First Course on Partial Differential Equations - II
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Lecture - 08
Conservation Law

Welcome to this second part of our video lecture series on PDE 1, the first part was covered in 20 and odd lectures and the similar number of lectures will be delivered in the continuation second part. I am P. S. Datti from former faculty from TIFR Center for Applicable Mathematics Bangalore. Professor Nandakumaran my colleague and friend for more than 30 years and myself are giving this set of video lectures like in the first part, who will again focus for most part of this lectures on classical theory of PDEs.

However, there are some exceptions so, we include a discussion on Hamilton Jacobi equation and conservation laws both are examples of first order quasi linear equations, which we dealt with in the first part of the course. And in the first part of the course, we also mentioned that in the quasi linear equations in general the singularities are discontinuities in the solutions after some finite time will not be able to obtain solution in the classical sense that is we cannot obtain solutions which are differentiable for all time.

And that necessitated the inclusion of so called weak solutions, because these problems are of great practical importance and the solutions for all time need to be defined, you will need a discontinuous maths and that brings us to the formulation of weak solution. So, what is meant by weak solution of these 2 important equations and those things will be dealt with in detail as we move on. There is also an inclusion of some introductory lectures on weak formulation of PDEs.

And these are meant to arouse some interest at least in some of you who would like to pursue their higher studies in this area of PDEs. So, these modern techniques are inevitable if you want to pursue research in the subject of PDEs and hopefully those few lectures introductory lectures on weak formulation will be of some help. So, I begin today a discussion on scalar conservation laws in the next few hours, you will be listening to this in some detail.

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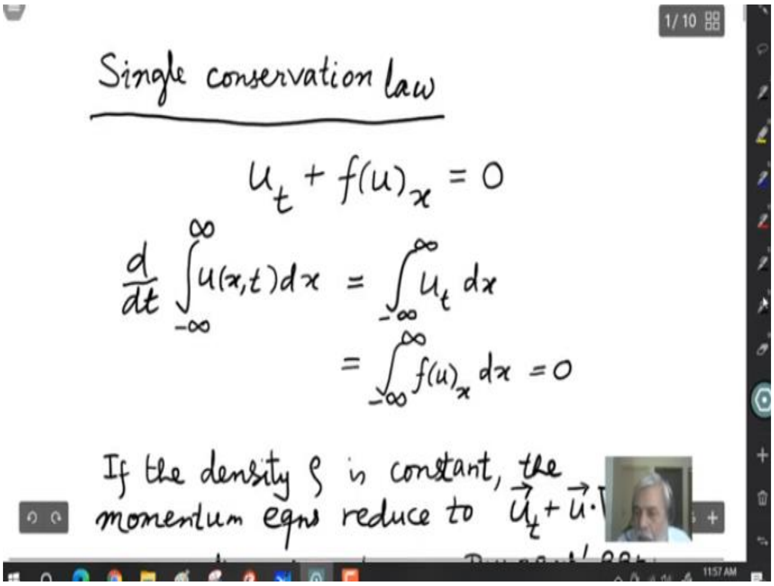

A single conservation law is given by this equation, first order quasi linear equation u sub t + f of u x = 0. So, as you go along we will put further conditions on this nonlinear function f for the time being just a general discussion why it is called the conservation equation and you just look at this integral minus infinity to infinity u x t dx. So, this is a function of t, so let us differentiate that with respect to t. So, everything is formula here so this integral of finite etcetera.

So, suppose you know this differentiation under the integral sign is a lot so again, you just do that, so we will get integral of u t and that is the minus sign here, minus f of u x sign is there is a derivative. So, assuming that infinity, they just was so that will get 0, so there this quantity is a constant, so it is conserved in time and as you go along will give more examples.

