First Course on Partial Differential Equations - II
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Lecture - 06
Hamilton Jacobi Equation

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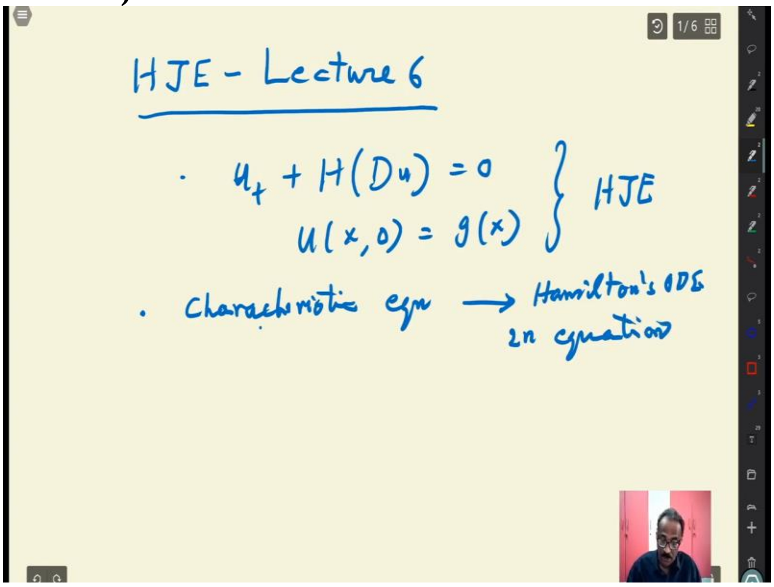

So, this is the last lecture on Hamilton Jacobi equations we will move on to other lectures after this lecture. So, in this lecture will be proving basically the existence and uniqueness of the Hamilton Jacobi questions and the solution given by the or function given by the Hopf Lax formula will show that it satisfies the Hamilton Jacobi equations and uniqueness under certain original assumptions we will not be able to prove the results.

So, basically we do the idea about the theorem and additional conditions about it but before that let me consolidate what we have done in the last 5 lectures. So, we have started with the Hamilton Jacobi equation, Hamiltonian given by H of X D u = 0 with the u at x 0 = g of x. So, this is your Hamilton Jacobi equations and then after this we have derived the characteristic equations is a system of Hamilton's ODE we call it later we have seen that Hamilton's ODE comes from elsewhere Hamilton's ODE. So, this is system 2n equations your defined the characters you can write down the characteristic equation.

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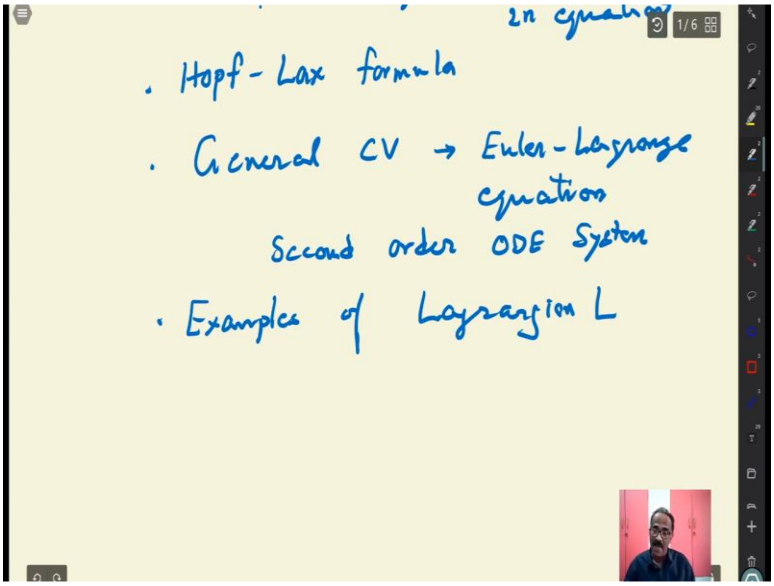

After that we have derived what is called the Hopf Lax formula corresponding to a calculus of variation problem and where you want to minimize over trajectories and Hopf Lax formula gives you that minimization can be converted into a minimization of the Euclidean space. And hence you can get an explicit formula for the often after the Hopf Lax formula of course we left it Hamilton Jacobi equations at that state.

After that we have studied general calculus of variation problem and then we also derived what is called the Euler Lagrange equations that is what we have seen that. So, the optimal solution satisfies a second order ODE system and then we have seen many examples of Lagrangian the minimizing function at a under the integral sign is called the Lagrangian we have seen examples of Lagrangian L and one of them is our classical mechanics L. So, this is the theory to bigger than that classical mechanics Lagrangian L we have seen Brakish problem, catenaries problems on all that.

