being the determinant where the first vector is the first row, the second being the determinant of the matrix where the second vector is the first row.

And similarly, if I take the determinant of the matrix, t a t b c d this is equal to t times the determinant of a b c and d. So it is linear in the first row. But very important is that determinant of B plus C is not equal to determinant of B plus determinant of C in general. So here, it is true. When B and C differ only in one row, then determinant of B plus C is equal to determinant of B plus determinant of C. But in general, it is not true. And further determinant of say t times A is not equal to t times determinant of A. So only when you scale a single row the determinant scales by the same factor t.

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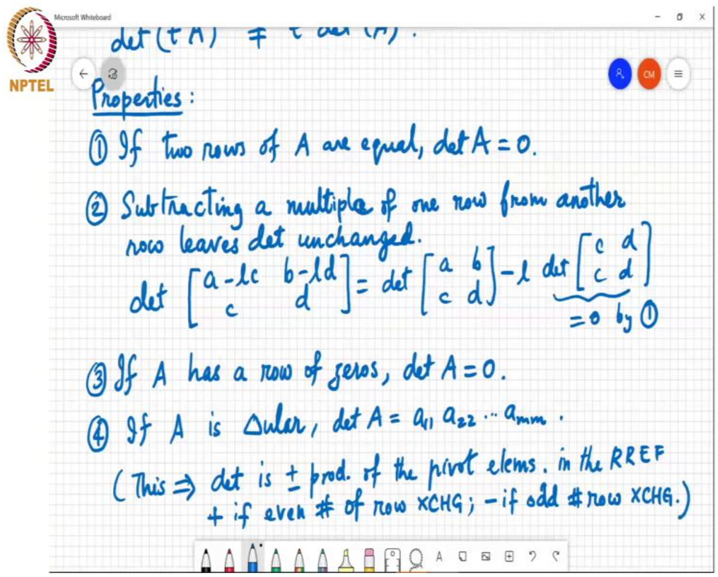

Okay, so now we will discuss several properties of the determinant. 1 if two rows of A are equal, then what happens to the determinant?

Student: 0.

Student: 0.

Professor: Exactly. The second property. So why is this true? It is because if I exchange a pair of rows, if two rows are equal, let me exchange those two rows, the matrix remains the same, but the determinant is supposed to have change the sign. But the only number which when you change its sign is, remains the same number is 0.

So that is why the determinant is 0. Subtracting a multiple, so if you do a like an elementary row operations, subtracting a multiple from another row, it leaves the determinant unchanged. So subtracting or adding, its a same thing, a multiple of one row from another row leaves the determinant unchanged.

Student: Hello.

Professor: Yes, please.

Student: Sir in the previous section. You specifically mentioned about the first row, if it is linear with respect to first row, what about sir other rows are they will be same?

Professor: So what do you think?

Student: Sir I think same (())(23:58).

Professor: Yes. And why do you think so. So all you do is, if you are, if you take a linear size, if some other row is a sum of two vectors, what you do is you exchange that with the first row, the determinant changes the sign. Now you apply this rule, now you get two different matrices. Now exchange the rows back, it changes sign again and goes back to the original determinant.

Student: Yeah, so minus minus will be plus again, so it will be determinant.

Professor: Exactly.

Student: Okay sir, got it.

Professor: So, that is because if I take like this, what I have done is I (mult) I have subtracted l times the second row from the first row. Now I can use my rules namely that if I look at a matrix where the first row is the sum of two terms, then I can split it as the sum of two determinants. So I have written the first determinant here a b c d. And second determinant is like determinant of minus l c, minus l d, c d but the first row is being scaled by a factor t, which is equal to minus l here, so I can pull that minus l out and write this as determinant of c d and c d.

And this determinant is 0, because two rows are identical, and so then the determinant of the matrix remains unchanged. If you subtract a multiple of the second row from the first row. See, I am showing you this proof for the 2 cross 2 case. But please keep in mind that showing a proof for a 2 cross 2 case is not really a proof. I am showing it for the 2 cross 2 case in this example, because the same exact thing can be done for a 3 cross 3 or 4 cross 4 or 8 cross 8 or any size m cross m case. Okay, the same idea exactly holds. And so there is no harm in this particular case to consider a 2 cross 2 case and say what is happening.

