First Course on Partial Differential Equations - II
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Lecture - 05
Hamilton Jacobi Equations

Good morning and welcome once again to the Hamilton Jacobi equations. So, we will be deriving or we will be giving 2 more lectures in Hamilton Jacobi equations. So, we will not be able to prove every theorem, but we will state some of the important theorems and some of the proofs are given here and those who are interested can study that in detail. So, let me recall what we were doing yesterday. Yesterday we are having a minimization general minimization problem and then if w bar is an optimum.

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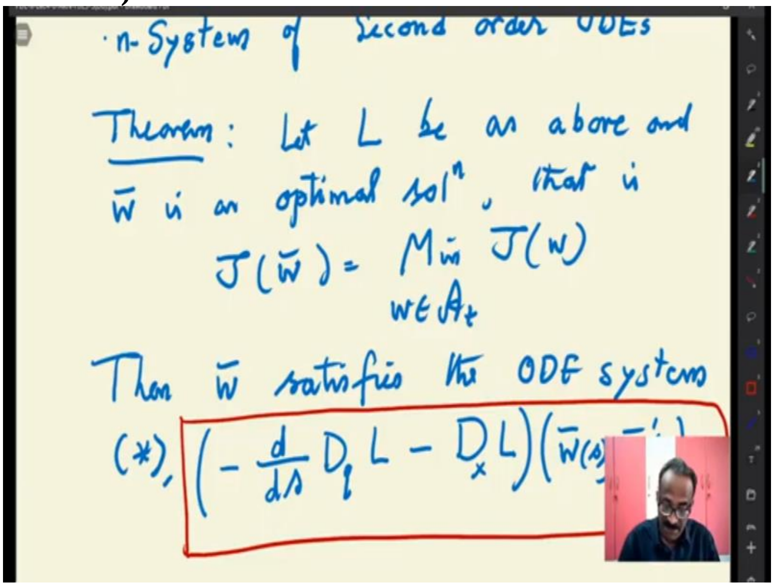

So, you have a functional given by a Lagrangian L that J w and w bar is the optimal or a minimal solution then w bar satisfies the second order ODE a system of L second order ODE is what is called the Euler equation. So, we will have Lagrangian equations, which is a system of n equations.

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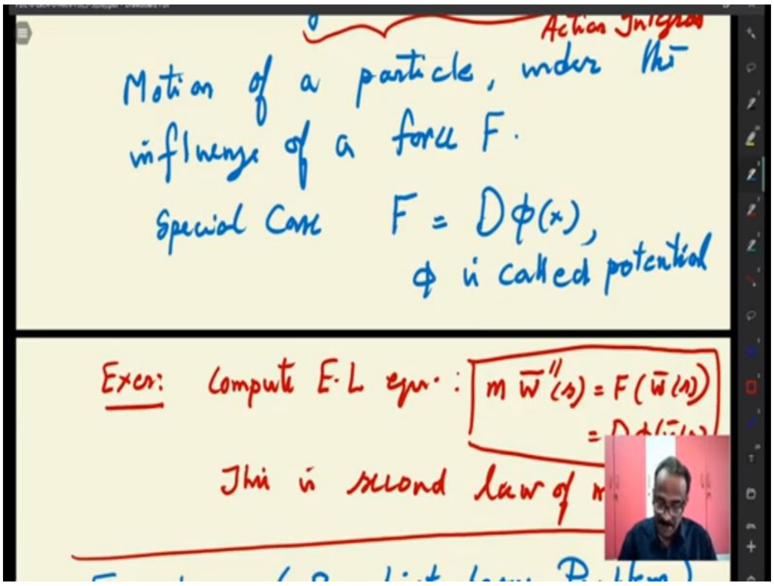

Then we are given specifically 3 examples. One example says the familiar classic problem from classical mechanics, where you have your L x q is the least action principle follows the least action principle of minimization and L is nothing but the difference in kinetic and potential energy and which is something that the emotional particle under the influence of a force F and what we are discussed here is that OLE Lagrange equations is nothing but the Newton's second law of motion given by m w double prime is equal to F of the w bar that is d phi of the w where phi is the potential.

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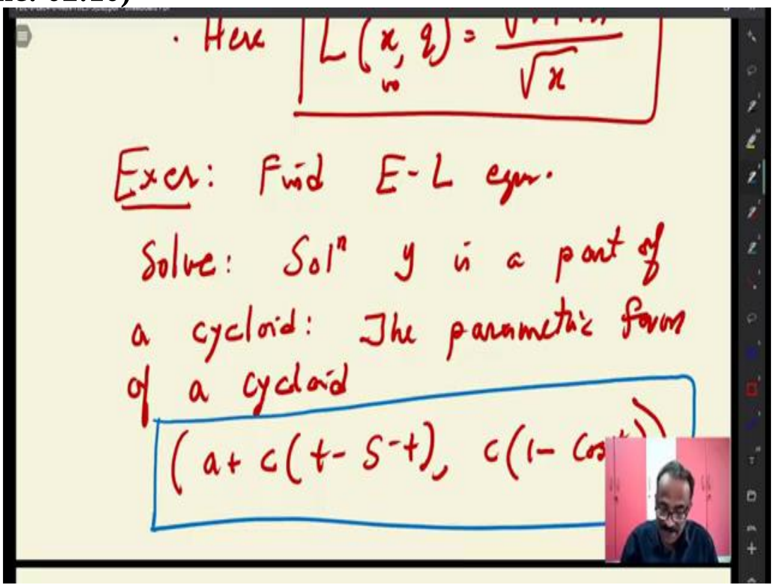

Then we have the brackish token problem that discuss how do you travel under the influence of a gravity and what is the minimum apart, which takes the least time to reach from a point to another point down along a vertical line and we have seen the corresponding l here you use the conservation of energy that we have defined the L, L is given by this functional and then the exercise left out to us to write down the thing and then you solve it then you solve it you see that the solution is actually a cycloid.

