**First Course on Partial Differential Equations - II**
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**Lecture - 09**
**Conservation Law**

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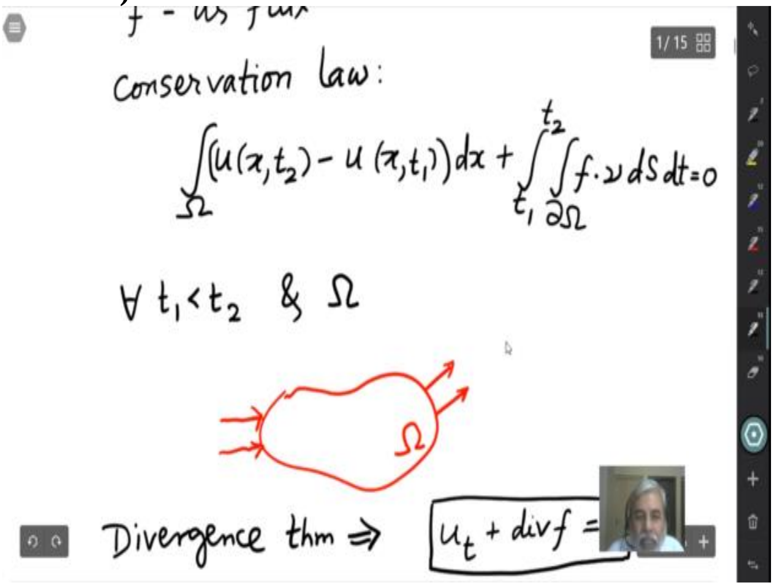


Hello, everyone, welcome back so we will continue the discussion on the conservation law. So before we go further, so we will discuss some examples and let us start with what is meant by conservation law. So, again this is motivated by fluid flow problem. So, imagine a fluid flow and so, there is some imaginary domain within the fluid omega. So, at any time the mass of the fluid within omega is a constant and that is termed as conservation of mass and motivated by that.

So, the mass comes from the density so, we have some general definition of conservation law. So u is density of that substance and f is its flux. So here that fluid, so suppose the fluid is in this direction, the arrow of direction, then fluid enters this domain and some fluid leaves this domain and at any time interval, so, we expect this conservation law to hold. So, this is density so, this gives you, if its density of the fluid, it gives you the mass of the fluid in omega that is present between 2 times t 1 and t 2.

And that must be equal to the whatever the amount of fluid that has entered the region and left at t 1 it has entered t 2 it had left and so, there is actually this should be equal to minus so

I am taking that so, f is the flux and nu is the outward unique knock as usual. So, this is a generalised statement of a conservation law. And when most of the time we are dealing with 1 dimensional thing, so, this omega is replaced by an interval.

So, we will see an example of traffic flow that is 1 dimensional. So, this is the conservation law in integrated form so, this for example, from fluid dynamics gas dynamics. So, this conservation of mass, momentum and energy they all appear in the integrated form, under appropriate assumptions on the continuity of the functions involved. So, we get the conservation law in the differentiated form, so, that is a differential equation.

So, first you divide by t 2 - t 1 and let t 2 = 0 and that gives you this u t here, so, omega integral omega u t and this side you have only this surface integral this when divide by t 2 - t 1 and let t 1 to t 2. So, that will give you a surface integral and then you apply the divergence theorem. So, that becomes divergence of f. And then if you combine these 2 you get integral omega of this quantity u t + divergence of f = 0.

So, for that is true for all omega so, once you shrink omega so assuming that this quantity is continuous. So, we derive the conservation law in the differentiated form. So, in deriving this mass, momentum and energy conservation equations, so, we need to take different u's and different f's.

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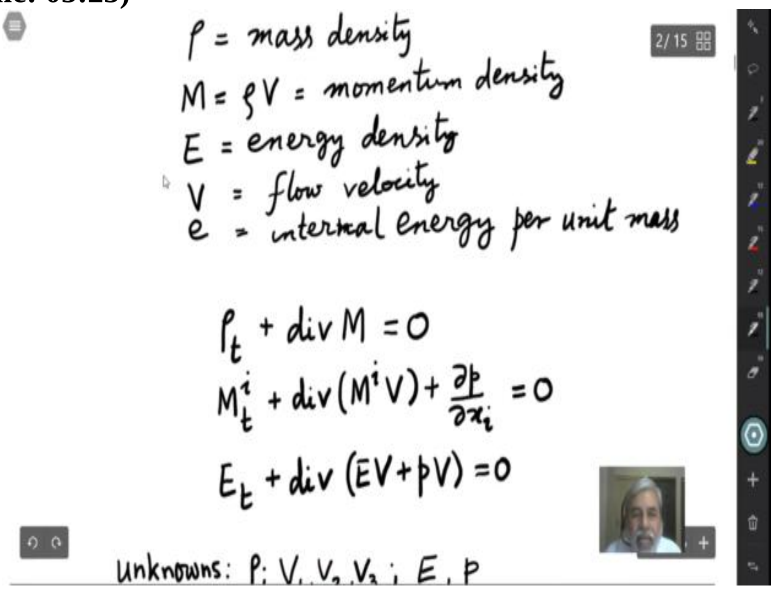


So, here I just written some terminology, but for more details you should consult a good book on fluid dynamics. So, now coming back to a fluid flow problem, so, these are the usual notations, so, this is rho is mass density and capital M is rho times V, V be the velocity, so, that is a vector again. So, there are 3 components in Cartesian coordinates, so, that is called momentum density and then the energy density and internal energy.

So, all these quantities, so, if you want to know more about the physical significance, you should read some good books fluid dynamics. And then using this conservation principle we derive these system of equations and then of course, there are 1, 2, 3, 4, 5, 6 unknowns but there are only 1, 3, 4 only 5 equations.

