First Course on Partial Differential Equations - II
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Lecture - 05
Hamilton Jacobi Equations

Good morning and welcome to the lectures on Hamilton Jacobi equations. And in the last lecture we were discussing some calculus of variations and the corresponding Euler Lagrange equation.

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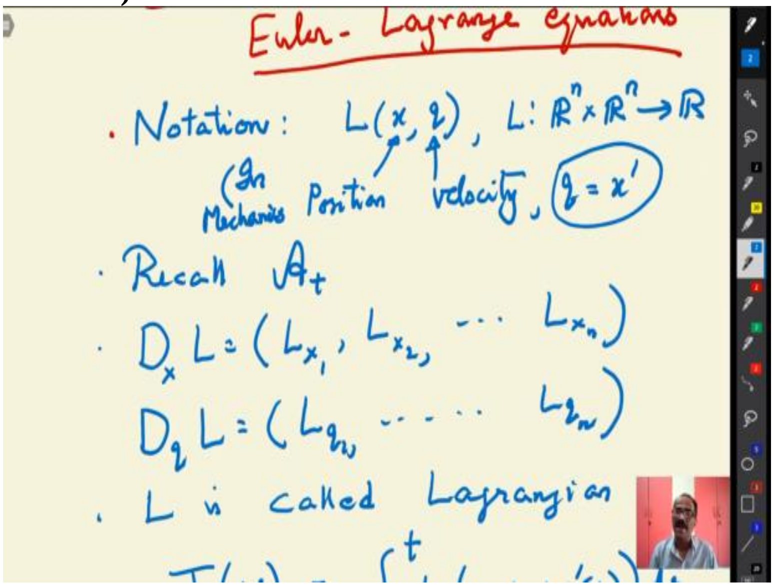

So, let me briefly recall what we were doing in the last lecture. So, we are trying to consider more general Lagrangians of the form L x, q, where L is from R n cross R n to R n and then you have your similar A t, it is a set off all admissible trajectories and we have these notations introduced the D x L and D q L. So, these are all vectors because x is a vector q is a vector and then L is called the Lagrangian and this is a J w is a corresponding function and we want to minimise w.

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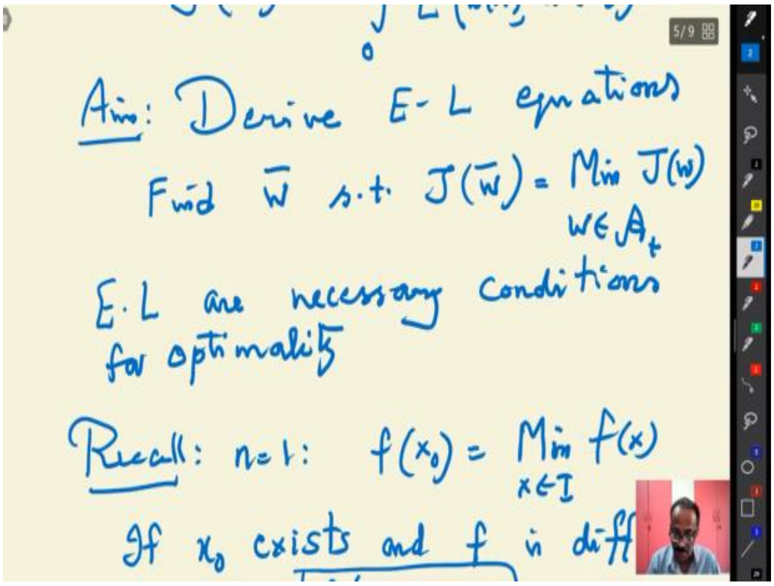

So, we want to derive the necessary conditions that is where for optimality and that means, if the w bar is an optimal solution, that is minimal solution to that one. So, what is the corresponding necessary conditions and this is not something new the only thing is that what we are going to do it in infinite dimensional we know infinite dimensional effects not minimises a functional and then if x 0 such an x 0 exists and then if f is differentiable.

So, there is you need a notion of differentiability in the infinite dimensional Banach spaces or so, Hilbert spaces and then f prime of x 0 = 0 which is a necessary condition. So, by solving f prime of x 0 you could possibly find all the minimising point if exists.

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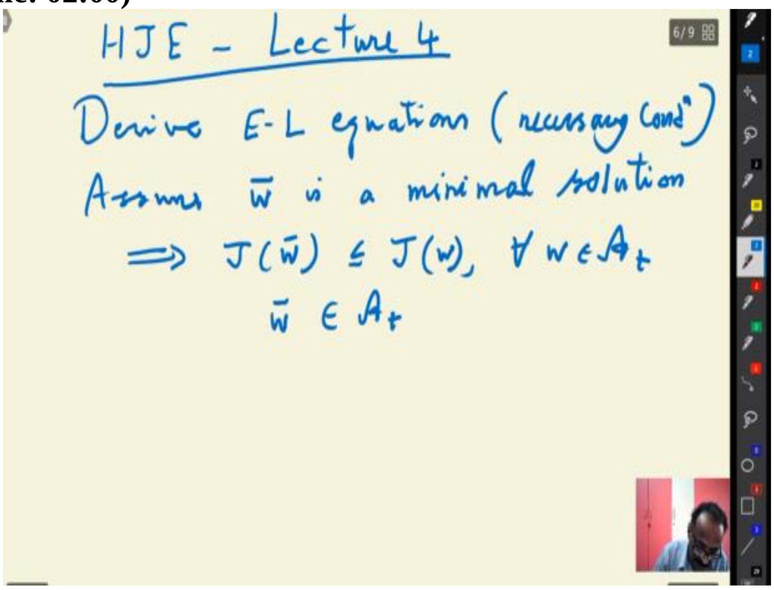