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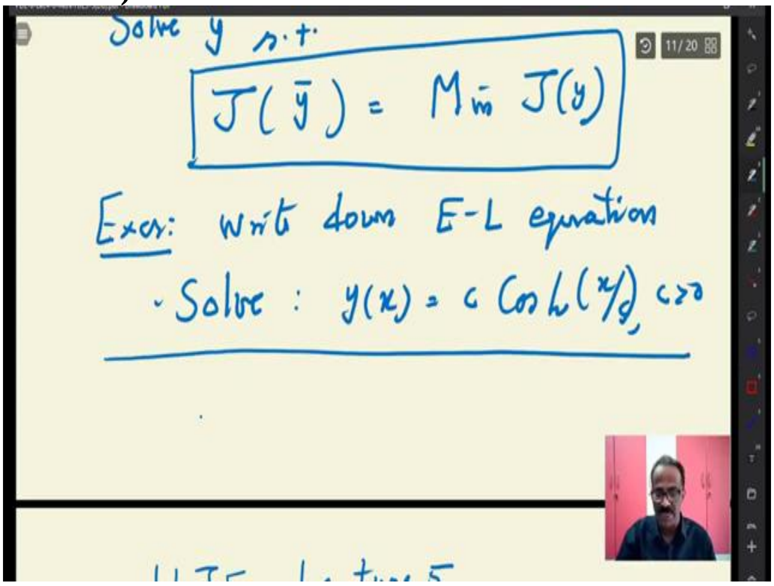

The third problem we have discussed over catenary where you have a hanging a chain freely hanging chain and then you want to find out the shape of that curve and then you turned out to be it function.

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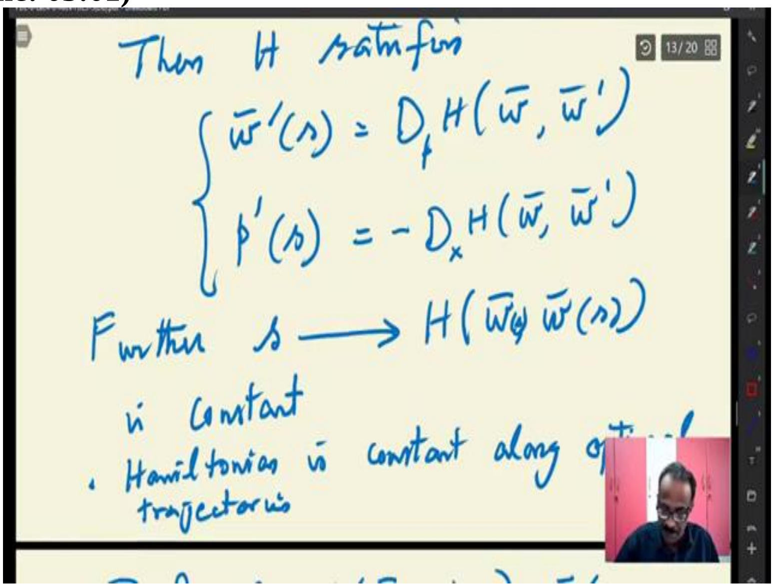

And there are 4 the minimizing function as is potential energy and the potential energy functional is written there and you want to minimize it, we did not write down the Euler-Lagrange equations left it as an exercise and then we will show I explained that what you solve it you get the solution the curve is nothing but the cos hyperbolic.

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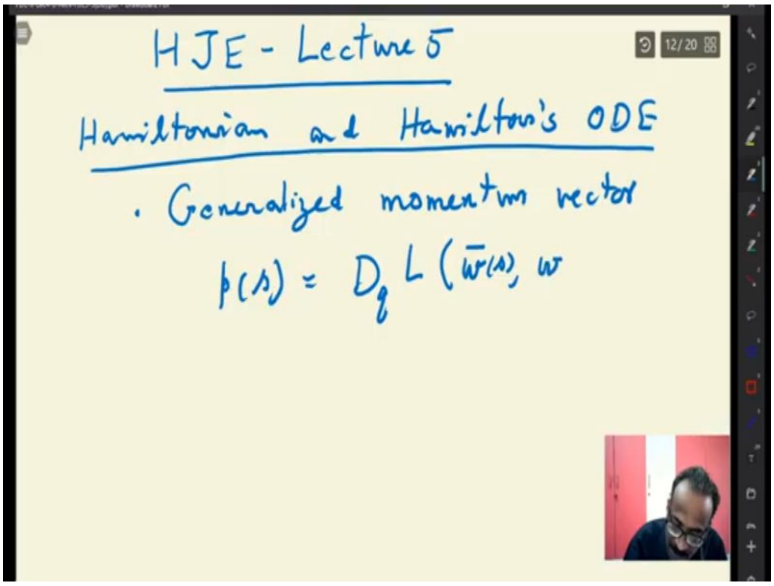

So, what we are going to do today is we are converting that 2 / 2 system into what is called a 2n first order systems and which is called the Hamiltonian. So, we are going to slowly introduce now Hamiltonian and Hamilton's ODE. So, there is a big solvability issue, I will discuss it here. So, let me introduce first a term what is called a generalized momentum will shortly tell you why we call it a generalized the momentum vector where ps is equal to I use the standard notation like in classical mechanics.

It is nothing but D q of L an optimal solution w bar is your optimal solution. So, we are again assuming all that w bar of prime of s. This is your generalized momentum but why it is called a generalized momentum vector.

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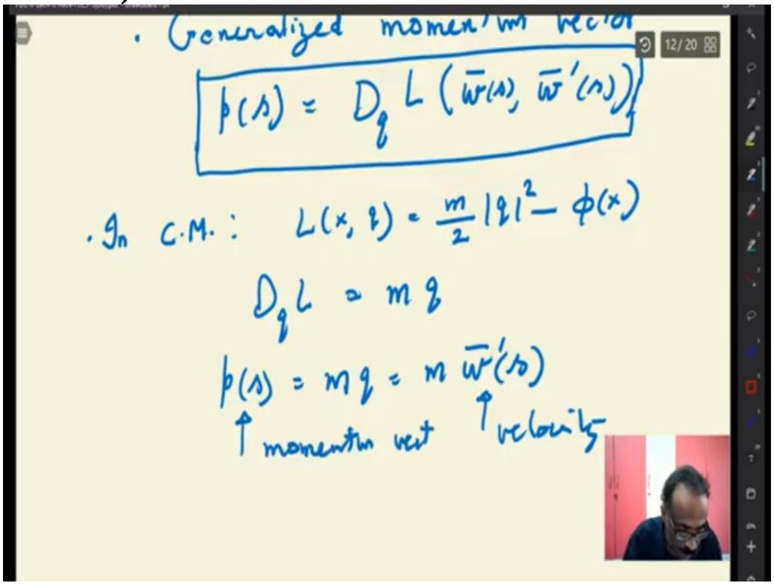

Because in classical mechanics, what is your L x, q is equal to the kinetic energy minus the potential energy. So, what is your D q of L here, D q of L is nothing but m into q. So, D q at L evaluated upper I told you that mq is nothing but represents the velocity. So, you are ps in this case math into q is w bar of s at that point mass into the q that is equal to mass into w bar or s. So, you see that this mass into velocity.