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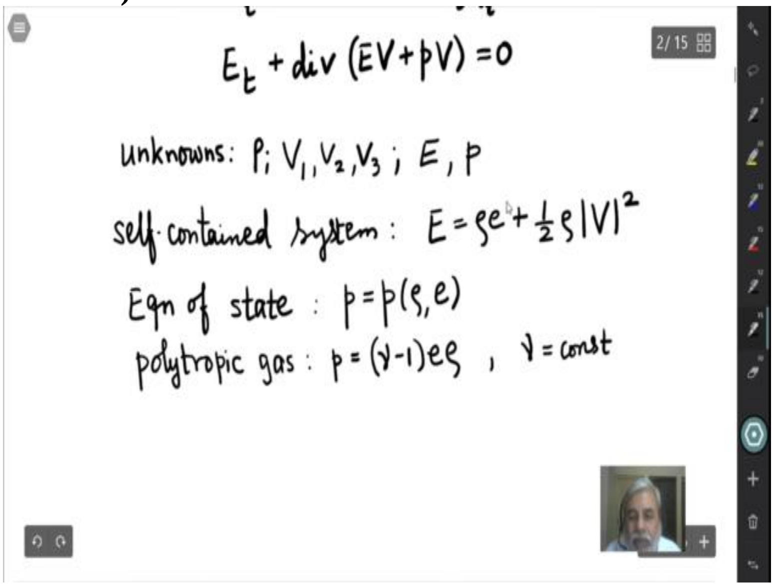


And to make it a self contained systems. So, many assumptions are made in the context of fluid dynamics, so, one usually gets this system of Euler equations so, you just go through some good book on fluid dynamics and in fact fluid dynamics gives you lots of models for this conservation loss that is main source in fact, that is the main source.

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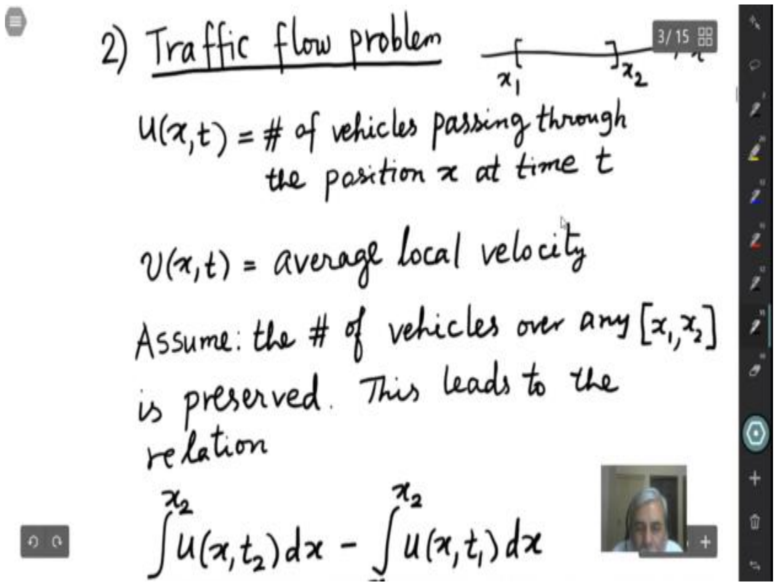


And the second example is traffic flow problem. So, this is again when we make that so, here so, this is long highway. So, that is along the x axis so, this is what we are proposing here this model is a single lane problem. So, one can also incorporate multi lane so, big highways are now that are multi lane highways, so, that is more complicated. So, here this is just a single lane and that too in a unique direction, so, if you include again both way traffic, this will be more complicated.

So, here is the simple model so, let u x, t denote the number of vehicles passing through the position at x at time t. So, this is a whole number, we treat this as a continuous variable. That is usually what is done in most of the modelling problems, you want fluid flow problems. So they are actually finitely many particles, but then there is continuum hypotheses etcetera which makes it possible to consider has a continuum.

So those are some of the philosophical questions. So similarly, here, we are doing that. And let v x, t denote the average local velocity of the vehicles that is when local speed. And let us assume that at any into x 1, x 2 and this highway. So at any time, t, the number of vehicles within this interval, x 1 into 2 is a constant. So there is  the assumptions of conservation and that leads to just take a look at this so that u number of vehicles is our density now and the flux will be u times so u is the average velocity.

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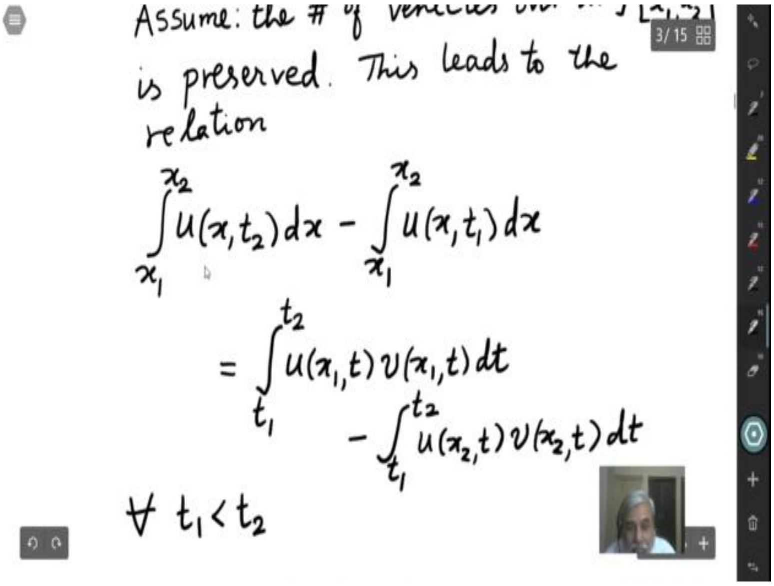


And so, that relation is in the present context. So, this at any 2 times t 1 and t 2 assuming that this number of vehicles in any interval is preserved, we get this conservation relation.