So, we start with the same thing here. So, you want to derive Euler Lagrange equation that is what we are going to do it today as I said these are some necessary conditions. So, you assume w bar is a minimal solution minimising the function given a minimal solution. So, we are going to of course, w satisfies the condition w 0 = y and w at t = x. So, we are going to do that one that means, this implies your J of w bar is less than or equal to J of w for all w in A t. And of course, w bar is also in A t so, that is the meaning of minimal solution. So, I choose special test functions to derive a necessary condition.

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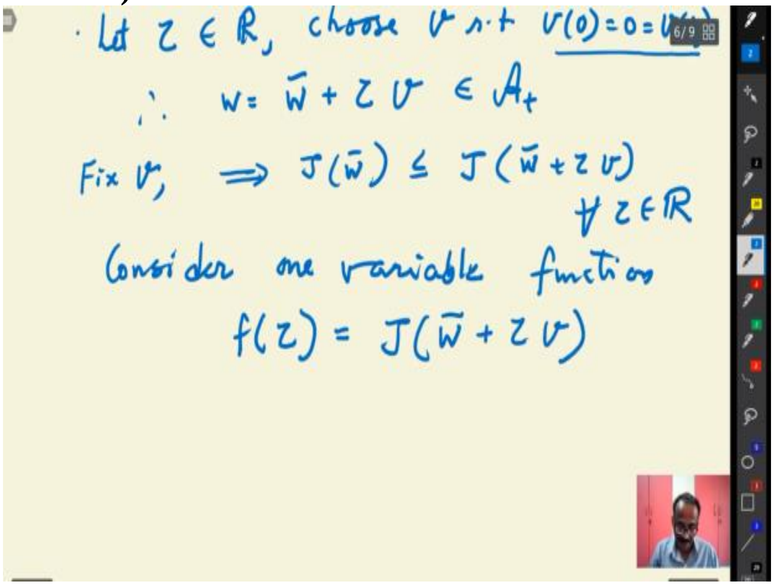

So, let tau be any real number and you choose a V not in A t choose V such that V at 0 = 0, that is equal to V at t so, that it is a trivial boundary conditions. Therefore, if I add choose w = w bar + tau V, and V is smooth V is C 2 function. So, V is also smooth, but V does not contribute any boundary condition. So, it will retain the same boundary condition so, this is equal to A t. So, I fix t for the time and later I need it.

So fix V t also there w bar is already chosen which is fixed. So, fix V and then that implies J of w bar is less than equal to J of this w that is nothing but w bar + tau of V this is true for all tau in real. So, that means that now you will try to do a differentiation process. So, I consider you have 1 variable function in top variable function f tau is a variable functioning thing because w bar and things are fixed J of w bar + tau V I fix that one and what is my f 0?

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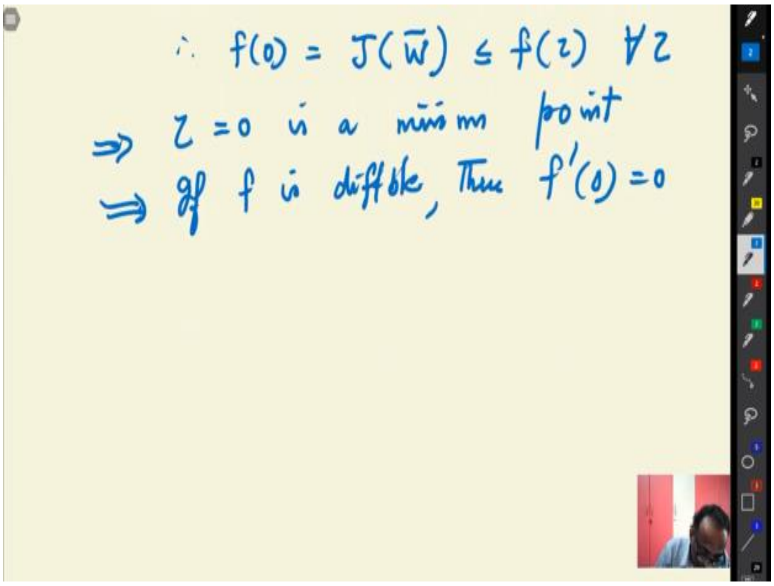

Therefore, f 0 = J of w bar less than equal to J of f tau because by the previous thing this is true for all tau that implies tau = 0 is a minimum point and one dimensional minimum point and that implies if f is differentiable of course, we will show that f is differentiable, and if f is differentiable, then your f prime of 0 = 0. So, I want to compute I will do something in between so, we want to compute f prime of tau. So, to compute f prime of tau you write down this definition and you do the integration under the integral sign.