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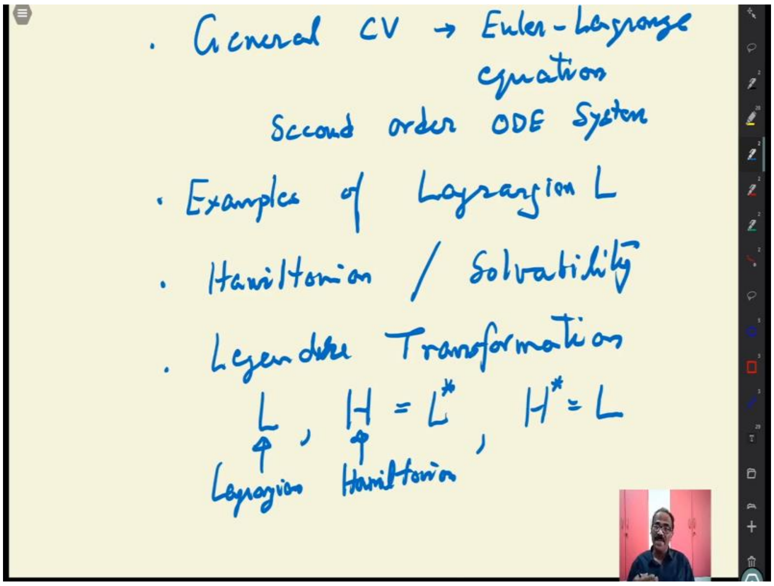

After that we have introduced to what is called the Hamiltonian and you see that there is a solubility issue. A Hamiltonian solubility issue when in the classical mechanics this solubility issue does not arise because it is automatically satisfies the solubility but in general like Lagrangian the solubility may not be there. So, you have to make it as an assumption and after that we introduced what is called the Legendre transformation.

Because there is a connection between L and H, L is the Lagrangian and H is the Hamiltonian. So, in classical mechanics you can go from one to another Lagrangian formula some took Hamiltonian formalism, Hamiltonian formalism is a system of 2n ODE's Lagrangian is a second order system of n ODE's. And in classical mechanics Lagrangian describes the position and velocity Hamiltonian it is position and momentum.

So, we use keep on using these things and what in general theories under the assumption of solubility we have seen a connection between that we introduced a Legendre transformation what is called L star you will see that Legendre transformation of L is nothing but the Hamiltonian and Hamiltonian is H star it is also of course you make assumptions on Lagrangian L which is continuous cohesive and convexity.

So, we are in the convex situation and H also satisfies the same property. So, it is a duality between the Legendre. So, it is a 2 problems are dual problems basically this is what we have done here. Now what we are going to see is that the u given by the Hopf Lax formula satisfies the actual Hamilton Jacobi equations where H is given by the dual of n. So, the u is defined via L and then you define H thing and we can do the other way given H you can defined.

So, if u given Hamilton Jacobi equation is given a satisfying the cohesivity and convexity continuous condition then you can define L and accordingly you how the minimization problem one way to other we can go from being under the solubility assumption.

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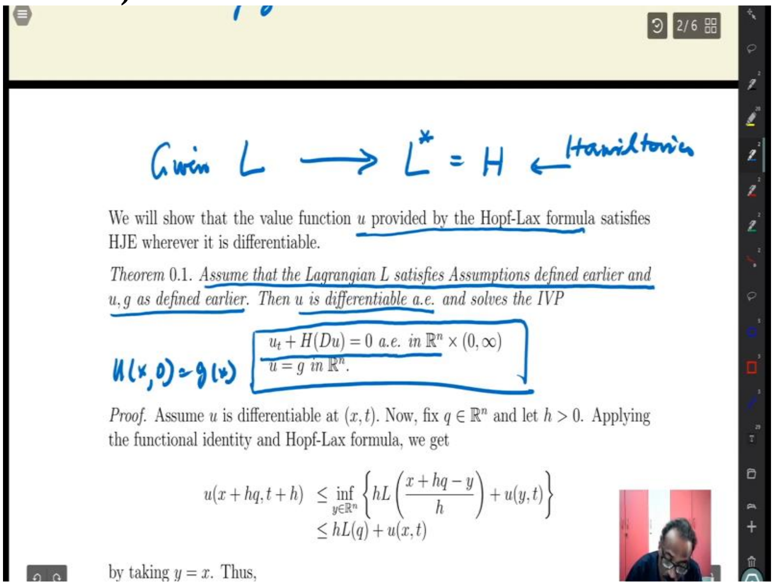

So, now as we are it is a little more technical let me give a printed material here. So, what we are stating which we will not give you the entire proof because I their proof may take each proof may take half an hour and we do not have that kind of time. And you may have those you are interested should go through the proof and that is very important to learn mathematics. So, what will see the u provided by the Hopf Lax formula so you start with a L.

So, you say so given L under the assumptions gives you see L star which is H you're your Hamiltonian this is Hamiltonian and using that you define u via Hopf Lax formula and then you also prove that it is Lipschitz and Lipschitz where by the redementia theorem once u is Lipschitz it is differentiable are mostly well so that is what you are excitedly assuming. So, assume that L satisfies the assumption and u, g as defined earlier.

Where u is the Hopf Lax formula minimize problem which is also satisfies the Hopf Lax formula and g is also Lipschitz continuity you can replace by a different condition then u is differentiable almost everywhere this comes because whenever u is defined by the Hopf Lax formula then it is Lipschitz and redementia theorem tells you that it is differentiable almost every day.

So, whenever u is differentiable all that differentiable point you satisfies this equation and u = g we already proved in 2couples of classes back we proved that u satisfies g on R n that means u = u x 0 = g x. So, you whenever u is differentiable use this equation u t = H of D u is satisfied and it is almost everywhere because u is almost everywhere differentiable.