Student: Hello Sir.

Professor: Yeah.

Student: Sign of determinant also get change when we exchange the columns?

Professor: Yes, and that will be obvious in a second. Because it turns out determinant of A is equal to the determinant of A transpose and exchanging columns is the same as exchanging rows of A transpose.

Student: Okay sir.

Professor: And so this is obvious again, because if he has a row of zeros, all you have to do is to add one other row to this row of zeros. And because adding a row or a multiple of a row to another row remain, leaves the determinant unchanged, the determinant does not change, but now you have a matrix with two identical rows, then by property 1 the determinant of A is 0. The other way to do is think of it is to think of the cofactor expansion formula, where you are expanding along that row, which is the row of zeros.

And if you are expanding along the row of zeros, then the coefficient of every term in the cofactor expansion is 0. So it only, it can only add up to 0. This is a property I already said which is if A is triangular, then determinant of A a 1 1 a 2 2 up to a m m, okay. So basically, for triangular matrices, the determinant can be obtained by just taking the product of the diagonal entries.

So in particular, remember that the, you get the row reduced echelon form by doing elementary row operations. And these elementary row operations involve either subtracting a multiple of a given row from another row, or exchanging rows. So if you exchange rows, the sign of the determinant will change. If you multiply, subtract a multiple of a row from another row, then the determinant will remain unchanged.

And so when you do these elementary row operations, you will get a matrix which is ultimately upper triangular. And for upper triangular matrices, the determinant is the product of the diagonal elements. And therefore, one way to compute the determinant of a matrix is to first perform the, find the row reduced echelon form, and then take the product of the diagonal elements.

And then you also keep track of how many row exchanges you had to do in the process of finding the row reduced echelon form, if you had to do an even number of row exchanges, then the determinant is simply the product of the pivot elements. If you had to do an odd number of row exchanges, then the determinant is the negative of the product of the pivot elements. So basically, this implies determinant is of the pivot elements in row reduced echelon form. Okay, and plus, if even number of row exchanges minus odd.

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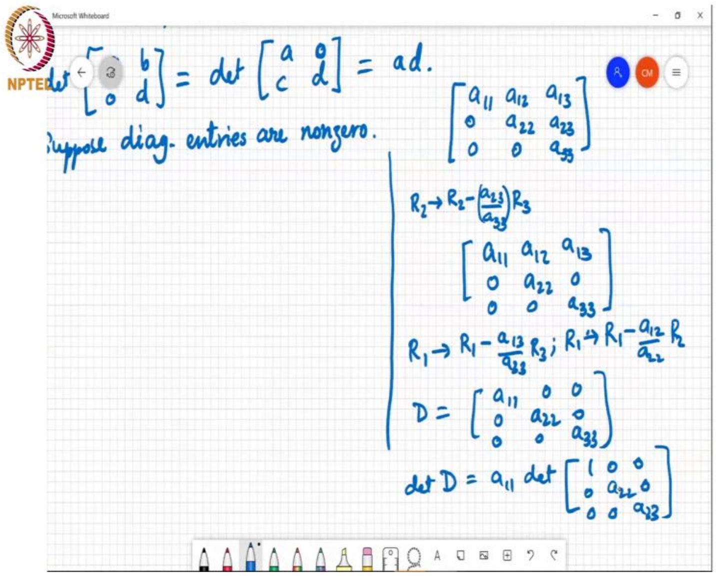

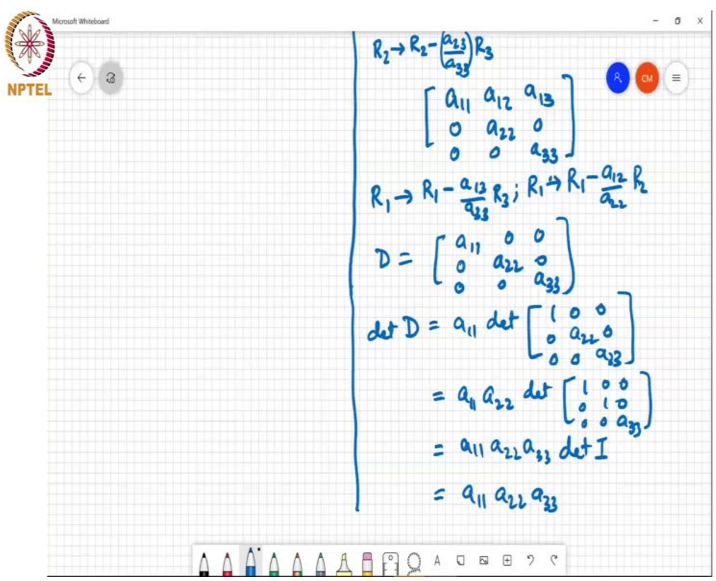

So basically, maybe to just write this determinant of, in the case of a 2 cross 2 you can see it immediately a b 0 d.