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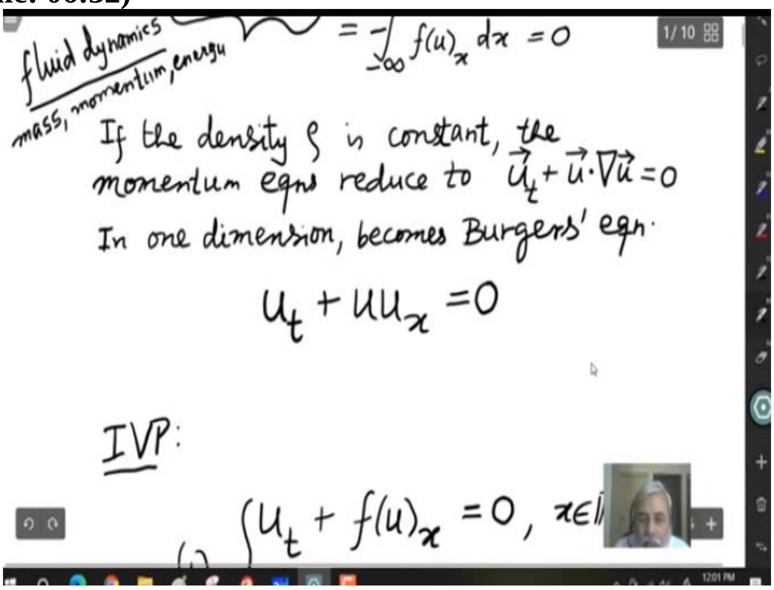

So, another example is from this gas dynamics in fact, most of the conservator equations are derived from fluid dynamics. So, we will discuss some of these things, but not in detail so that you can refer to any introductory book on fluid dynamics. So, there are 3 equations mainly so conservation of mass, momentum and energy, so other assumption are the fluid in question, with some simplification so if the density of the fluid is a constant, then the momentum equations reduced to this vector equation.

So, there are 3 equations here and this if we consider just 1 dimension, so that becomes Burger's equation. So, this we have studied even in the first part in great detail and this was the original idea of Burger's to take a simplified model in order to study the turbulence nature of the fluid dynamics. And since then this equation has played an important role in the development of this theory of conservation laws. This is a single equation there whereas, if you look at the linear equation they are systems.

So, system and single equation there is lots of difference in fact many things are still not complete as far as the system of conservation laws are concerned. And this has remained along with the Hamilton Jacobi Equation an active research area for many years and it continues to be one.

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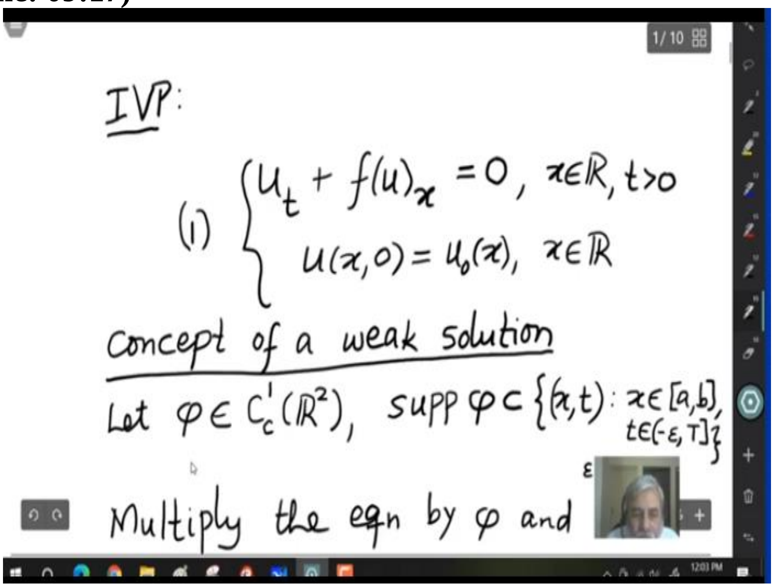

So, most of the discussion on this conservation law is confined to this initial value problem. So, u t + f of u x = 0 and some initial condition is given. So, as you go along we will precisely tell you the assumptions on the non linearity f and also on the initial condition. And as we have observed through the example of Burger's equation and with many initial

(())(10:00) smooth initial conditions, we have already seen that the differential solutions will not exist for all time so, t is time variable. So, there is need of considering non differentiable functions as solutions of this equation.