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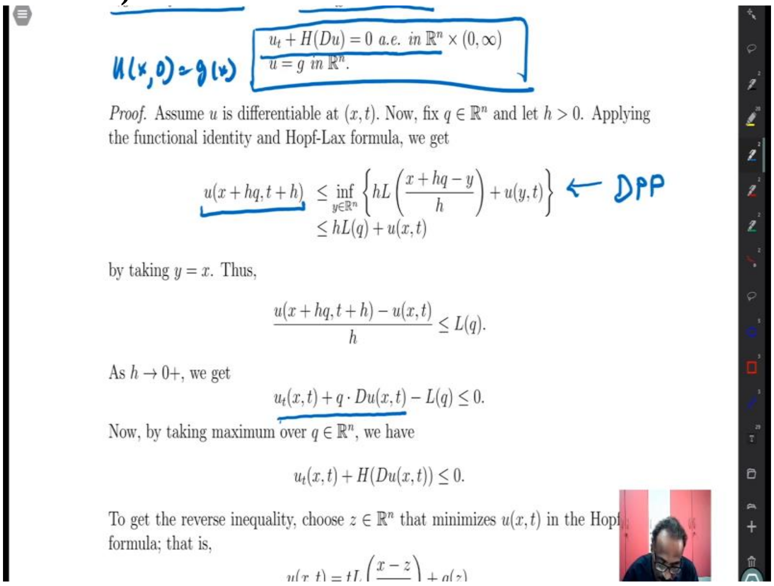

So that is what you so basically you try to compute your derivative of u wherever it is differentiable. So that is what you are and then you prove these inequalities. So, you apply the Hopf Lax formula very cleverly plus the functional identity you proved 2 things on u one is the functional identity, the second one is the Lipschitzness. So, you how to use which is a dynamic programming principle maybe will come into more using that.

So, you want to compute the derivative of u both with respect to x and so you how to compute something like this, this is a term we want to compute your thing you apply the you are this is a functional identity this is nothing but your dynamic programming principle. So, you look at it you have dynamic programming principle and then you can do this one and this result is this is infimum.

So, you can choose y = x and then h, h cancel you will get this equation u xt. So, if you taking u xt by h is less than or equal to L of q so if you take the limit as h tends to 0 so you will get the full derivative you see. So, you have u t so as h tends to 0 you get a t because there are n variable is the derivative with respect to u evaluated at q that is how you define derivative or direction derivative minus L q.

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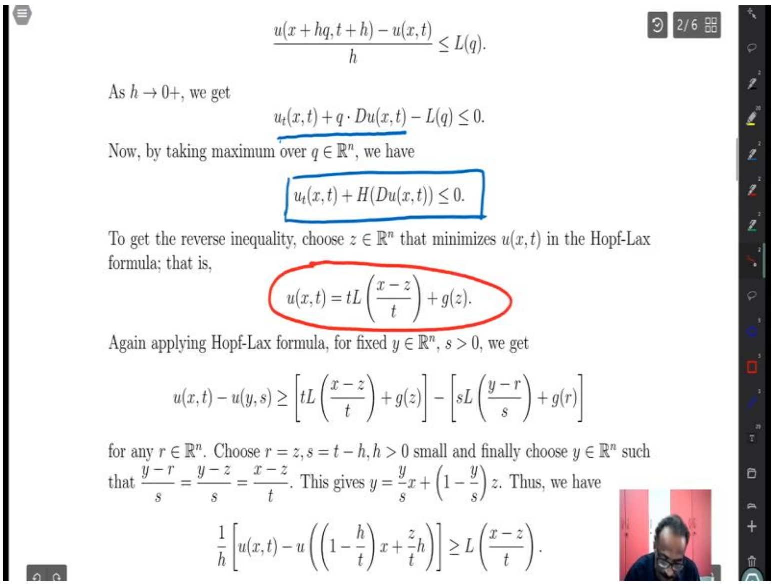

So that implies your one way of inequality whenever so the first inequality is slightly easier the second inequality is the same trick we have used earlier in the Hopf Lax formula and the functional identity. So, similar to you how to use it so you have to get a reverse identity for this one. So, you have to have a reverse identity for that one and then you start with the Hopf Lax formula so you can always choose an s here. So, you can choose a z here to get to this formula using so you can get that one and then you do some computations.

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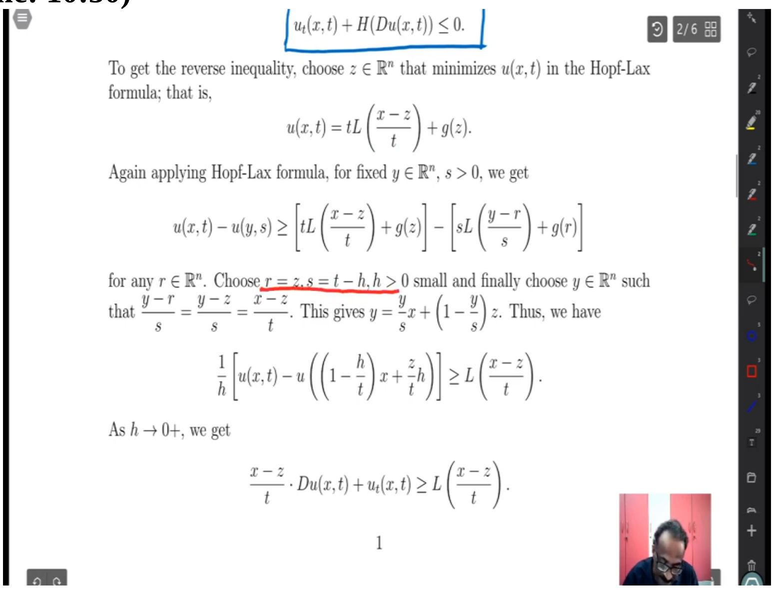

So, you have to apply this one apply Hopf Lax formula and you see this is infimum so you can always bound by y thing but here it is a minimum so minus sign is there within infimum. So, you have to have this exact point at this thing because it is an infimum you cannot bound immediately by this one. So, you need this exact point and then this is the infimum because of the minus sign you can bound by these thing.