Student: Hello Sir.

Professor: Yeah.

Student: Sir can you please explain the point number 3 using properties 1 and 2.

Professor: So, if A has a row of zeros, then to do is to add one of the other rows to the 0 row, that does not change the determinant, then, now you have a matrix where two rows are identical. And by property 1 determinant of A is 0.

Student: Okay. Thank you sir.

Professor: Okay, now, for this property to see that if A is triangular, then determinant of A is 0, then one way to see that is suppose the diagonal entries are nonzero. Okay, then you have a matrix. So I will just show this in a picture to give you the idea. But or actually, I will just write this as in sort of example, to give you the idea, but the arguments hold in the general case. So I have a 1 1 a 1 2 say, upper triangular, a 1 3 0 a 2 2 a 2 3 and 0 0 a 3 3, then what I can do is, I can subtract a multiple of the third row from the second row such that this entry a 2 3 becomes 0.

So, I will just write that here. So I will replace R 2 with R 2 minus a 2 3 over a 3 3 times R 3 then this gives me a matrix a 1 1 a 1 2 a 1 3 0. Now, a 2 2 will remain unchanged. And this becomes 0 0 0 a 3 3. Then I can do something similar, like replace R 1, with R 1 minus a 1 3 over a 3 3 times r 3. And then I can again, replace R 1 with R 1 minus a 1 2 over a 2 2 times R 2 then what happens is both these terms get eliminated, and you will be left with a 1 1 0 0 0 a 2 2 0 0 0 a 3 3.

So basically, if it is an upper triangular matrix, or even a lower triangular matrix with the diagonal entries being nonzero, then you can do one more round of elimination and reduce it down to a diagonal matrix. And in this process, I only subtracted multiples of a row from another row. And so that does not change the determinant, I have not changed the determinant throughout this process.

So the determinant of this matrix here is the same as this determinant of a diagonal matrix. But for a diagonal matrix, the determinant is very easy to find. What I do is I first recognize that I can take out a 1 1 from this because it is linearly dependent on the first row, it is linearly dependent on a 1, 1 1. So if I take this matrix And let me call it D, then determinant of D is equal to a 1 1 times the determinant of this matrix 1 0 0 0 a 2 2 0 0 0 a 3 3, then what I can do in this matrix is I can exchange the second row and the first row.

And then I know that it is the determinant has changed time, but I will keep that in mind. Now in the first row it is linear in a 2 2 so, I can pull out an a 2 2 and then I can once again exchange rows 1 and 2. So, the determinant will once again change sign it will be back to the same old sign as here. So, I can write this as a 1 1 a 2 2 determinant of 1 0 0 0 1 0 0 0 a 3 3 and then I can pull out a 3 3 and write this as equal to a 1 1 a 2 2 a 3 3 times the determinant of the identity matrix, which is equal to a 1 1 a 2 2 a 3 3. So, it is the product of these diagonal entries.

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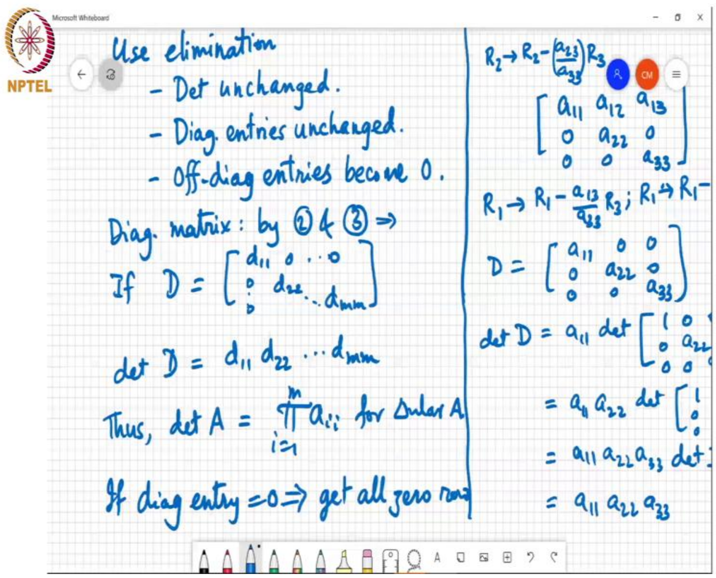