This is the velocity mass into no this is mass into the w q is equal to w prime. So, this is velocity and hence, this is the momentum vector in classical mechanics. So, what we have introduced is a momentum vector. So, you have your momentum vector.

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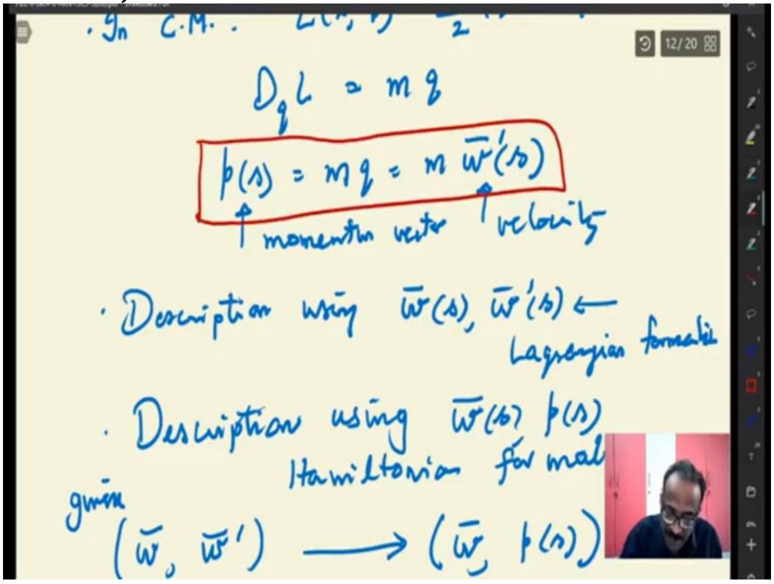

So, one of the issue is that so, if you understand the classical mechanics well, so, description in classical mechanics, description using velocity now, a position and velocity that is w prime that is nothing but your q s that is the Lagrangian formulation, these are all typical Lagrangian formulation. On the other hand, description using the position this is position and momentum is your Hamilton form Hamiltonian formalism. These are well known in classical mechanics, formalism and you can go.

So, if you are given w and w bar, but w bar prime given this you can compute your position is already there and p and conversely w the other formulas and there is suppose w bar and p is given then you can have the w bar is there and you can solve this equation you see, you can solve this equation and given p you can compute w bar.

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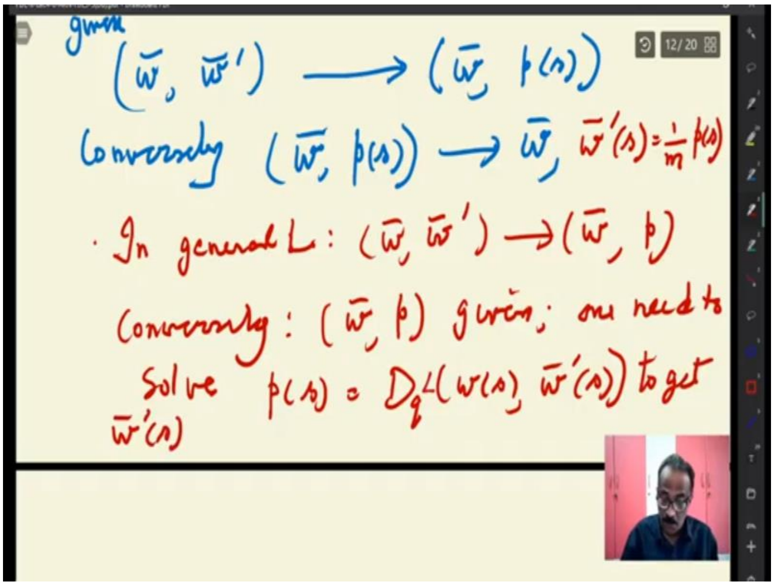

So, you can solve it here w bar prime you will see that is equal to nothing but 1 over m into 1 over m into ps. That is that thing, but in general so, given in general L, if you take it given w bar and w prime, this gives w bar p because p is defined like that. So, you see, you have defined like this p but conversely converse part is not generally to conversely w bar and p given one need to solve this equation. You will need to because your p is defined like this need to solve this equation p of s is equal to D q w of s, w prime of s, need to solve to get the w bar prime.

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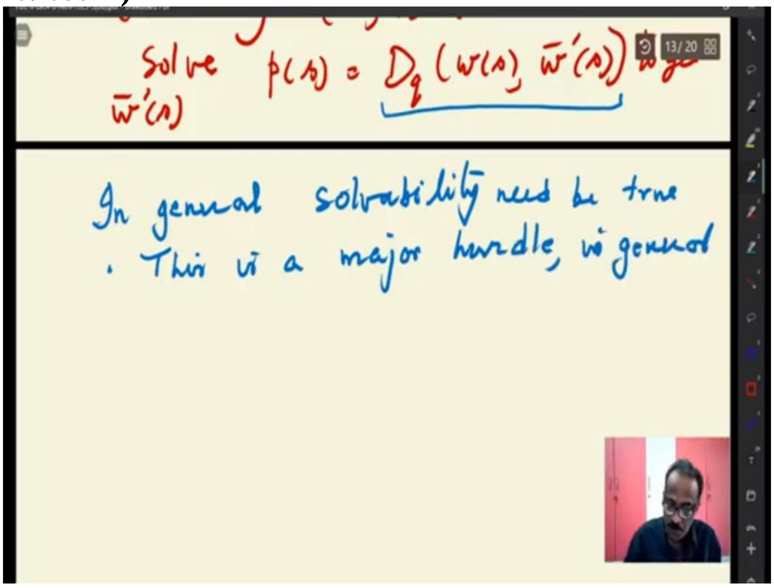

So, in general so, let me use the in general for solvability may not be true, because it is an algebraic solvability and you know that an algebraic equation you cannot solve this is a nonlinear equation in general and the other one happened to be linear and you have a very precise say, in general solvability need not be true this is a major hurdle in general.