And then you choose very cleverly this r is etcetera and then do some analysis. So that is what is prescribed here so your minimum knowledge is enough you do not need a very modern mathematics to do that. So, you do the computations of apply functional identity clever choice vary because you have infimum so you have to choose the correct values there.

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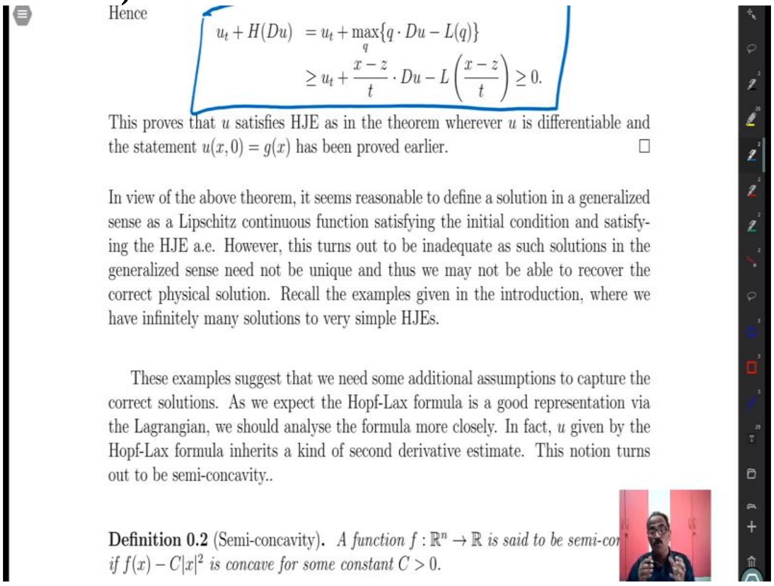

And the Hopf Lax formula if you do all these; you get your final equation and that shows that the Hopf Lax formula given. Now there is some serious issues to tell you already seen in the beginning. So, you already obtained a solution for your Hamilton Jacobi equations that is u given by the Hopf Lax formula so if you start with a Hamilton Jacobi equation corresponding to H you can define L.

And you can define using L you can now the minimization problem corresponding to minimization problem you can write down your Hopf Lax formula and that Hopf Lax formula will give you the solution to Hamilton Jacobi equation but then that solution is in this sense of Lipschitz condition. So, it is natural that to define a generalized solution as a Lipschitz continuous function satisfying the initial condition.

And satisfying the Hamilton Jacobi equation almost everywhere so you can always define a weaker sense of the solution Lipschitz function is said to be a solution of your PDE since it Lipschitz it is derivative x is almost everywhere and then it satisfies the equation wherever the derivative x is but then you have seen in the beginning if you have Lipschitz funding his functions this solutions may not be unique.

So, in this generalized sense you may get solution but then you may also get unwanted solution you see so but then naturally you expect your Hopf Lax formula coming from physical thing. So, you are the solution provided by your Hopf Lax formula should turned out to be the correct solution but then this generalized concept of solutions will produce spurious solutions over unwanted solutions which you may not require. So, how do you remove all these conditions? So, you have to look for special properties of the solution given by the Hopf Lax formula.

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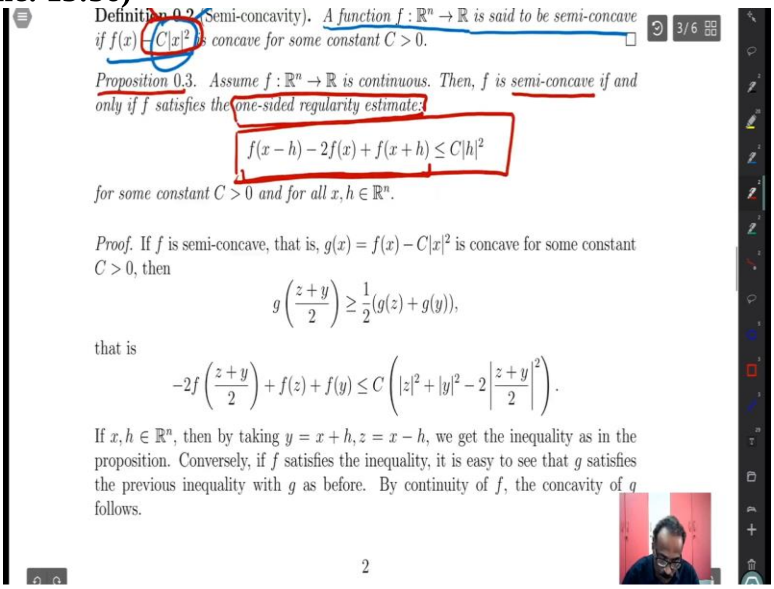

And that is what and this turned out to be what is called a semi concavity. So, I will introduce what is semi concavity here any function f that is said to be semi concave you see this concave look at here this function these for C positive this is convex. But then if you add this minus sign this is concave. So, with your function you are adding a quadratic concave function to make your f is concave.

That may not be the kind of fully motivating thing what is more motivating is this proportion. The proportion tells you that f starting with a continuous function the semi concavity is equal to this estimate this is what is called a one sided regularity estimate. One sided second derivative if you look at it here this term this is something is a kind of if everything is nice this is a kind of approximation with a 2 / 2 h of whatever is an approximation to your second derivative.