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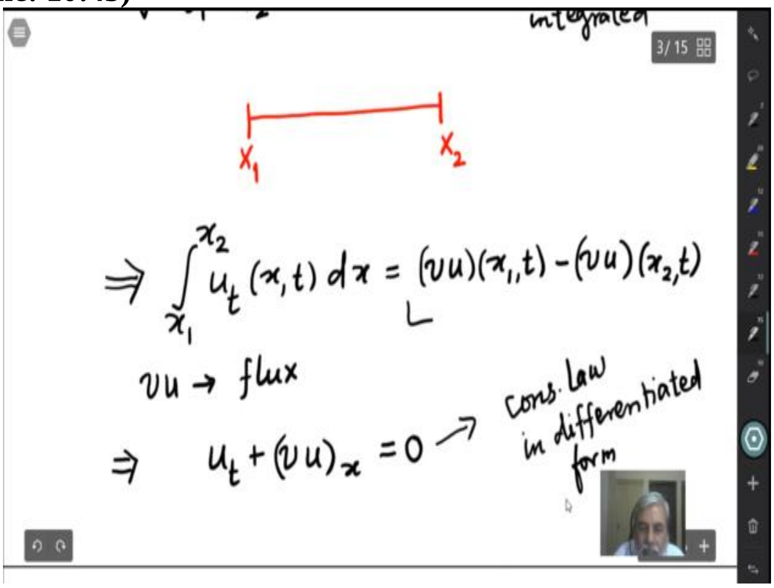


And again you divide by t 2 - t 1 and let that go to 0 that difference so, we get left hand side we get integral of u sub t. So, that is the derivative and right hand side to get vu x 1 t, vu x 2 t. So, in this relation we easily see that vu denotes the flux. And again you will divide by x 1 - x 2 and let that go to 0. So, then we get so, this is the actually this one is the conservation law in integrated form and provided the quantities in question are continuous.

So, u t should be continuous and this vu should be continuous and this is the conservation law in differentiated so, that is a different thing. So, again first start with order we get first order partial differential equation.

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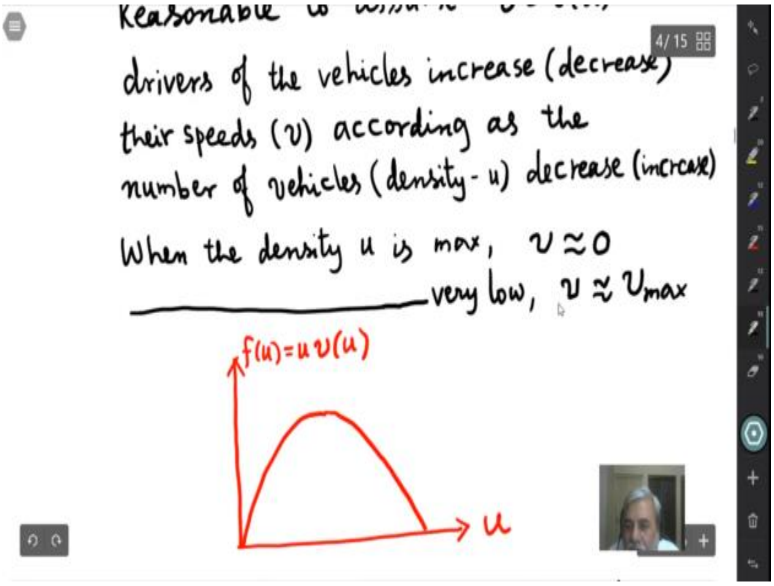


And now make some reasonable assumptions on the local velocity. So, obviously, we can assume that this local velocity is a function of u. So, as the experience shows the drivers of the vehicles increase or decrease their speeds, then that is v according as the number of vehicles and that density u decrease or increase. If there are less number of vehicles on the road the speed will be increased and if there are more number of vehicles obviously, the speed will be reduced and when the density is maximum.

So, almost all vehicles will come to a standstill and that represents almost 0 velocity and in any even if the density is very low, usually there are speed limits. So this velocity cannot peak cannot exceed certain limits. So, when density is very, very low, so, v is almost that a load limit that will be x. And so that represents the speed max there. So, when u is so the density is 0 obviously the no vehicle so speed is obviously 0 and again when u with that maximum possible density.

And then again v 0 and so there is a maximum speed. And in this case, looking at the graph of this a few we see that f is a concave function. And so this conservation law is a concave conservation law. But most of the theory that we are going to study applies only to convex. That is the main adjunction convex conservation law that is f the flux convex but that is easily turned into a convex law by changing the sign. So, if f is concave minus f is convex so, that is not a problem at all. So, this traffic flow problem also very well fitted to the general theory of convex conservation law.

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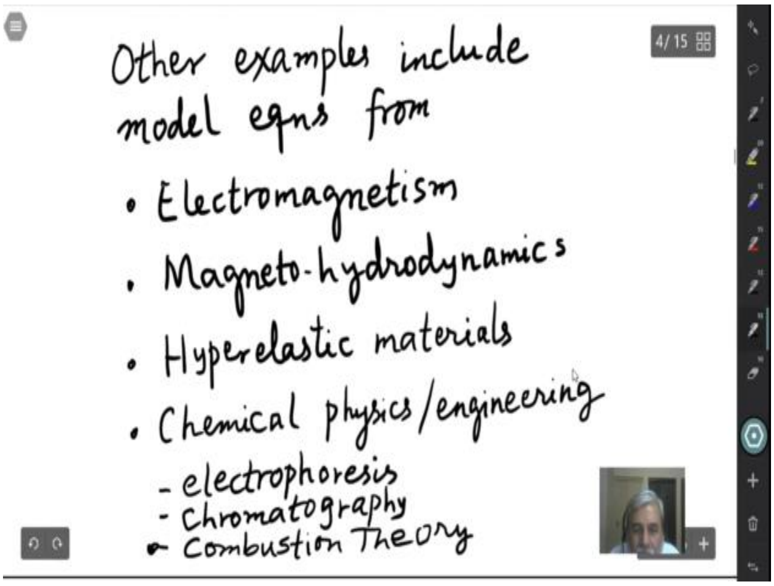


And I just mentioned here some other examples the model equations from many fields come into this form of conservation laws and these fields include electromagnetism, magneto hydrodynamics, hyperelastic materials, chemical physics and engineering, there are processes called electrophoresis and chromatography. So, these are again very good models and these how benefitted lots of industries and there is also combustion theory.