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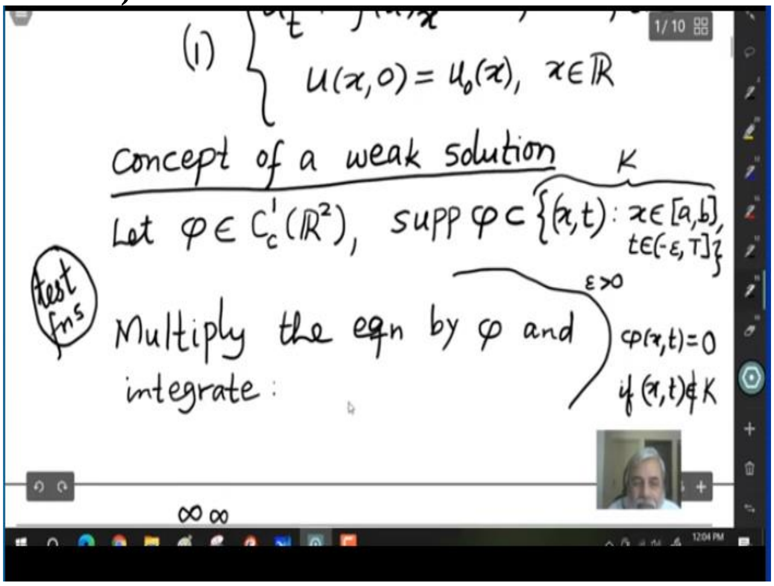

So, let us start with a discussion on what is meant by a weak solution there is concept of a weak solution. So, this is the motivation so, suppose we have a smooth solution that is a differentiable solution to this problem 1. And now you multiply this equation phi a smooth function so, this C c 1 that subscript c denotes compact support. So, this phi as a function of 2 variables x and t, it has support in this such a set of code this set depends on phi.

So, as you change this phi this support also changes, but the structure is the same and so this what that mean? Again recall so, phi of x t will be 0. So, for the time being just call the set as some K if x t does not belong to K. So, that is meant by the support of phi. So, you take any such function and these are called test functions. So, multiply the equation by phi and integrate.

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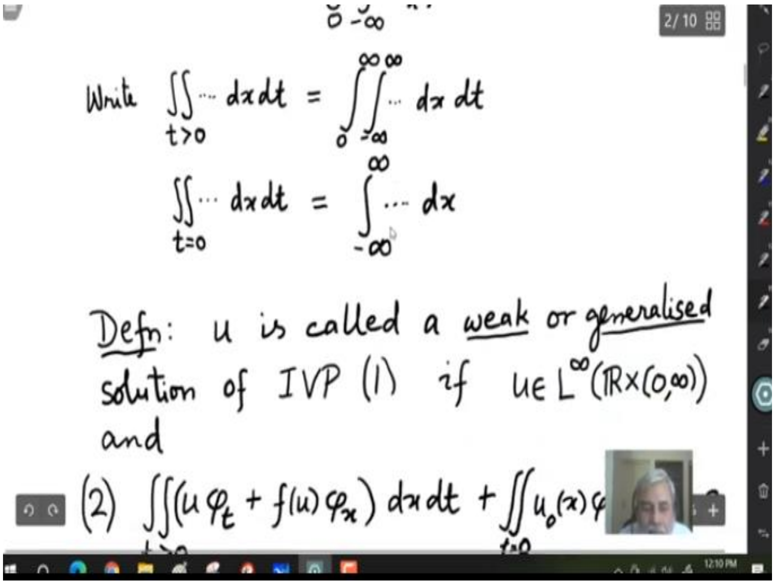

So, since this is 0, so I start with left hand side 0 and you integrating you are integrating with respect to t from 0 to infinity. So, remember, we are interested only t from 0 to infinity. So, with respect to dx 0 to infinity and with respect to x all of real line and now you integrate by parts. So, with respect to t here I do so, this minus sign after integration with parts. And that will produce since this is a finite quantity there namely 0.