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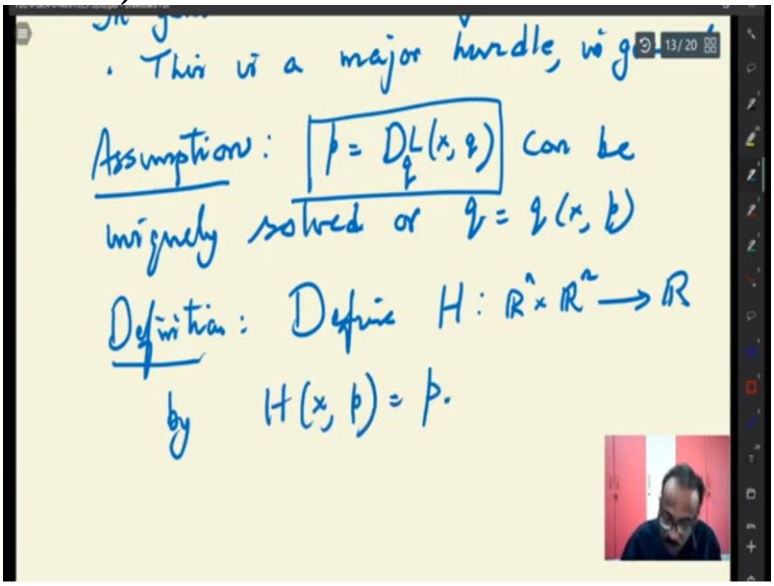

So, we make an assumption now. So, we are making an assumption immediately that p = D q of x q D q of L x q can be uniquely solved for q and q will be q x p, so, q is a solution to this equation and you are defining your Hamiltonian now, you will see that one definition you can define H now you are defining so, for H I use the variable p again it is from R n cross R n to R by H x p is defined to be you will see later more motivation.

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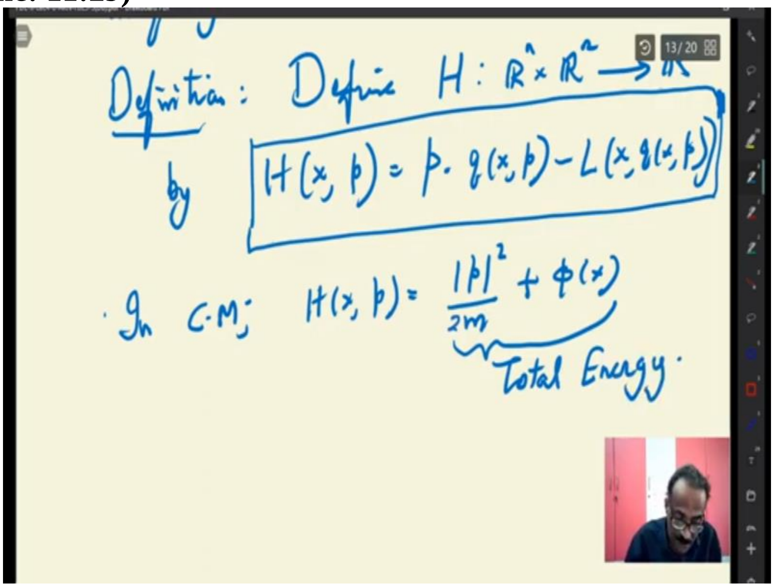

Why I am defining this way, p into q x p see q is the solution to that -L of x q x p. This is my common Hamiltonian for this equation. So, you see, so, you have your Hamiltonian equation defined like that and even in classical mechanics that why the motivation came, H x p will become it will become mode p square in terms of p / m plus you will get phi x. This is same mass kinetic energy this is the total energy. So, L was the difference in energy for the least action principle. So, H you get happened to be the total energy, so, you will define Hamiltonian like that.

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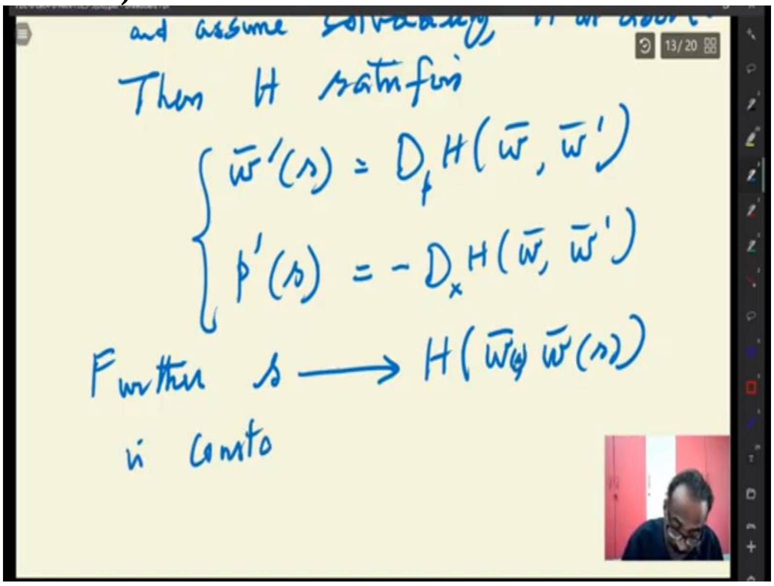

So, you have your assemble theorem the proof is not difficult. So, I may or may not give. So, assume let w bar be the optimal solution and H as above assume solvability and H as above then it satisfies the w bar prime of s = D p of H w bar w bar prime and this is a system of n equations here and then you are on p you are writing the system. So this is a system in w and p the earlier one was w now, the Euler-Lagrangian equations for w and w prime.

It is a velocity equation and this will be – D x of H and w bar w bar prime further is going to this Hamiltonian going to H of w bar and the w bar prime of s so H is constant that means that Hamiltonian is constant along the optimal trajectories is constant.