So, these all fall into this subject of conservation laws. So, these are some important class of examples of course, major part of the problems they come from fluid flow problems. So, with those examples now, we proceed to discuss the solutions of the conservation laws.

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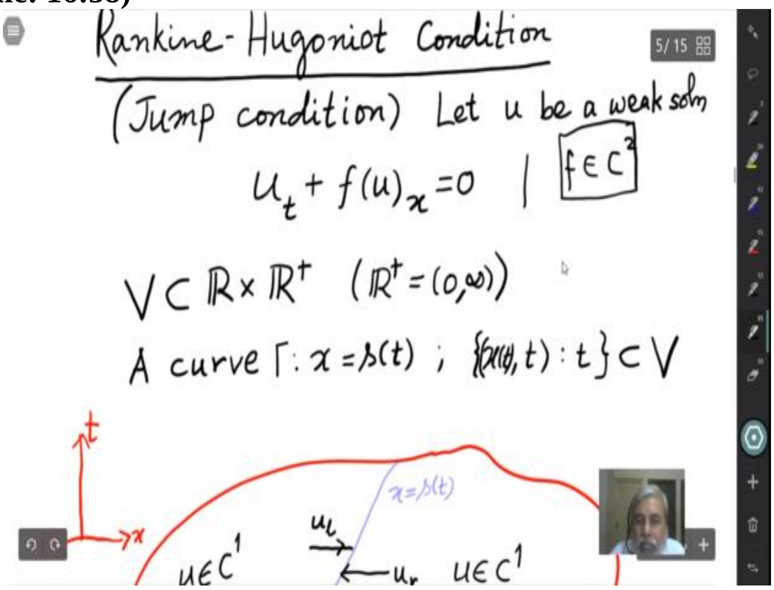


So, let me again start with this so, usually we assume this f is at least c 2 more assumptions as follows and this is an example of quasi linear first order equation. So, this can be solved by method of characteristics as long as the characteristics do not meet each other. So, that we have already seen in first part. So, let us now see if the how they discontinue develop how should this solution behave across a discontinuity.

So, let u be a weak solution of this conservation law and you take any open set in R cross R plus, R plus here is the positive real axis and let this gamma be a curve sitting in v. So, we assumed that this weak solution as we discussed in the previous class, so, the definition of weak solution gives us an integral relation in terms of the test functions.

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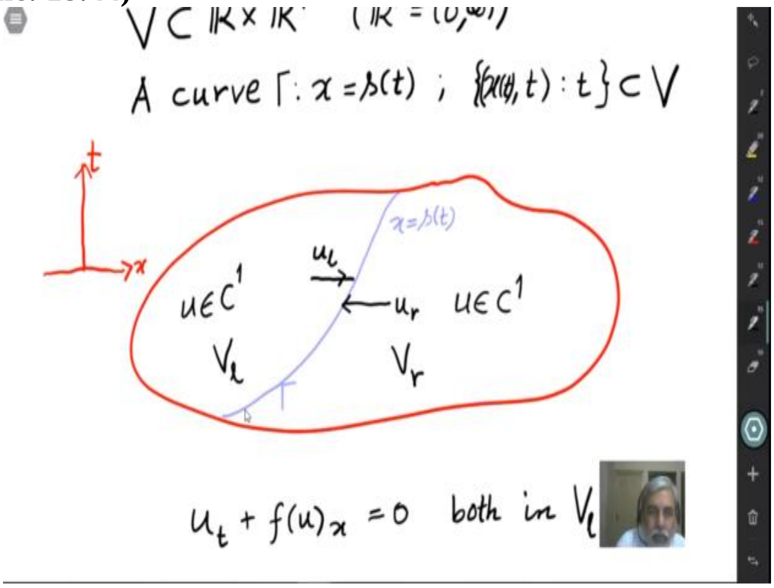


So, suppose we assume that so, this is our V and this curve divides this domain into 2 parts V l and V r, V l for left hand, V r right and assume that this weak solution is C 1 in both the these parts, so, left and right and gamma is the curve of discontinuity for the solution. So, we also assume that this limit of u when it approaches from the region l approach is curve gamma has a finite limit.

And similarly when it approaches from the region V r that is also finite and call them those limits, u l and u r, of course, depend on this weak point V r on this curve. So, u l, r functions of t across this gamma and u l is not equal to 1. So that is discontinued and since we are assuming u is C 1, in both V l and V r, as we saw in the previous class, this, u satisfy this question both V l and V r though it is a weak solution, but since it is C 1 here, so at those points, it satisfies the differential equation point wise, both in V l and V r.

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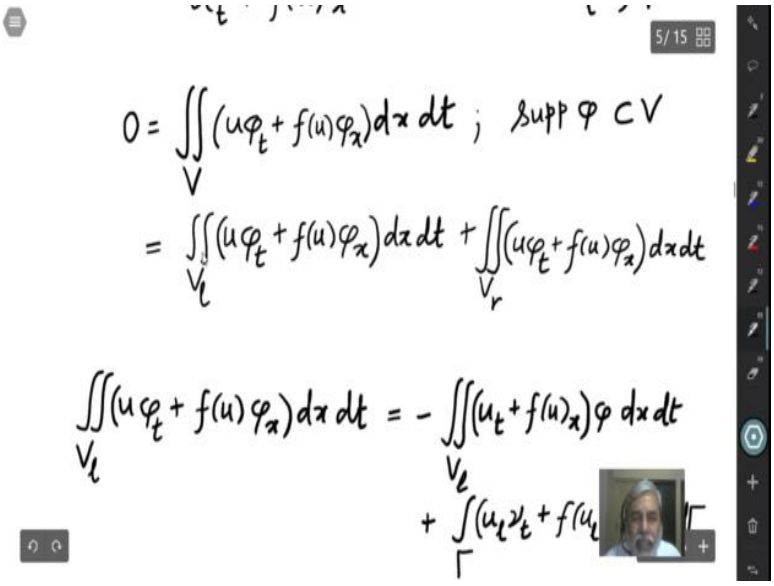


And now you start with the definition of the weak solution. So the origin is the weak solution. So this holds true for all test functions phi. So there is another term coming from the initial data, but if we do that suppose phi is in V, and 0 is not included there. So that phi 0 is 0, so other V is not there and since this V the union of V l and V r. So, that is the advantage of the integration.