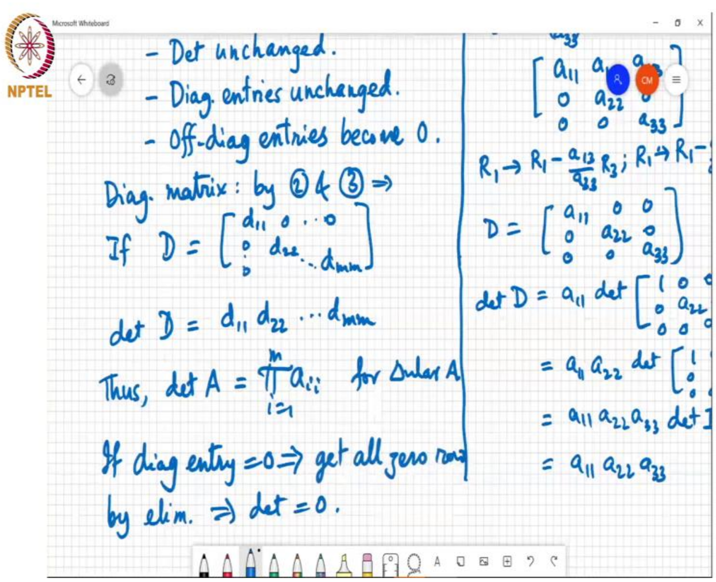

So, suppose the diagonal entries are nonzero, then I can use elimination so that means that the determinant remains unchanged also the diagonal entries remain unchanged and thirdly the off diagonal entries are eliminated. So, now we are left with a diagonal matrix. So by properties 2 and 3 determinant of, okay let me write it a little more systematically, if D is d 1 1 0, d 2 2 d m m, then determinant of D is equal to d 1 1 d 2 2 d m m.

Therefore, is the product i equal to 1 to m a i i for triangular, of course, if the diagonal entry is 0, what happens if you do this elimination step you will get an all 0 row which means that determinant is 0. Okay, so, let us continue.

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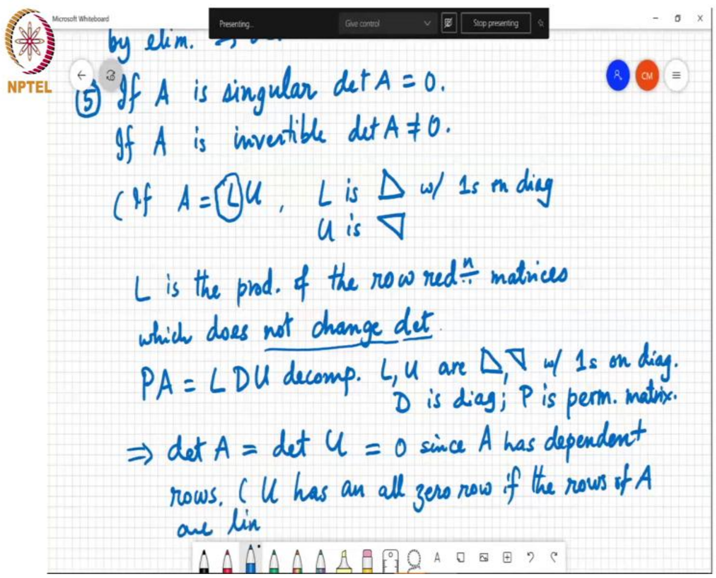