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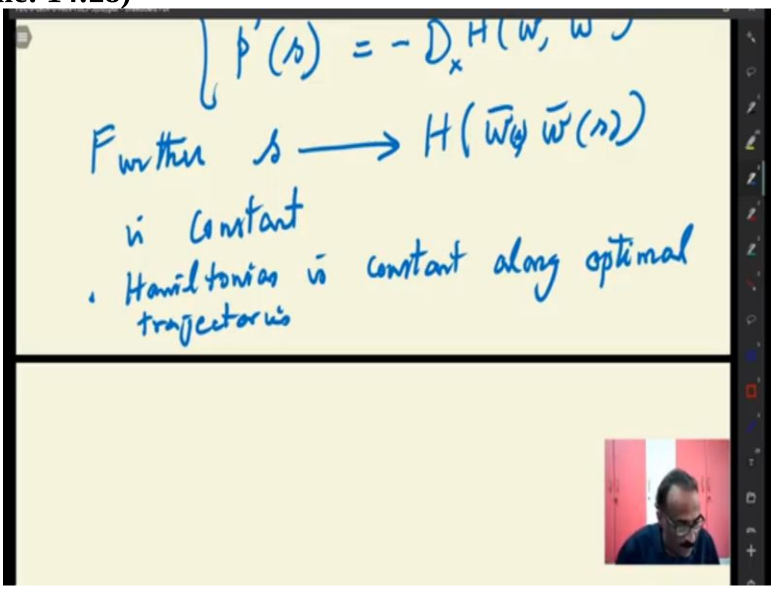

And if you recall in the introduction, when I introduce the Hamilton Jacobi equations into corresponding characteristic curves, along the characteristic curves which you have in so, the characteristic curves along the characteristic curves here right now, you do not have a Hamilton Jacobi case and we only introduced the Hamiltonian, but along the solution if you really recall at the end of it.

You can see you have the Hamilton Jacobi equation and the corresponding characteristic equation is along are the characteristic equation that is nothing but the Hamilton ODE in classical mechanics, s is the total energy and what it shows that along the characteristic curves and H is the total energy the total energy constant, in other words, the characteristic curves are nothing but the constant energy curves. So to have that kind; of interpretations when you finally end up with your Hamiltonian.

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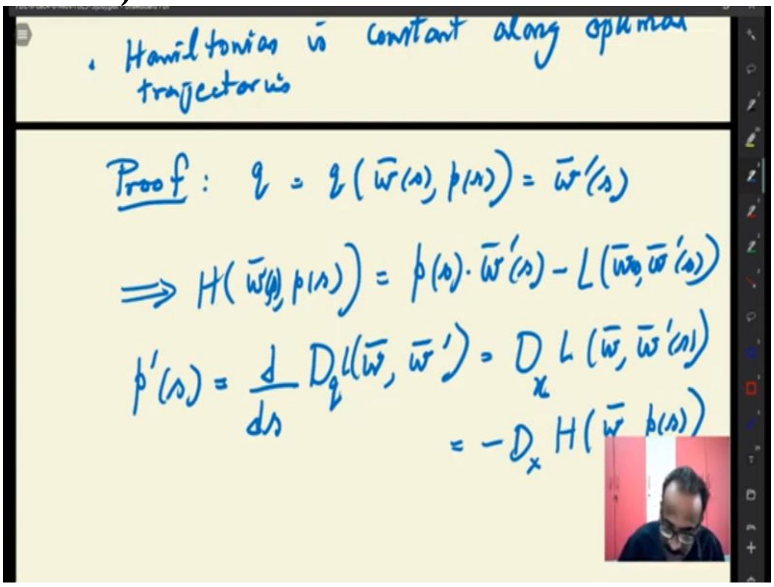

So, the proof is not that difficult. So, maybe it will give a cube proof, but I want to do something more here today, the proof is fine, because by definition, if you look at it, q is nothing but the q you solve these q is obtained by solving these p of s, but then this is obtained by nothing, but by definition of this one, this is nothing but w prime of s. So q is obtained by solving.

But then, if you look at here, the way the p is defined p is this one D q L of x of s q of s and that is how you solve these p. So, q is solve that one, but then your p is introduced like that? You are if you go back here, so you see, p is introduced like that, so that on the side part of q, if you look at it, you have these you will see. So, this is what you are solving. So, the q is nothing.

This is the equation you are solving. So, the q is nothing but w prime of s. So, you can see that one once you have that, that implies H of w bar w p s, you are always right w bar of s p s is nothing but by definition p of s w bar prime of s that is now that it is defined q. This is q minus L of w bar, the w bar of course, s is there all the time. So, I will not be writing s and again and again.

So, if you define this one, so, you will have this definition. So, therefore, p prime of s is equal to so, this immediately gives you p prime of s d / ds of by definition, D q p is that one D q of the w bar w bar prime, you will see that, but then from the Euler-Lagrange equations, this is nothing but D x. So, you see, you already have your Euler-Lagrange equation w bar w bar prime of s and that is nothing but you are –D x of by definition from here.

If you take your H here, so, if you differentiate with respect to the first variable, this is nothing but the x of L. So, you will see the previous equation tells you this is nothing but the x of H w prime and p of s.

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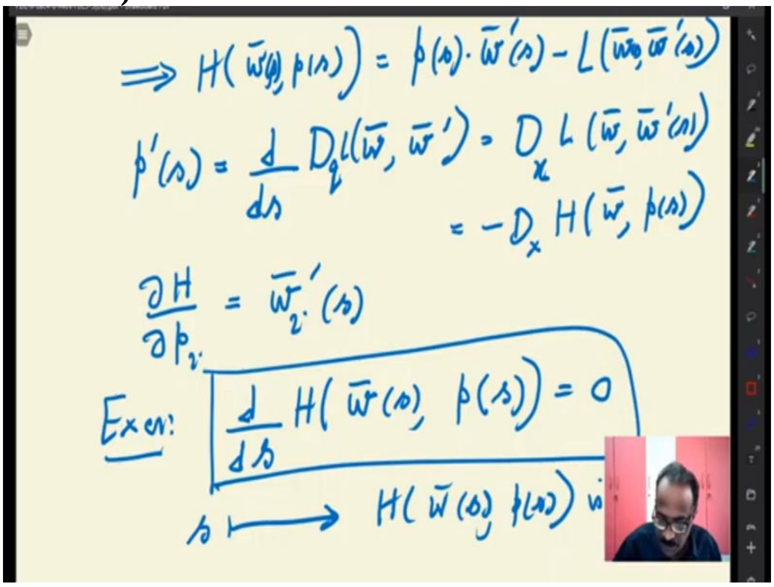

So, you have w prime and you have your p of s and that is what you want to prove it one equation p prime of s is equal to the L D x w bar p of s is equal to one. So, you have that to the x of L is nothing but that equation and we can also prove that if you compute your D H / D p i, you can easily show that the i prime of s and the exercise you can check this exercise I have done little bit quickly but you go through our book for any w bar of s, p of s you can show that this is an exercise. So that s going to H of the w bar s p of s is constant.

So, now we want to discuss something else what we call it a Legendre transformation. So, this is what Hamilton ODE which is a system of 2 equations and then we want to introduce in general case what is this Legendre transformation. These are all some advanced topics, so, it is important that you study properly you go through it very carefully on the Legendre transformations.