So, we can take that as sum of 2 integrals one over V sub l and another one from V sub r. So, consider for example, this V sub l integral, integral over V sub l and here so, since we are assuming u is C 1 in V sub l, so, we can integrate by parts. So, when you integrate by parts, we get that first term and then there is a boundary term so, just remember the boundary here. And what matters is this values on this gamma and values on the other part of the gamma are V l are taken care by the support of phi so, they will not be there.

So, only that integral over gamma appears. So, you apply Greens formula you get that and then V l we already seen that this for that vanishes. So, this integral over to V reduces to this line integral.

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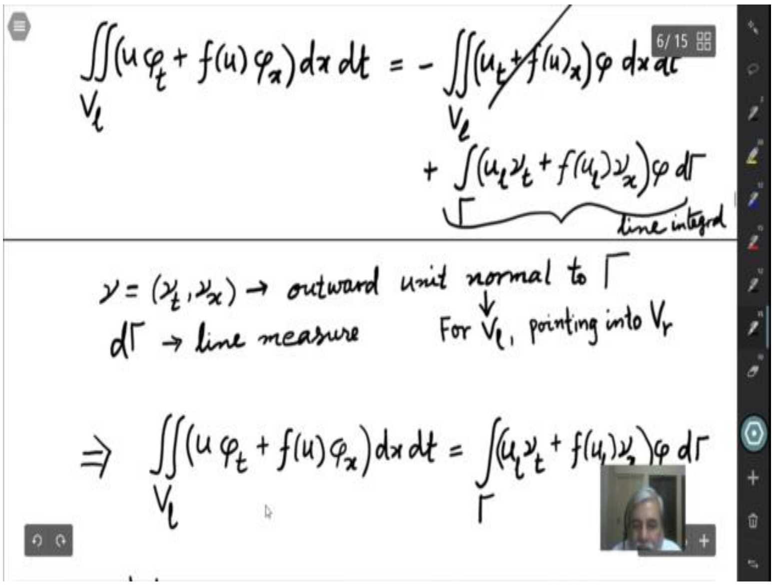


Where this nu t and nu x this components of the outward unit normal to gamma chain that this is gamma, so, since it is outward unit normal, so for the part V sub l this normal point towards V r, and for V r it points towards V l, so there is a sign change so that should keep in mind and this is the line measure you want to call that, it is just this is a curve. So this is line integral deca just that is notation. So, this is the line integral just remember this is from the Green's formula or divergent theorem and that gives us so, this term vanishes so this integral over the region l just given by this line integral.

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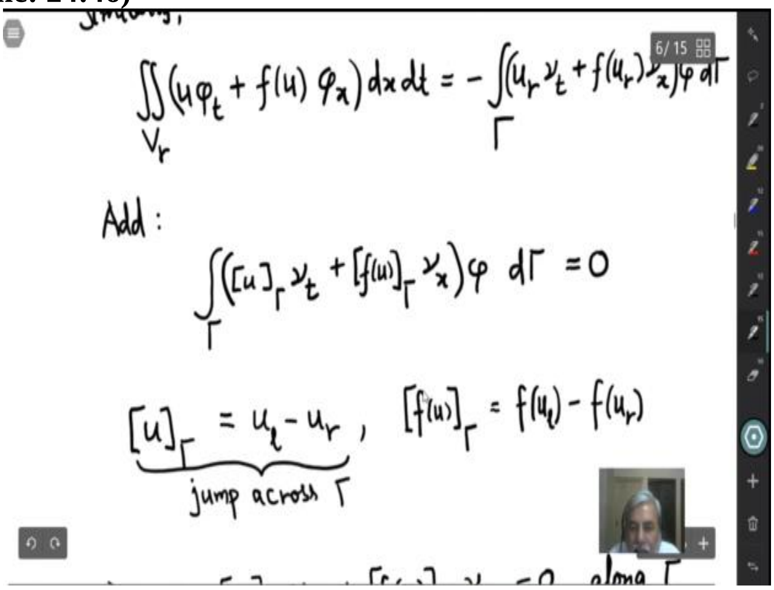


And similarly for the region V sub r is again line integral and as I said this sign changes so we have minus here because that direction of the unique normal changes, so get this minus and now we add these 2 relations. So, this integral over V l + integral over V r is equal to integral over this line integral over gamma and line integral over gamma. But the left hand side if we had integral over V l and V r and that is 0 that is how we started with this sum of these 2 integrals if 0.

So and that we get so, this u l - u r nu t f of u l - f of u r nu x whole multiplied by that test function phi is 0. So, here are denoted by the square bracket. So, these are jump across and that jump induces a jump in the flux here. So, this is jump of u and jump of f of x just by adding these 2 relations we get there and again since phi is arbitrary test function we should have this relation so, this integrant vanishes.

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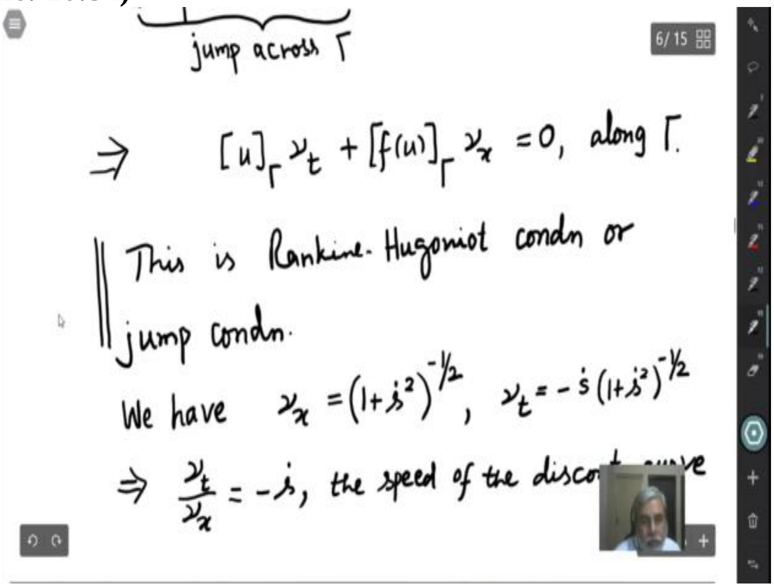


So, this of course along the gamma this is referred to as Rankine Hugoniot condition or simply jump condition. So, any weak solution of the given conservation law that is first order ODE. So, any weak solution of this conservation law having a discontinuity along the curve and smooth on either side of the discontinuity curve must satisfied. So, this is a necessary condition must satisfy this Rankine Hugoniot condition.