So, that will with respect to t variable it will produce a boundary term and that you again you integrate with respect to x so, that is phi of x 0 u of x 0 dx and with respect to x variable, so which is case t of phi of x f of u dx dt and there are no boundary terms here because of this reason. This 5 vanishes outside a finite interval so, there will be no boundary terms here.

So, there is only boundary term here that will also be 0 phi of x 0 0 but in general that phi of x 0 is not 0 by our this nature of phi we have taken. And for brevity let us write this double integral t positive this one 0 to infinity, minus infinity to infinity and t = 0 it is simply this integration with respect to x and we are putting t = 0 there. So, this will be just double integral t = 0. Here it is a definition.

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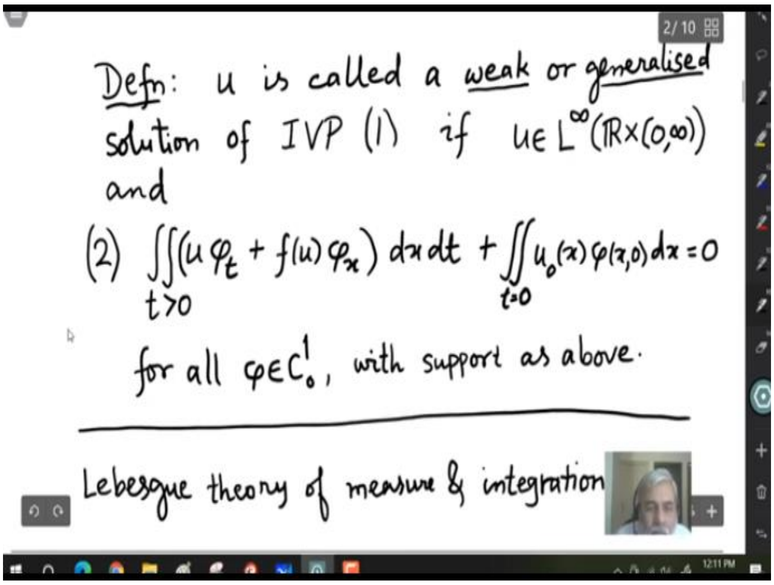

So, u is called a weak solution or generalized solutions of the initial value problem, again just remember that that will be the IVP will be dealing with throughout this series of lectures on conservation laws. So, if u is a bounded function so, this u L infinity R cross 0 infinity and this integral relation is satisfied and that integral relation is precisely this one. So, 0 is equal to minus minus minus so, you just make that plus.

So, u of phi t f u phi x dx dt + u 0 x and that is coming from the initial condition u 0 x phi x 0 dx for all phi which are test functions so, the test function means it has compact support and support is in such a rectangle in the x t plane. And this one remark here so both in the study of Hamilton Jacobi Equation as well as conservation laws we will be using some lebesgue integration theory and lebesgue theory in general which we are not theory of measure and integration.

So, some arguments crucially depend on this theory and better recall from your study of measure and integration some of these arguments. And this is the only place we are using this modern theory of integration. Otherwise, for most part of the course, we will be dealing with only smooth functions and this is. So, this u belong to L infinity R 0 infinity means u is a measurable function and it is essentially bounded in the lebesgue theory you call that essentially bounded except on the set of measure 0.

And that will be sufficient for this integration to make sense because this integration is only a bounded set of the x t plane because this phi has supported in bounded itself. So, these integrations they make sense. And for this definition, you also notice that we do not require any differentiation assumption on the solution. It is just a measurable and essentially bounded function. The only requirement on that function is these integrals R phi and that is taken care by the boundedness assumption.