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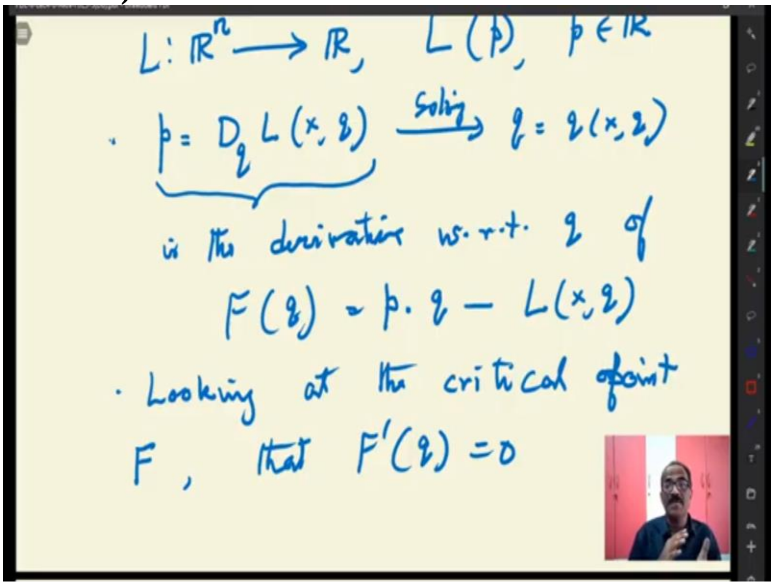

So, look at way so you have your L p s, so you are considering L as a function of in this case so, we are considering L is only as a function of R n to R n. So, that we use the parameter p with the p in R n. Now, look at this the way the p is defined here for your p is given p you solve this equation. So, look at this 1 the solving you get it solving, you get it q = q x p look at this equation. So if you look at it this equation.

You are solving this 1 this equation is the derivative of derivative with respect to q of course with respect to q of some F of q = p dot of q - L of x q that is an observation this is observation. So, this that implies q is a critical point of this 1 p is equal to is the derivative. So, you have p - D q x = 0 that means you are looking at the critical so you are looking at the critical point of F.

That is F prime of Q = 0 that is what is you have a exact solvability. So, naturally when you have p = D q to this particular critical point of F and then you are actually F is maximizing or minimizing. So you can see that it is exactly maximizing.

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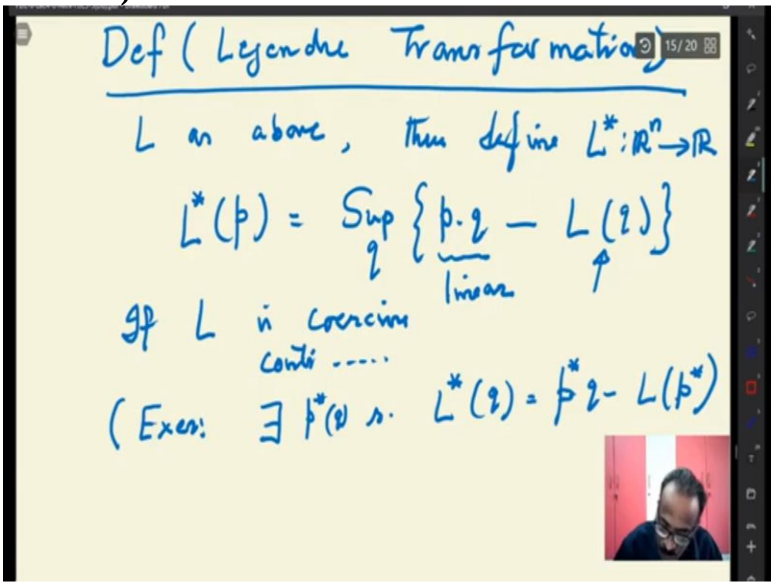

So, this motivates us to define the definition. So, you define as a definition Legendre transformation. So we have previously you are solving so, given p x p so we look at it given x p, you define you solve for q which depends on p of course you can depend on x that is fine because now we said we are not a having this explicit expression. So, you do not need x q = q p and H is equal to precisely the maximizer.

So you when you define H of q you are basically defined p into q of p - L of x q of p so we really look at it actually what we are doing is that in the previous case you have this equation p = D q of L is the critical point of F of q and H is the nothing but the maximum value so this is a maximizer it maximizes that q = q of p and H q is the maximizer of this 1 that is how we are previously done and this is exactly motivating to define what is the Legendre transformation which we will critically defined. So, so let me do this one.

So L as above so, of course we are assuming L as above then define L star which is a dual that is R n to R n only L star. Later we will see soon that L star is nothing but the L star, I can always define this one L star so, I will use the dual variable only L star of q is equal to supremum over p or p dot of q these are all vectors – L of p this is what you want to understand the supremum so, if L satisfies the condition, so you see if L satisfies the coercively if L is coercive.

And continuous and all that properties convexity etcetera so, you have this property of L and this is a bigger growth then this is a linear growth once q is fixed before p this is a linear growth. So, basically this goes to - infinity this term and supremum exists as a finite quantity. So, that is where so the with these assumptions, so I can leave it as an exercise for you there exists p some p star such that L star of q there are exists that not this is attained your L p star of course depends on q.

So there, H is a L star of q you can easily show that there will be p given q q star of p - L of this. So, you have that. So, again the supremum is attained that some p star not q star. So let me. So, it is attained for any q this is attained some p star of q so this is attained. So p star q – L p star you will see so, you get always such a p star, so that you can do it. So, what do you what I am trying to do is that you have L star q of p I made a mistake here because I use the f or I am using the dual variable. So, let me use this one p variable here and this is the q variable here. So, for L we are using q.