So, it is a kind of approximate so you do not have first derivative second derivative you have only contiguity so you have one sided derivative in a smooth sense if you know that when you expand it you can write down that is an expansion if it is differentiable 2 times or something like that. So, at working is that the semi concavity is a one sided derivative estimate so you do not have the full regularity but you have some sort of a one sided second order regularity estimate.

So that is one sided only one side you get that one and you are proof is given here. So, the semi concavity given as a very nice definition if f is semi concave f - C mod x square is concave that means you are adding a concave term with the large concave term with a quadratic but a large term can make it if you can do that that is called concave. And but you can never construct that many examples I do not want to give it a proof here but then semi concavity is equal to this estimate.

So, what you are going to show is that our u given by the Hopf Lax formula satisfies such an estimate. And then under that estimate you have your unique solution and the proof is given here if you want some of the proofs are represented here but I do not want to even read that you can actually go through it and it is not very difficult.

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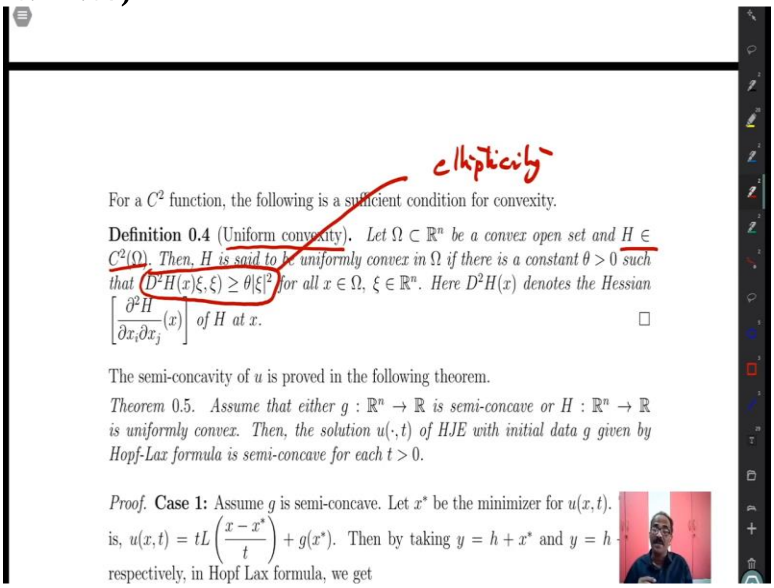

So that proof is not there but there are some more difficult proof some of them presented here some of them you can refer the book there is also another notion of convexity because you need to assume conditions from g if you do not have g concave so get the concavity for u you need concavity for g starting initial condition it is natural but you can also give a in terms of a uniform convexity.

So, the function H is called uniformly convex H is a C 2 function so you can understand it is Hessian. So, this is some sort of an ellipticity kind of ellipticity. So, uniform convexity is that ellipticity a function H called uniformly convex if this condition is satisfied the D square H is nothing but Hessian matrix the same matrix satisfies this uniform ellipticity theta is a constant H is true for every psi in R n. So, D square H is says are called semi convex.

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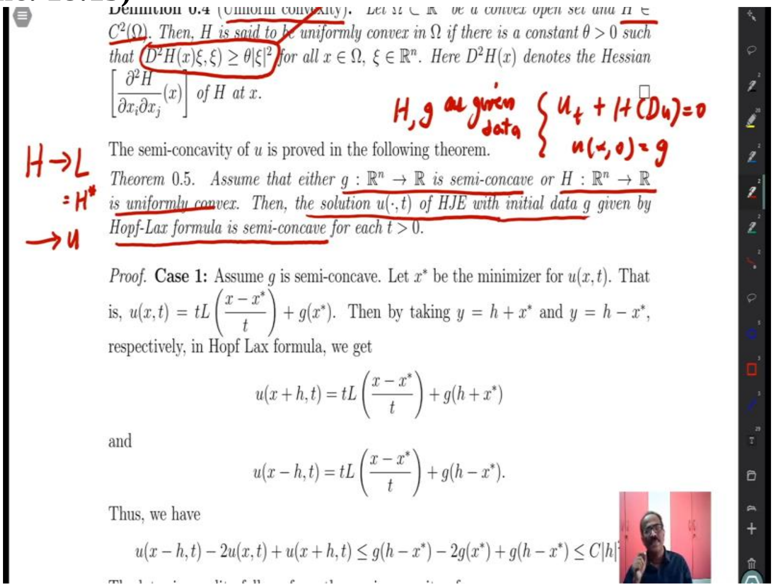

And here is your next theorem which I want to state if either g is semi concave or your H is uniformly convex. So, you recall these are the 2 data given so you have now you are H of D u = 0 and u at x 0 = g. So that means H g are the given data so you are making assumptions on the given data either this initial condition is semi concave or the function involved is uniformly convex then the solution u dot t is Hamilton Jacobi equation this is the Hamilton Jacobi equation with the initial data g given by the Hopf Lax formula is concave.

So, you see now you have your Hamilton Jacobi equation is given with that Hamilton Jacobi equation H you define H gives you L because H is you are assuming all that condition uniformly continuity, cohesivity all that gives your L star L = H star you can define that and that gives you u by the Hopf Lax formula which is semi concave. That is what and the proof is slightly longer so as I said I do not want to read again the proof of this here.

So, you can go through this proof as I said you have to take the minimizer properly because it is given by the Hopf Lax's formula naturally you have to take that it is a clever way of choosing your minimizer points because that is the best solution for that when you have a minimum of something you are always looking for at what point the minimum is achieved. And that is the point you will be looking all the time and then you have to prove this estimate.