So, this is the weak formulation of this scalar conservation law. In fact, this is the procedure followed for most of the linear and nonlinear PDEs. So, u multiply by a test function suitable test function so, this again class of test functions differ from problem to problem and you throw all the derivatives on this test functions by many integration by parts and you generally end up in the definition of weak solution now, given linear or nonlinear PDE.

In case the problem is posted in the setting of a Hilbert space, so, instead of integration, one uses the elements from the dual space, that you will see in a few introductory lectures on reformulation. So, this is the definition of this weak solution. So, our first task is, suppose u is a weak solution and supposes it also happens to be a differential function is it true that it satisfies the equation and the answer is yes in fact that it should be. So, it is a genuine extension of this definition of weak solution is genuinely extension of the classical solution.

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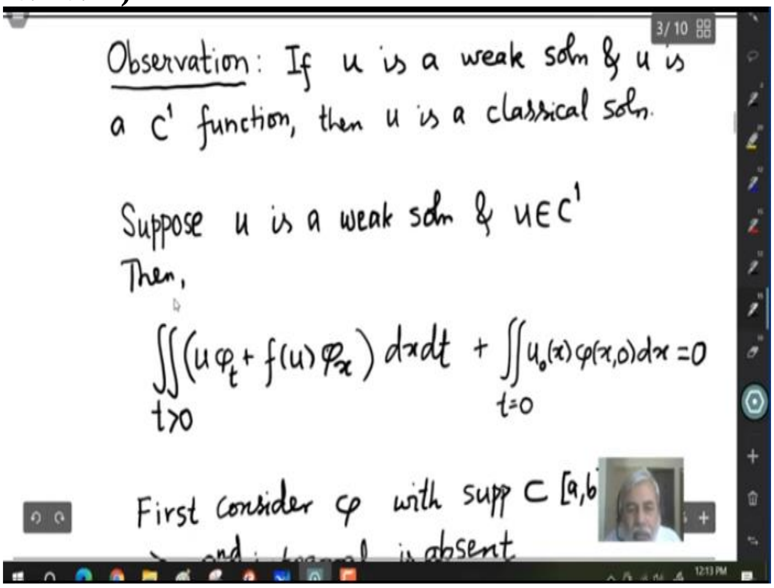

So, here that small result here so, if u is a weak solution and u is a C 1 function then u is a classical solution that means, so you also satisfy this point wise and this initial condition. So, for this of course, you need not be C 1 for all time. So, whatever time it has that differential measure and that will do so, by the fact that we have to just restrict the support of phi. So, suppose u is a weak solution and u C 1 do not do that u t + fu x is 0 and u x 0 = u 0 x. So, by definition we have this u of phi t + fu phi x dx dt + integral u 0 x phi x 0 = 0 for all test functions phi.

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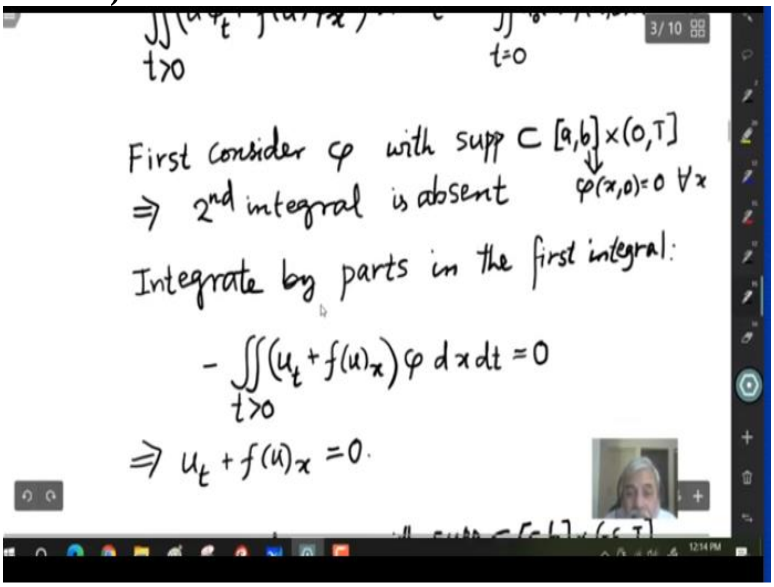