So, now, we can rewrite this in more neater form if you assume that this smooth curve is given in terms of a function namely x = t. So, we can express the components of the unit normal in terms of that s. So, this so, gamma is described by this curve called x = st and s dot denotes derivative with respect to and so, that denotes the speed of the this discontinuity curve in terms of s we have new sub x the component of the unit normal unit is one plus your s dot square to the minus half and the t component is minus one plus s dot square minus half.

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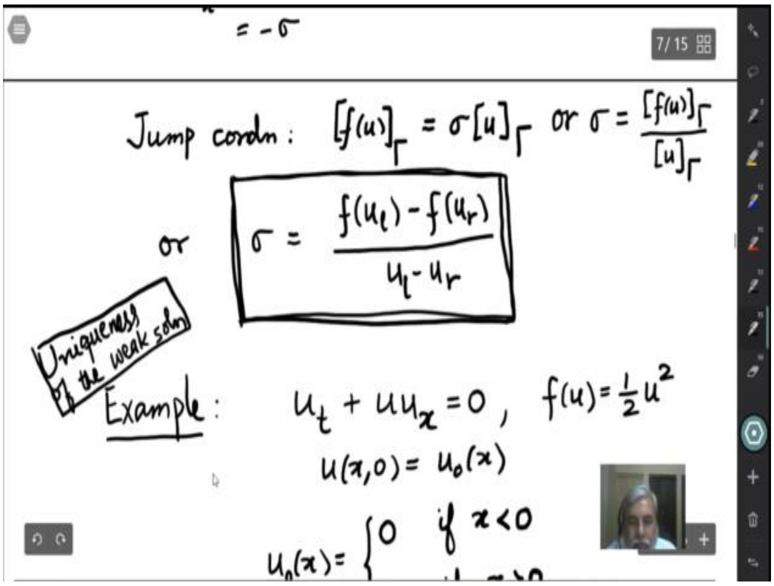


So, if you plug in that in this so, there is a new t and x and do some simplification what you get is f of u the jump across gamma is given in terms of jump of u multiplied by that sigma. So sigma so, this is minus they are functions of t just remember that. So, this speed of the discontinuity curve is expressed in terms of the jumps of the solution across that discontinuities.

So, this is an important relation that every weak solution must satisfy, our next discussion is regarding the uniqueness question. So, this example is with regard to uniqueness of the peaks of solution whether one can have more than one weak source and this example is towards that. So, consider this Burgers equation u t + uu x = 0. So, in terms of the f so, here in this case here f of u is half u square and let the initial data be given by this u 0.

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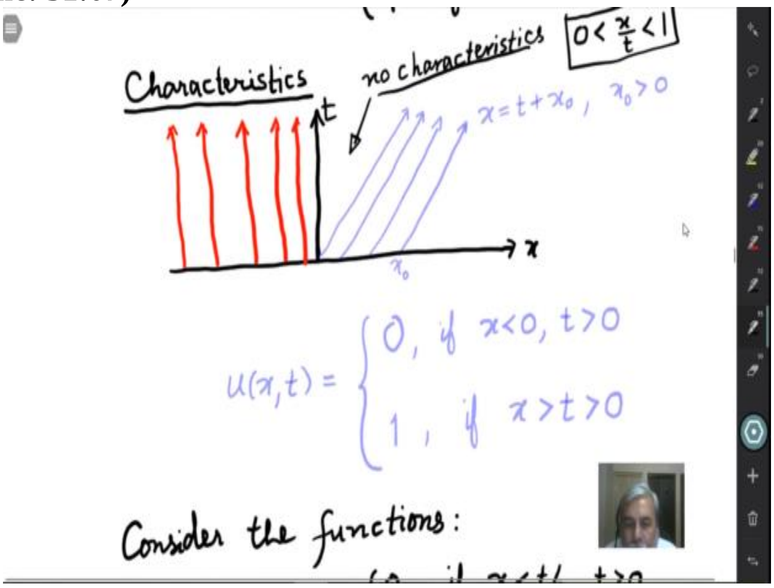

So, u 0 has discontinuity at x = 0, so, it is 0 for x less than 0 and 1 for x be 0. So, the characteristics in this case are very, very simple. You can recall from what we have done in the first part,. So, for x so these are in the plane so, for x less than 0 the characteristics have 0 speed here that comes from this 0 initial data there. So, they are all just straight lines so, x equal to constant.

So, I have put some arrows here, that is just to show that so, in this case, of course, they are constants so, there is increasing time. And for x bigger than 0, they all have the characters have all speed one, so, they are just given by so, these are the curves. So, any general curve here is x plus some x 0. So, this is x 0 they are all have slope 1 these are the characters and that leaves this portion this portion what is that portion? These portion no characteristics.

So, the method of characteristics does not give any value of the solution in this-region and what is that region so this is just 0 x / t that is the region. So, the method of characteristic just gives you this u of x t = 0 if x is less than 0. So, we are interested only t positive testing that and one again if x is bigger than t greater than 0, so, this leaves that region 0 less than x less than t. So, the method of characteristics would not give you any solution. Now, we try to construct solution weak solution for all x and t.