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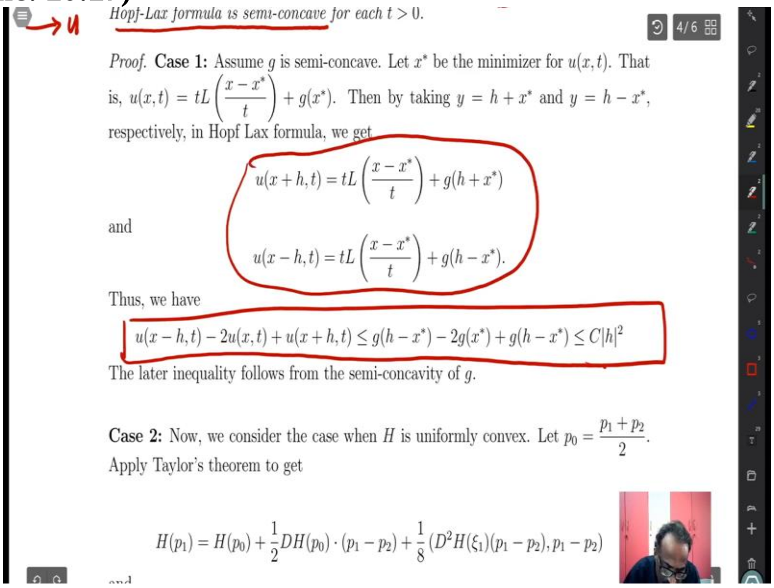

So, finally what you get is that you get some sort of g is semi concave. So, first case is when assumed g is semi concave using the semi concave you prove this semi concavity of concavity. So, you see so you are estimating this with respect to x semi concavity with respect to H for every t that is what you are going to do that. So, you will since g is semi concave you have this estimate so it is just choosing you are correct. So, you are just computing that because it is given by the Hopf Lax formula. So, you are basically choosing this x star here you see that Hopf Lax formula here and that is all you are choosing that.

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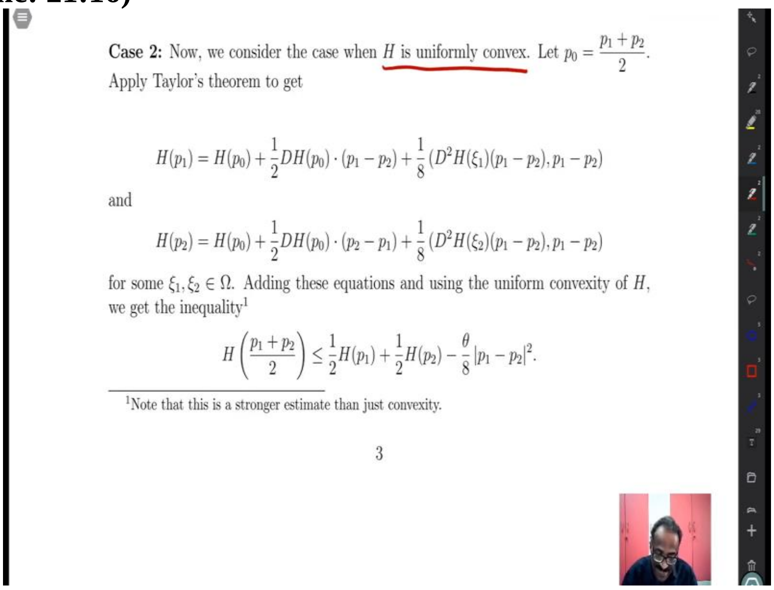

And you also given a proof in the case of when H is uniformly convex. So, you have a little bit your to show that so as this using the uniform convexity of H you first to prove some tricky inequality. So, this involves a little more work because you apply Taylor's Theorem because H is a twice differentiable function using that you have to derive some estimates on H.

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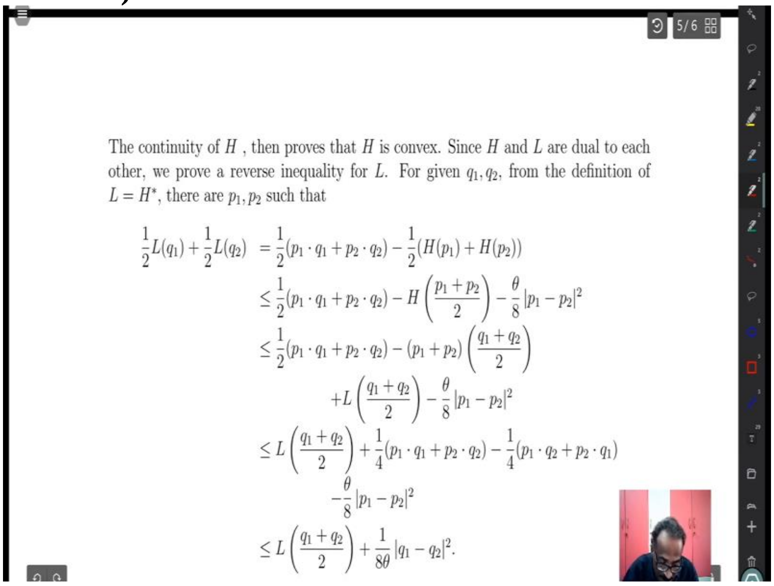