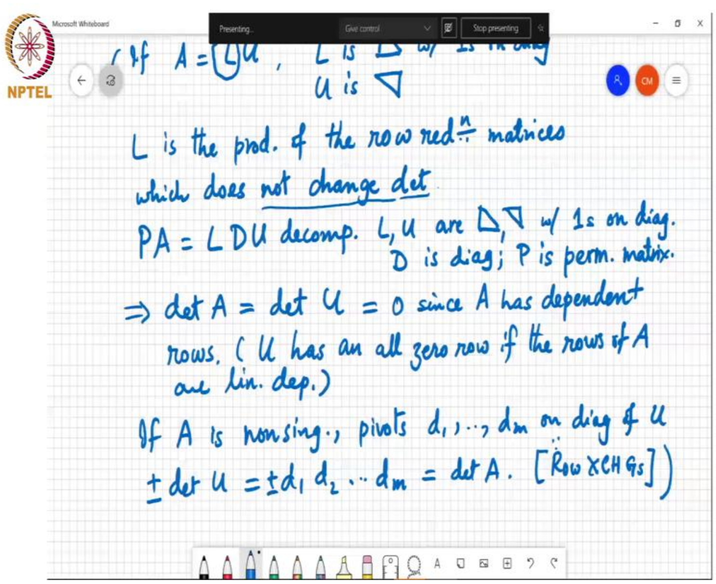

So, the next property is that determinant actually tells you if you can evaluate the determinant it tells you whether the matrix is singular or non singular. So, if A is singular. So, basically, what is a singular matrix, a singular matrix is a matrix which is not invertible. So, if A singular then determinant of is.

Student: 0.

Professor: 0, is invertible, determinant of A is nonzero, okay. So basically, the idea is that if A is singular, then it has, it has a rank which is less than m, which means that it has dependent rows. And so we will study this later. But basically, you can compute what is known as an L u decomposition of a matrix, where L is a lower triangular matrix and it is the row reduction matrix, which has ones a longest diagonal and U is an upper triangular matrix.

So, basically, I will just make some small note on this, if A equals L u, where L is lower triangular with ones on the diagonal and u is upper triangular, then L basically represents the row reduction matrices, L is the product of the row reduction matrices which does not change the determinant.

Student: Sir.

Professor: Yeah.

Student: Sir should u also have all the diagonal element as 1.

Professor: No, so u has the pivot elements along the diagonal. But there is another decomposition which we will study the I am not going into the detail here, because we are going to study these decompositions of matrices separately later on. The other way to think about it is you can also do something like and also I have not considered the case where you are using, you may have to do row exchanges in order to find this L u decomposition, the more general thing is there is actually PA equals L D u decomposition where L and u are lower triangular and upper triangular with ones on the diagonal and D is a diagonal matrix, and P is a permutation matrix.

Okay, this is the more general form. But for now, I am just giving you a little feel for why it is true that determinant of A equal to 0 for a singular matrix. So if you consider this, if you consider the process of doing the row reduction, the sequence of steps involved in the row reduction can be consolidated. Each one, each row reduction step is actually a linear transform. So what you are doing is a sequence of linear transforms.

And the effect of all those linear transforms is the product of all those linear transformation matrices. And that is a matrix L which has ones on the diagonal okay, and it is a lower triangular matrix. So, basically, since each step involved is a lower, is a row reduction step, it does not change the determinant and as a consequence, determinant of A is actually equal to the determinant of u which will be 0 if A is singular since A has dependent rows.

So, if A has dependent rows, then u has an all 0 row. So, if this is true, let me put it this way, u has an all 0 row. Okay, so then determinant is 0, but if A is non singular, then you will have the pivots. Say I can call them d 1 to d m on the diagonal of u. So determinant of u will be equal to d 1 d 2 up to d m which is equal to determinant of A.

So, I want to be a little careful here and so I am going to write it in this way, plus or minus this is equal to the determinant of A in the sense that when I started out here, I assumed that there is no row exchange operation that was performed. So, determinant of A is either determinant of u or minus determinant of u, depending on row exchanges.

Student: Sir.

Professor: Yeah.

Student: Sir when you call this pivot element so just another name for diagonal elements?

Professor: Yes.

Student: Okay. Okay sir.

Professor: So in the context of matrix factorization, the factors that come out when you do the row reduction operation, when you do the row reduction operation, the diagonal entries of the reduced matrix are called pivot elements. So, maybe one way to think about it is that I can ask what are the diagonal elements of any matrix. But these diagonal elements may not be equal to the pivot elements. Okay, but the diagonal elements will be equal to the pivot elements for triangular matrices.