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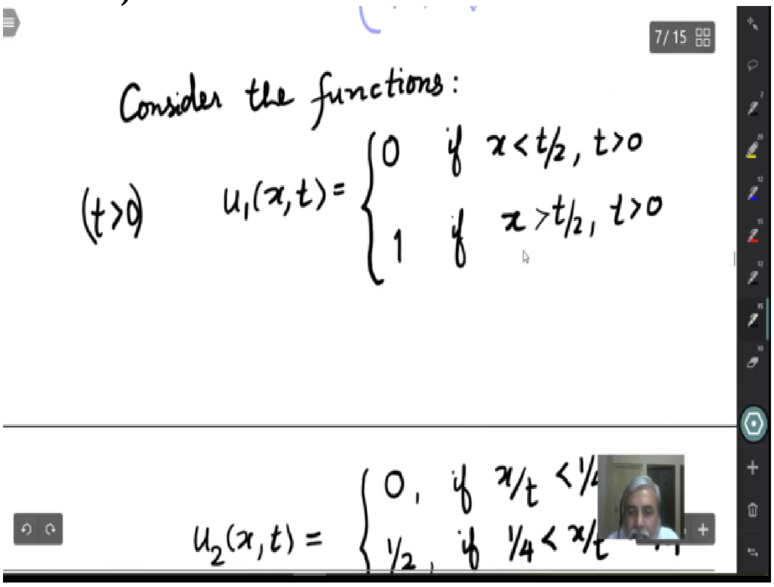


Now consider these 3 functions so I am defining 3 functions here. So, u 1of x t I am defining for all t is positive so that is I want to define x t positive t = 0 we have given that. So u 1 of x t is 0 if x is less than t / 2 and 1 if x is bigger than t / 2. Of course, any constant is a solution of the given equation, so, only it needs to satisfy the initial condition. So, in all these cases, we are just gluing appropriate constants to the left of the line and to the right of the line.

So, here you can see so, even t 0 can take the limit. So, x less than 0 we get 0 and x bigger than 0 we get 1 so, that initial condition is certainly satisfied and since they are just constants, so they also satisfy the equation. And so, here the discontinuity is given by x = t / 2 so we will see that.

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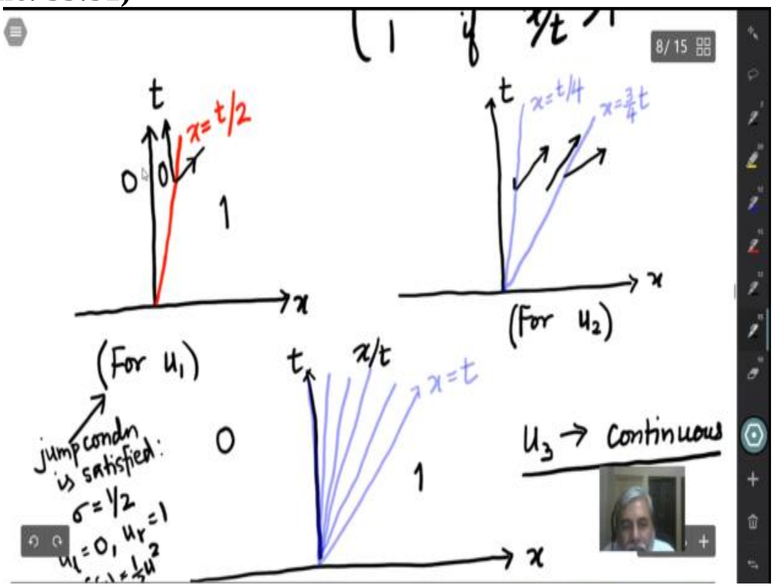


And again here so, u 2 also consists of 3 constants states, so 0, half and 1 in different regions. And now, this is if x / t is 1 by less than 1 / 4, and half, 1 / 4 and 3 / 4, and one gives x / t together 3, 4, and one more function. Let me define that and then we will analyse all the resources. So, u 3 0 if x / t less than 0 and the x / t if 0 is less than x / t less than 1and 1 x / t is bigger than. So, let us consider this first function u 1 of t, so x = t / 2 that is the line of discontinuities.

So, in this case, the curve is a straight line x =  t / 2 and so, we have on the left we have the value 0 and on the right we have the value 1. So, if you plug in this into this jump condition, you easily see that the jump condition is satisfied. So, in this case we have constant speed so sigma is just half x is t / 2. So x is t, x is t / 2 so if derivative is just half and as I said this u l is 0 and u r is 1 and our f of u is half u square.

So you get the f of the u l is 0 f of u r is half. So if you plug in now this value, so you get half equal to half, so jump condition is easily seem to be satisfied here. And you can even put in these values so these are just constant state, so it is easy to verify that is in fact that is not difficult at all this presenter consisting objects. And what about u 2, so u 2 we see there are 2 lines of discontinuity, namely, we x = t / 4, and x = t / 3 and again, which 0 here, half there.

So you are just joining this constant functions by these lines of this discontinuity and the as we did here, so you can easily check that the jump conditions across both the discontinuities. So here, we have 1x = t / 4 and here it is equal to 3 / 4t. So again, jump conditions are satisfied. Now there are 2 so, you can see what about u 3? u 3 there is no need to verify the jump condition though accurately you see that the x = 0 and x = t but across those 2 this u 3 is continuous.

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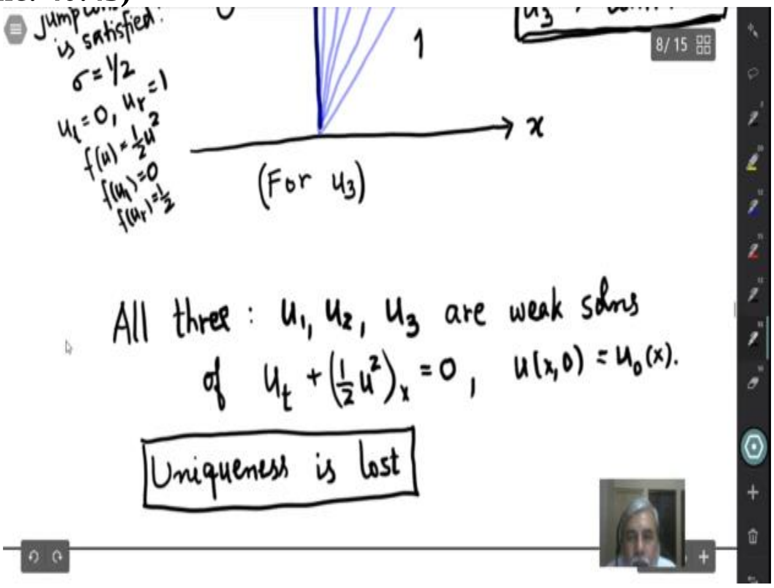


So, all 3 so namely u 1, u 2, u 3 in fact you can produce many more are weak solutions this will be x 0. So, certainly the uniqueness is lost but uniqueness is an important requirement in the study of any physical situation which will not like to have multiple solutions. So, which one to choose and which one to reject?