So, first we consider the test functions phi which have support in a, b cross 0, T that means that phi of x 0 so, this means. So, this implies phi of x 0 is 0 for all x so, here I just checked second integral is absent. So, we have to deal only with the first integral. And that when you integrate by parts that we can do because we are assuming u is C 1. And again the second integration parts and again there are no boundary conditions because we have taken the support of phi in such a rectangle.

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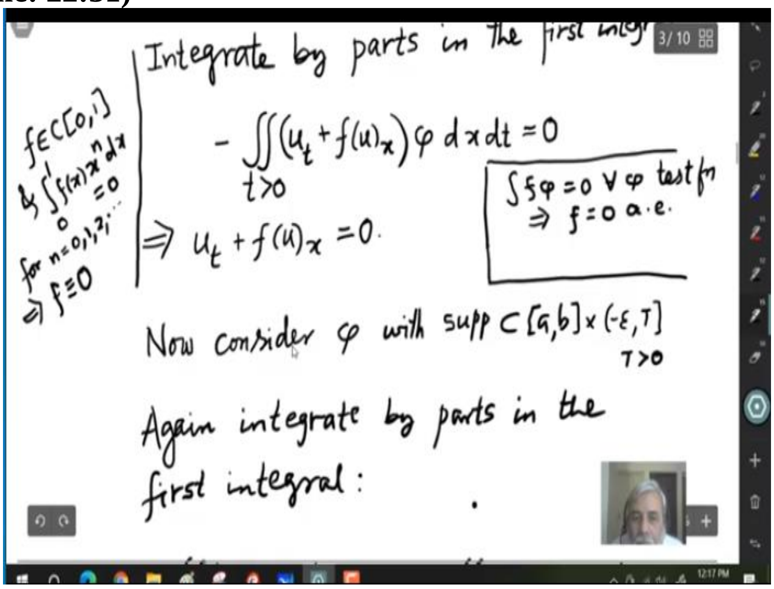

And so, this u t f of u x into phi dx dt = 0 and that implies u t f of u x = 0. And this is a non trivial result again from the theory of this, so integral f phi 0 for all phi test condition, so this is a non trivial result. So, this actually one showed that f = 0 almost everywhere 0 as of on a set of measures 0, but in this case so this is a continuous function, a continuous function which is 0 almost everywhere has to be 0 everywhere. So, this has the resemblance to this really simple result in analysis.

So, if f is a continuous function on 0 1 and 0 to 1 fx x to the n dx = 0 for n = 0, 1, 2, etcetera. So, these are called moments of f. So, if all the moments of f for 0 and f is a continuous function, then f is where anything. So, very similar to that result and this result follows from (())(24:36) theorem namely any continuous function is approximated by a polynomial. So, in other words, the polynomials are dense in this space.

So, similarly, this test functions are dense in many integrable class of functions and that gives you this result. So, this simplification is non trivial, so that you have to again consult for integration theorem that will be repeatedly using, so that is the first part. So, if u is a weak solution and u is a C 1 function then it satisfies the given differential equation namely u t + fu x = 0 point wise. So, what about the initial condition that is also now easily follows.