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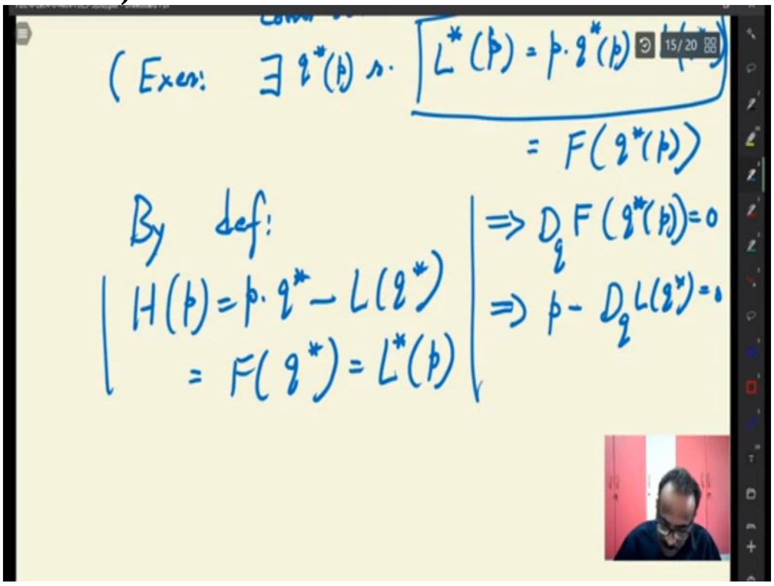

So, it is the same thing, but I want to use the variable. So in this case, I use q star. So, given p there exists q star of p and you are L star of p is here to use p here and then. So you have a q here so L star of p = p dot q star which depends on p - L q star so that is what you have; there exits that is because of the coercivity and other property so, you will have all these things immediately this is nothing but your F q star you see for you.

So you can prove immediately that like that so, immediately tells you that so, you are H is by definition, you know that by definition since this is a maximizing point so, that immediately this thing that because this is q star a maximizer. So, that implies you are D q of F evaluated at q star of p = 0. So, that is exactly tells you that so, that is maximizer q star is a maximizer to this one.

So, this has to be 0 that implies p - D q of L q star = 0 you have that by definition, your H of p is nothing but the p q star - L q star and that is nothing but your F q star and F q star is nothing but L star of p.

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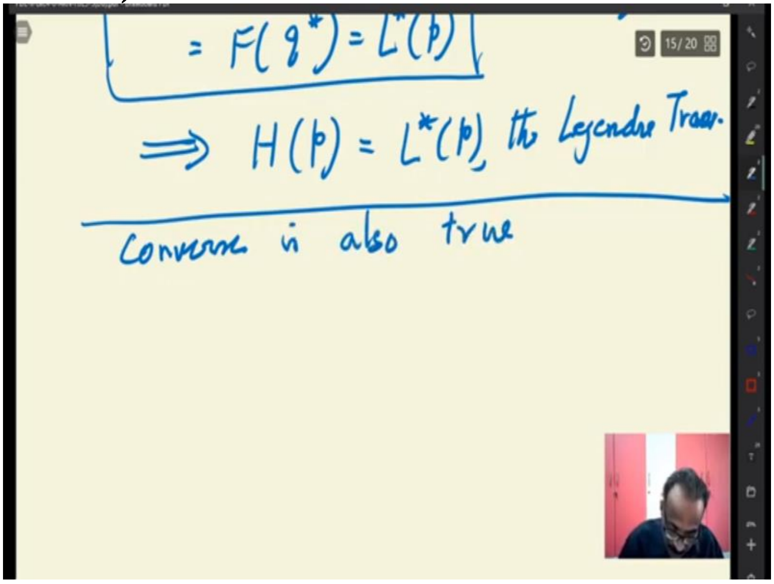

So you have to say so that implies that implies your Legendre transformation H of p = L star of p the Legendre transformation the Legendre transformation that is what am I having is that you actually have the converse is also so, let me state the theorem here.

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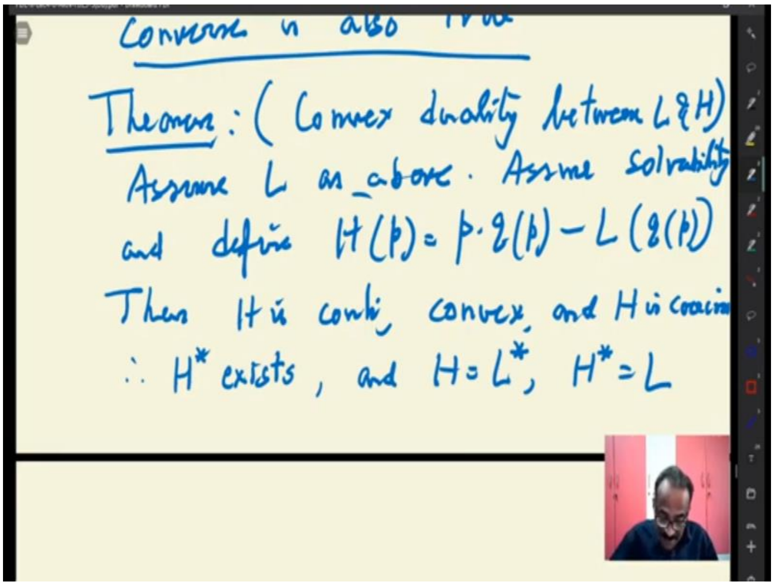

Converse is also true maybe I will try to state the theorem and try to give a proof of the theorem this is called the convex duality, convex duality between L and H assume L as above that means the L is convex continuous and coercive as above and satisfying all assumptions and H is defined the end as above assume solvability this is important. So, you are assuming solvability and define your Hamiltonian H as so H p exactly define and the solution p dot of q p q p is the solution corresponding to p - L of q p.

So, L is given to you and solve the equation, solve the equations mean solve the equation p - p = d q of d q of L of x p. So, that is L of p. That is why L of q, that is what you having to solve it. So, define H p like that then H is continuous convex and coercive. So, L H behaves in the same way that is what it shows that H is and H is coercive so, that means that one H satisfies the continuity, convexity and coercivity.

You can also define H star now, therefore and this therefore, star H is because it satisfies the similar property you take the supremum. So, H star exists and this you already know H = L star that you know always and H star is L. So, you will get back your L. So, L is defined you get a H = L star with you already proved it, but I am saying that H also satisfies a similar properties of continuity, convexity and coercivity and hence, you get H star you can define and that H star with be here.