For non triangular matrices, the diagonal elements will not be equal to the pivot elements, the pivot elements are obtained by taking the diagonal entries of the row reduced matrix. Does that answer your question?

Student: Yes sir, yes sir, it makes sense, thank you sir.

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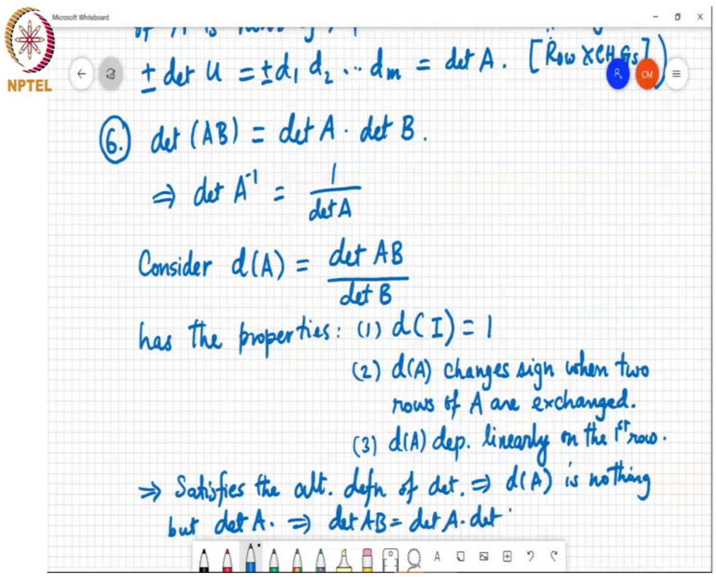

Professor: Property 6 is very important and also a property that is, again, to me really not obvious determinant of the product of two matrices is equal to what?

Student: Product of determinant.

Professor: Yeah. How does one show that determinant of A B equals determinant of A times determinant of B? Before I discuss that, one immediate consequence is that, if A B is the identity matrix, then B is the inverse of A, then the determinant of the identity matrix is 1. And therefore, we can say that determinant of the inverse of A when A is invertible is equal to 1 over the determinant of A because determinant, so determinant of A B is 1, which is because A B is the identity matrix and on the right hand side, I have determinant of A times determinant of A inverse.

So, the determinant of A inverse is 1 over the determinant of A. I will give you one proof here, there is another proof in the textbook which you can potentially look up on your own. So, one simple proof is like this. If I consider a function d of A, which I defined to be equal to determinant of A B divided by determinant of B. Now, I am assuming here that B is non singular, so that its determinant is nonzero, but that is okay because if B were singular then we know that the rank of a matrix cannot increase when you multiply it by another matrix.

So, if B is singular than A, B, or if B is rank deficient than A B is also rank deficient. And therefore determinant of A, B will be 0, and the right side is equal to 0, because it is determinant A times determinant of B, which is equal to 0. And so if determinant of B is 0, then the result determinant of A B equals the determinant of A times determinant of B is obvious. So we will consider the case where determinant of B is not equal to 0 and consider this function d of A defined to be determinant of A B divided by determinant of B.

Now, if I look at the alternative definition of the determinant, function, this is now a function of A, it is mapping A to a number, its mapping it to this particular number determinant of A B divided by determinant of B. Now it has the following properties. 1, d of the identity matrix, if I replace a with the identity matrix, I have determinant of B divided by determinant of B equal to 1, 2, d of A, changes sign two rows of A are exchanged.

That is because if I exchange two rows of A, then I also end up exchanging the same two rows of AB. Because exchanging rows can be thought of as left multiplication by a permutation matrix. And so exchanging rows of A is the same as in the effect of exchanging a pair of rows of A on the product A B is the same as exchanging those two rows of A B. And so d of A changes sign when two rows of A are exchanged, and 3, d of A depends linearly on the first row.

So, it satisfies the alternative definition. This you just think through a little bit, you will see it is true. It is because this determinant maybe here depends linearly on the first row of A. So basically, this means that it satisfies the three properties. I would say the alternative definition of determinant and so which implies d of A is nothing but determinant of A. And so determinant of A equals determinant of A B divided by determinant of B, meaning that determinant of A B equals determinant of A times determinant of B.