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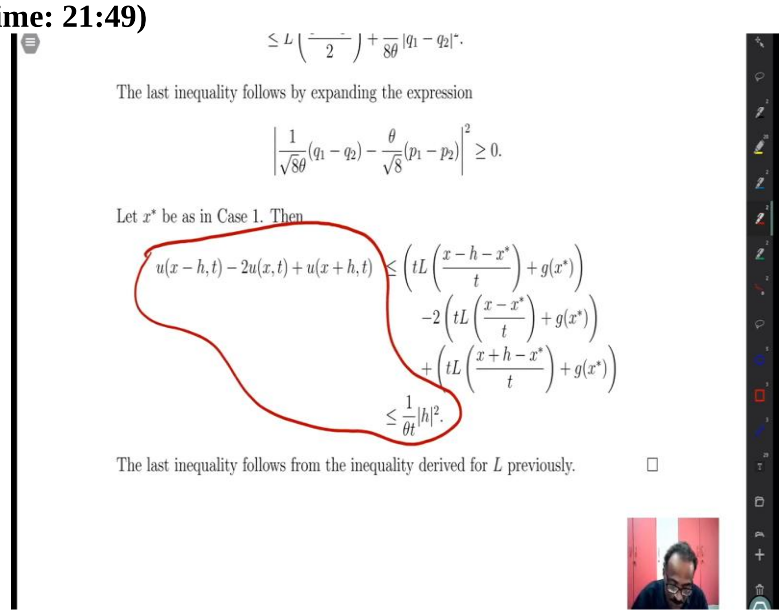

After that estimates you derive some estimates on L you see so with that you derive. So, these are technically a little more technical thing. And using that technical thing there are one more result which is highly technical but for this I am giving the proof here for other thing you will give here. So, you see you prove this of course this is what each t you are proving so you have the last inequality derived for L previously from here. So, you have you see inequality which is semi concavity you have proved it. So, the constant of course depends on t in general because you are proving your semi concavity for with respect to the x variable.

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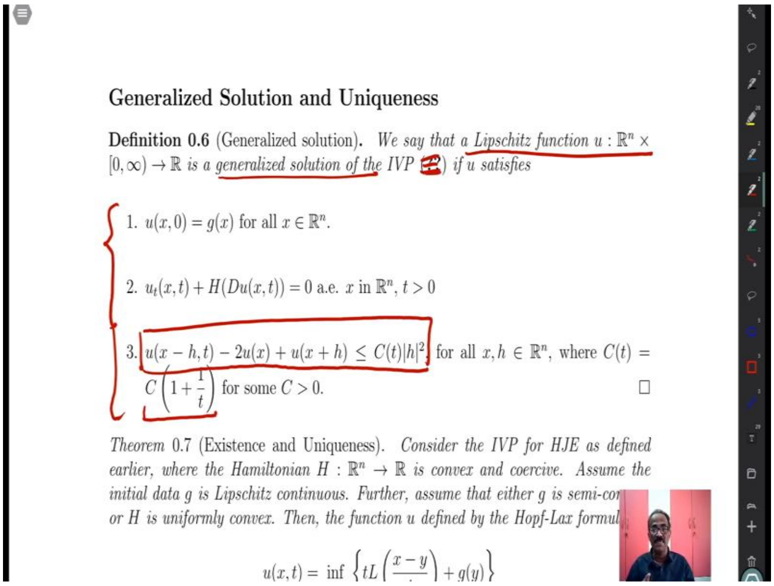

With that we are into a definition of the generalized solution and uniqueness why we need this additional condition the generalized solution as the Lipschitz solution will not guarantee you the uniqueness it is already given the existence but does not give you that uniqueness but the uniqueness proof very, very highly non trivial. So, I do not want to even tell anything about it in this class.

So, those who are interested and with you need to understand the analysis properly is to understand the proof of the uniqueness but let me explain to you the main thing. So, let me not the initial value problem let me not hear the said that a Lipschitz continuous function. So, really carefully Lipschitz continuous function u is a generalized to solution these are the standard condition because it is a Lipschitz continuous it is differentiable almost everywhere.

So, you want these to satisfy which we have already proved there is a uniqueness and then generalized solution you add this condition as well. So, this satisfies the one sided derivative estimate and you are given with the C first. So, if is function Lipschitz continuous function satisfying the estimate and satisfying the initial condition and then satisfying the Hamilton Jacobi equations in the all most everywhere sense then it is called a generalized solution.

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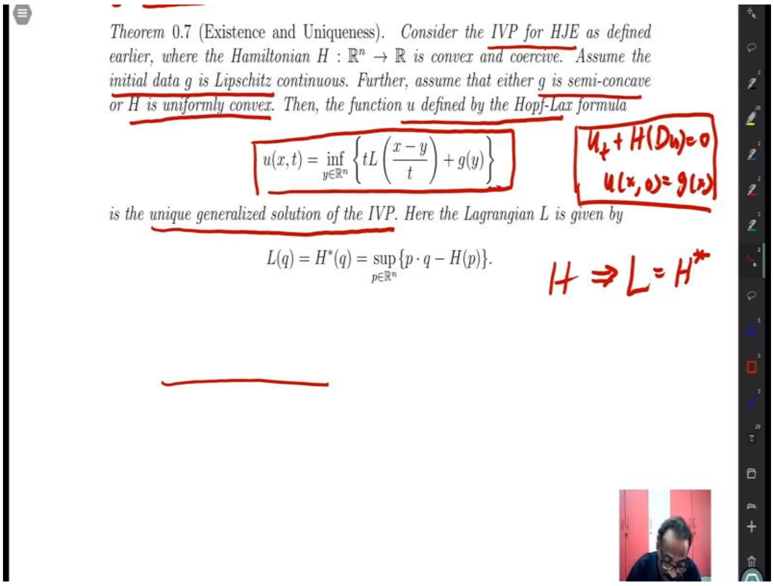

And what the final theorem which I will state today and the final part of my talk in Hamilton Jacobi equations you have your existence and uniqueness weak not weak existence, existence of a weak solution. So, you have your consider the IVP which we have described the IVP is a u t it again I am repeating so that you are equal to 0 and u at x 0 is = g x. This is your Hamilton Jacobi equation with H and g are given.