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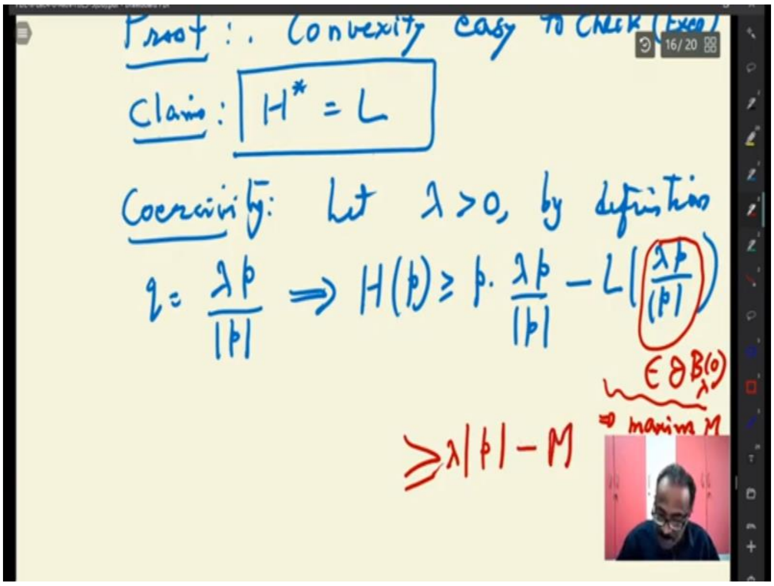

So, let me try to finish before the lecture even though it is delayed a little bit. So, let me give proof of that before ending this lecture and convexity easy to check. So, I will not do here convexity easy to check so, that is your exercise and then what you have to prove is that you want to prove a claim maybe I will prove one side you want to prove that H star = L, this is what they want to prove it the all other things are proved.

So coercivity also you can prove everything. So, maybe before that, let me see that coercivity. So, first let me do the coercivity H, I will prove it here. So, take lambda equal to positive take lambda is greater than 0 then by definition because H is supremum of that 1 take q = lambda p H star is supremum p mod p implies your H p greater than equal to p into lambda p is not taking me into lambda p because H is the maximizer.

So, for any choice of this one it will be smaller L of lambda p / mod p look at this vector this vector is a belongs to on the ball of radius lambda you look at it, this is the ball of radius lambda that means, it is on a compact set. So, therefore, this way it has a maximum this has a maximum you see so, you check these proofs, because it is an L varies over a boundary of a fixed ball this is a maximum. That means but so have a maximum n since this is a minus sign, I will have mod p lambda into p small p square. So, you have lambda into mod p minus some fixed constant here.

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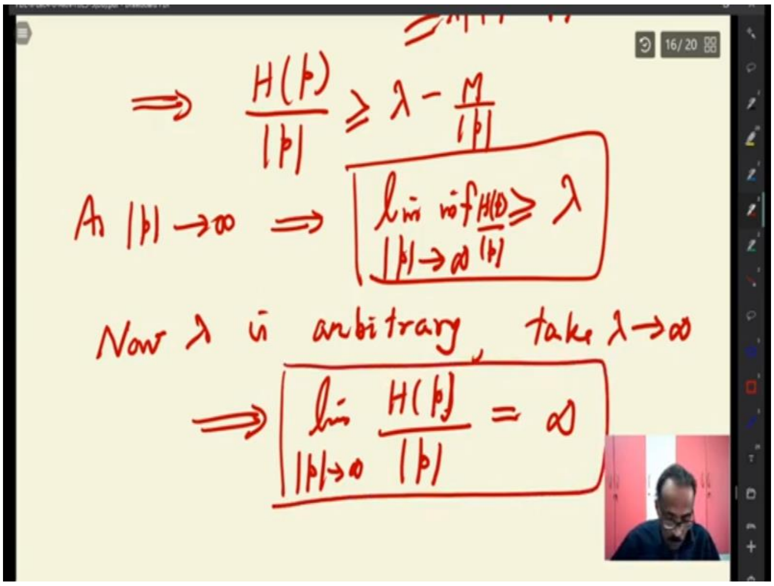

Because it is on the boundary of the board that implies H p by mod p greater than equal to lambda minus. So, you can see that when you take this is a mod p so you divide - m by mod p so, if you want to take it m is the supremum over that. So, if you take now take as mode p tends to infinity as mode p tends to infinity fixing lambda implies the limit may not exist, but I cannot call it limit infimum mod p tends to infinity greater than lambda see no lambda is arbitrary.

These are all some analysis technique no lambda is arbitrary take lambda tends to infinity no it m is not there. So, you can take lambda tends to infinity that implies this is H p / mod p you get that the limit x is and the limit H by mod p limited x is an infinite quantity mod p tends to infinity the limit in goes to infinity this will become infinity so, it is coercive.

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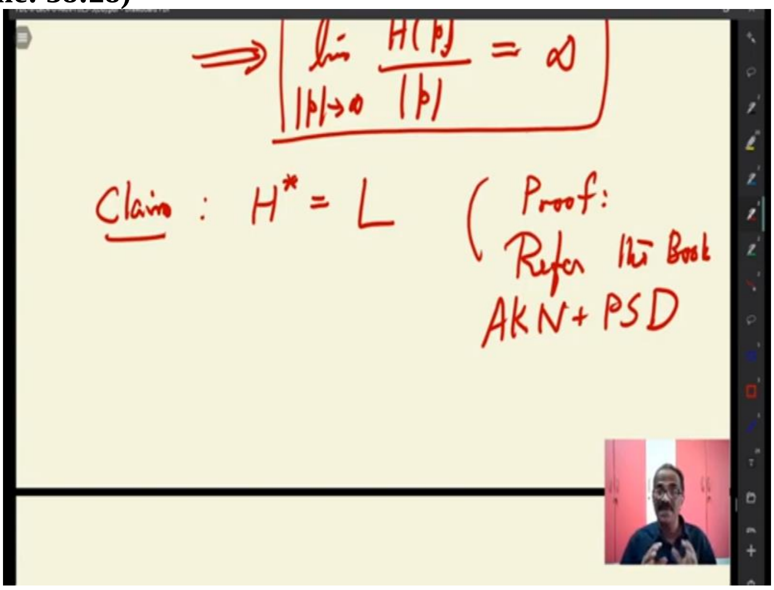

So, you have to prove now the claim main claim that one it H star = L so, since I do not know it may take some time. So, maybe I left I will skip the proof because so, proof refer the book it is not difficult refer the book AKN + PSD. So, that will take another 5 to 6 minutes probably I will skip the both because it is already 35 minutes. So, please go and have a look at the proof.

So, as I said that is why we will not be able to give all the proofs because if you want to give all the proofs and even if you want to give a basic introduction to Hamiltonian it takes maybe 10 hours, but at least 5, 6 hours, but what we are covered here is only some 3 to 4 hours or something like that in 6 lectures. So I will stop here and now with this material. We are going to basically state we are not going to prove anything in the last lecture on Hamilton Jacobi equations. We will be stating the existence and uniqueness, as I said, in a weak sense. Thank you.