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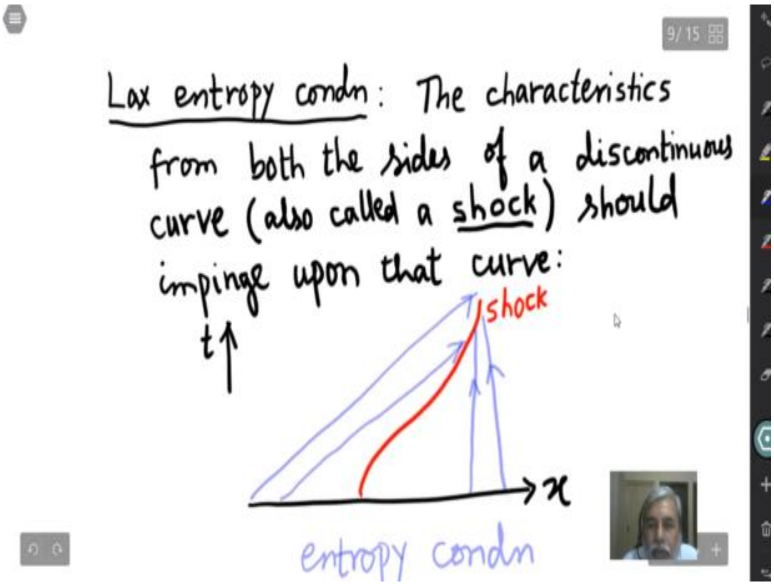


And that is where the physics has to enter into picture the physical situation and it was Lax who first proposed this particle Lax's entropy condition simply about the condition. So, let me write the condition and then we will see which one we should keep as a solution and which we should reject. So, here so, the characteristics so, this is in here looks a little complicated the characteristics from both the sides of a discontinuous and this is also called.

So, this also we said already also shock should impinge upon comment something like that. So here is x this is t so, this is say shock. So this entropy condition will suppose these are the characteristics from this side. So this is the top so if you look back again these examples. So you see, that is why I have drawn that yellow so, the entropic condition. So, in this case handouts will discuss.

So, for this reason this because of the central failure of the entropy condition, these 2 solutions are rejected, so, here the question of verifying the entropy condition does not arise because u 3 is continuous. And this has a name, so, this is called rarefaction, so it is a constant state and the left of this so, this is like a fan in fact, it is called fan wave. So, in this fan wave joins these 2 states 0 and 1 continuously. So at the left, it joins 0 and at the right, it joins 1. And over the years, this central precondition has gone into many different forms. So, obviously, this is somewhat tedious to verify in a given situation.

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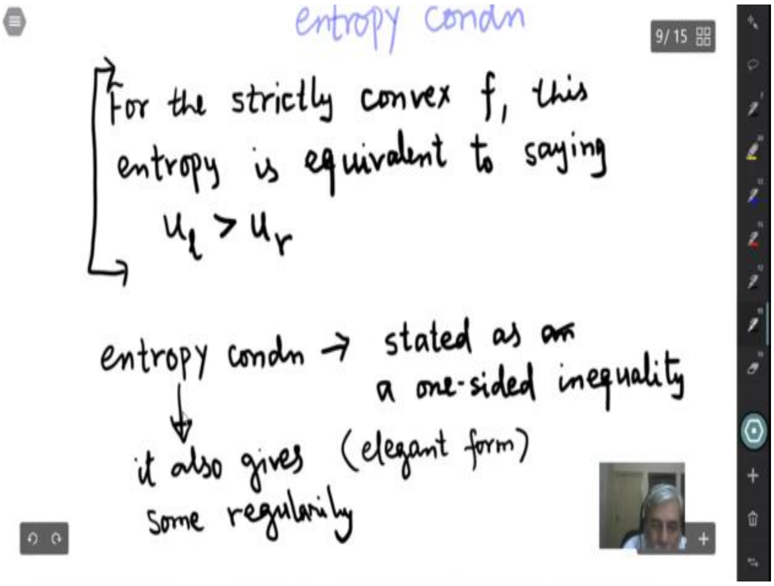


But for so that is what Lax observed so for the strictly convex so in that case, we call the conservation bias strictly convex conservation law. This entropy condition is equivalent to saying u l is bigger than u r. So remember u l in the limit of the solution as it approaches the discontinuity from the left and use a bar the value of the solution as opposed from the right and even in that, so, in this case this half u square is strictly convex, there is no problem with that.

So, here we have on the left it in 0 and on the right 1. So, that it is violated and again here on the left it is 0 and here it is half again that is violated and here on the left it is half and on the right it is 1 again that is violated. So, u 1 for the other region so this so this is somewhat simpler. But this applies only to this strictly convex of course, we are dealing with that and now it is this entropic condition.

So, this is the result of many years of research. So, this now stated in the form of an image one, one sided very elegant form. So we will discuss this after we obtain Lax-Oleinik formula. So, it combines the earlier versions into so, this is just elegant form and it directly involves the solution. So, it is in terms of the solution and as you see, it also gives rise to so also gives some regularity we see all these things.

So, in order to avoid this non uniqueness, we have to impose an additional condition now that is known as in entropy condition so, this is an easier version of saying that, so, but will soon see this can be stated as a one sided unique quanity, so we from here, we proceed next time. Thank you.