And define where the Hamiltonian is convex and cohesive you need that assumption either the initial data g is Lipschitz continuous assume that initial data is Lipschitz continuous and assume that either g is semi concave Lipschitz continuity you are starting or semi concave or H is uniformly convex then the function u define defined by the Hopf Lax formula and what is L? L is not given to u as I said H gives L now H gives L as the Legendre transformation.

And definition as H implies L and then your Hopf Lax formula is this one easy unique generalize to the solution of the IVP generalize to solution in the sense this is the generalized to solution in this sense. So, you have your generalized solution that one and your Hopf Lax formula given by this one is the unique solution. So, I will more or less finish here and maybe in a couple of minutes, I will just remark something.

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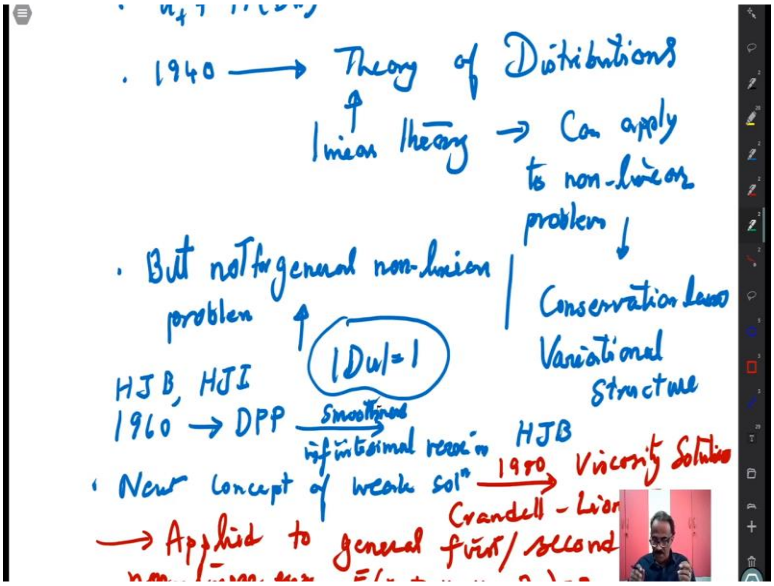

So, now we have a consolidator picture about it we already told you about the consolidator picture but then what we have proved is the defined in this lecture here defined the weak solution Lipschitz continuous function satisfying a one sided derivative estimate is the unique solution and that u given by the Hopf Lax formula is the unique solution to Hamilton Jacobi equations so, the remarks with some of the points which we have done.

We have only studied a very particular equation u t equal to in a very special case a general theory requires something more than that. So, as you know that in 1940's so the people have understood in the first part of the 20th century generalized the concept of weak solutions are important and one of the breakthrough in the 1940 started from 1900 is the theory of distributions.

But then this is typically a linear theory can apply to nonlinear problems as well but not for general nonlinear problems that is not possible because there is no way see if you want it for example you can apply it to conservation law you will see this in the next set of lectures which is a set of nonlinear problems. So, this has some sort of a variational structural that need some sort of integration by parts but if you look at the problems here even mod D u = 1.

And this does not have any variation structure you cannot do any integration by parts formula. So, General nonlinear problems is not easy to handle and this happened in the 1940s but for this especially Hamilton Jacobi equations from the calculus of variations but more general Hamilton Jacobi Bellman equations and Hamilton Jacobi Isaac regression these are all mainly started the theory became very, very important in the 1960s.

And one of the principle was that as functional identity type inequality what are called dynamic programming principle. And as I said if you have smoothness it infinitesimal version gives you Hamilton Jacobi equation so especially this optimal control theory became very, very important in the 1960s. So, you need to have a new type of variationly, so this distribution theory concept will not be helpful here.

So, we need a new concept of weak solutions we have defined in the Lipschitz thing that for that particular class in general these are all not you need a more stable week for solutions new concept of solution and that is what in the 1960s or 1980s came up what are called viscosity solutions we will not do anything here viscosity solution is by Crandell you can see there are some nice article hard to read Leones events and many others.

There are some good books in this direction and some of them are referred in our textbook you can see and there is a not mentioned in our book on Hamilton Jacobi equations and some references are given. So, this can be applied to general first and second order nonlinear equations very, very general equations like F, x, t whatever you want it D u, u t D u this is the first order equation and you can add the term if you want it here.

And something like this curve of u so first and seconds and that is a little more classic. It is not classical but some continuous how do you interpret a continuous function as a solution to such nonlinear equations and we should be stable small disturbance should not affect that one. So, I think I will stop here so this is only a kind of tip of an ice berg of the general theory of Hamilton Jacopo Hamilton Jacobi Bellman is execution a plenty of people work on this area. So, this is a kind of glimpse of that what is presented here in this small set of lecture 6 lectures. Thank you, thank you very much.