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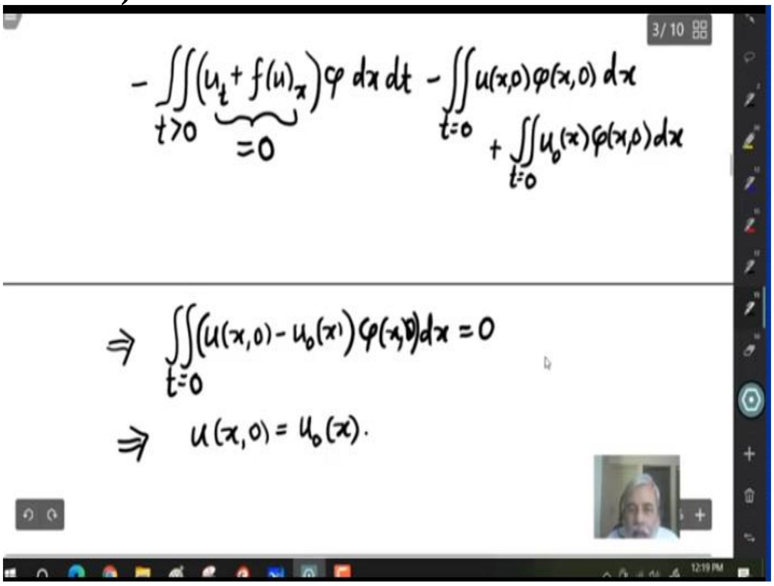

And now you take any phi test function again you integrate by parts and now we get a boundary term here. So, when you integrate by part in this first integral, now we get a boundary namely u x 0 phi x 0 dx and this were already shown to be 0. So, that implies this integral t = 0 u x 0 - u 0 x phi x 0 dx 0 and again the same argument here. So, this is true for all fine and that implies u x 0 = u 0 x. So, at least one verification that if u is a big solution and also a C 1 function then use a classical function.

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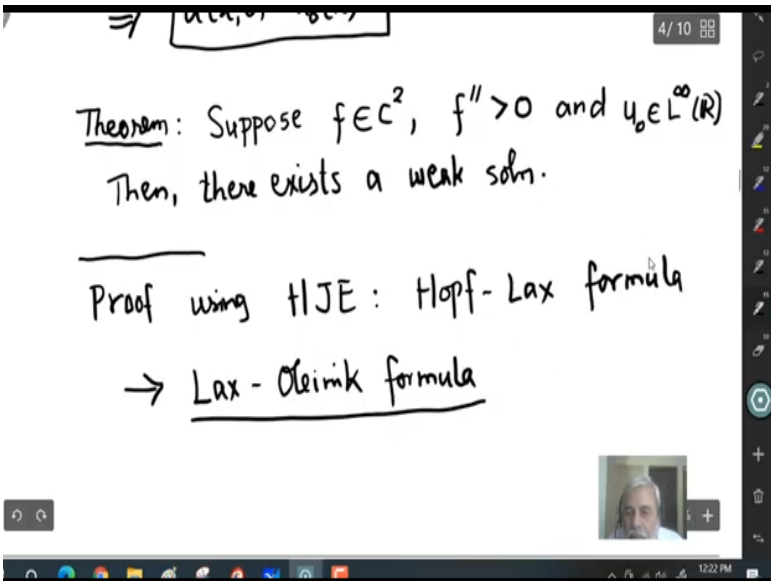

So, for most of the remaining lectures will see several examples and we prove this theorem of state that suppose f is a C 2 function. So, now assumptions are the nonlinear function f double prime is bigger than 0 and the initial condition is a bounded function just bounded function, then there exists a weak solution. And there this a weak solution in edition enjoys many important properties that we will discuss one by one. And now one set property also in place uniqueness of the solution so, it is not clear that whether the weak solutions are unique or not.

So, we can see to some examples and our approaches of proving this theorem so, this is an important theorem and there are many different ways of proving this theorem and our approach is so prove using Hamilton Jacobi Equation and namely, we will use this Hopf-Lax formula for the solution of the Hamilton Jacobi Equation and then derive the so called Lax-Oleinik formula.

And we verify so this will take us some time derive this Lax-Oleinik formula using the Hopf-Lax formula and then we verify the solution given by Lax Oleinik formula is indeed a weak solution of the scalar conservation law. So, this will take up in the subsequent classes. Thank you.