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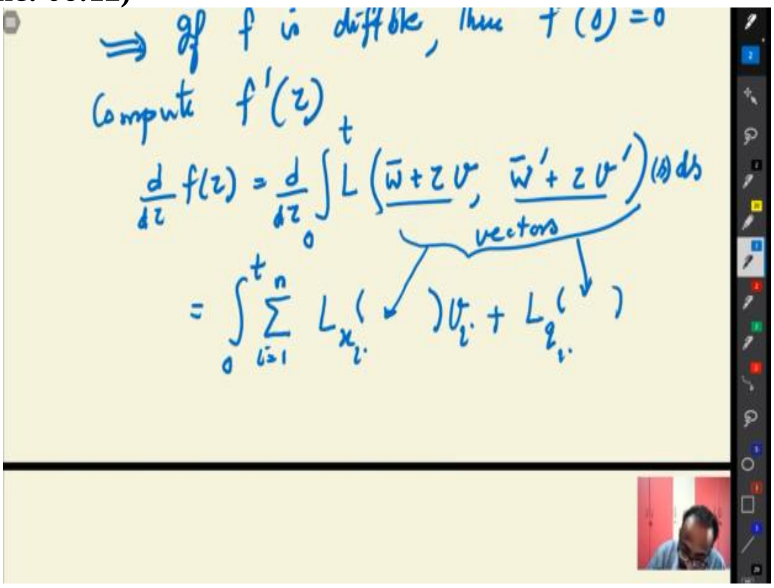

So, what is your F of tau so, d by dt, d by d tau, f of tau is equal to this one J of w bar + tau that is by definition. So, we call your J of w by definition definitely needs this one. So, you have to write the correct argument properly. So, d by d tau of f tau is equal to integral from 0 to t L of w bar + tau V, w bar prime + tau V prime that is the definition of s everything acts on s. So, it will be acting s ds that so, this is just by definition d by d tau of that.

So, you do the integration under the integral sign. So, if you do this integration and so, these are all vectors so, you have all the variable. So, if you want later I will give you a first time let me give you a summation notation i = 1 to n. So, you want to differentiate with respect to the first argument when you differentiate with respect to the first argument you will get L x i the first argument L x p, L x q it is. So, L x i into the same argument, so, I will not write repeatedly write here.

So, this write here, so, that is the argument and then you are to differentiate this term you want to differentiate this with respect to tau and that will give you V the V i prime the ith component with respect to tau if you differentiate you will get your V i and then you will differentiate with respect to the second variable that will be L q i and put the same thing here also into differentiate the second argument with respect to tau that will give you V i prime everything with respect to s, ds so, this is the summation.

Now, look at the second term so, you can do V i satisfies the 0 boundary condition so, integrate by parts. So, you have to do an integration by parts here and your V i and V i t = 0. So, and let me also write there, so, this derivative V i prime will come to L q i. So, that the derivative V i prime so, you can write down.

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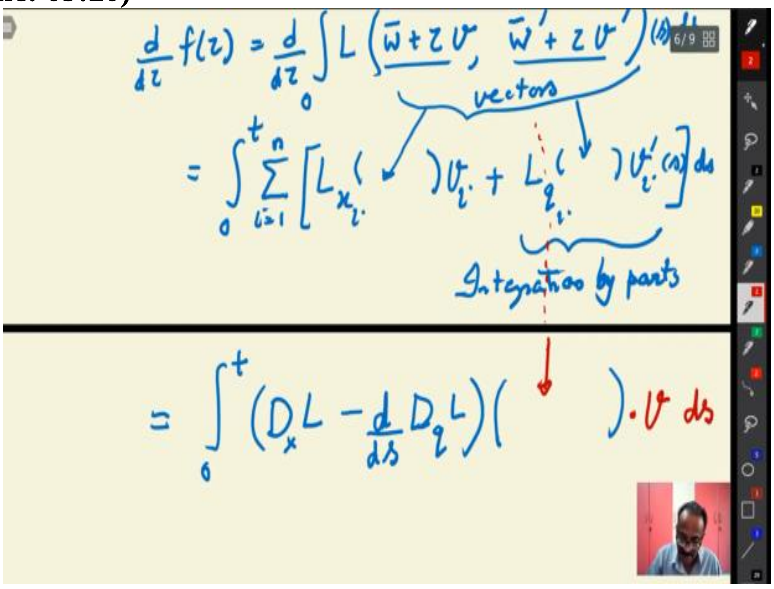

And let me do a combined notation so, this will become 0 to t and I will remove summation and I will write the D x of L and then integration by parts, so, there will be a negative sign coming here. So, there will be d by ds. So, there will be a d by ds of this ds prime coming. So, d by ds of integration coming here, you will have the D q of L of course evaluated at that variable. So again, there will be exactly that component, this component will be coming L coming here. And then these are all with the dot product that derivative at V i. So it will be dot product will be V ds that is a thing.

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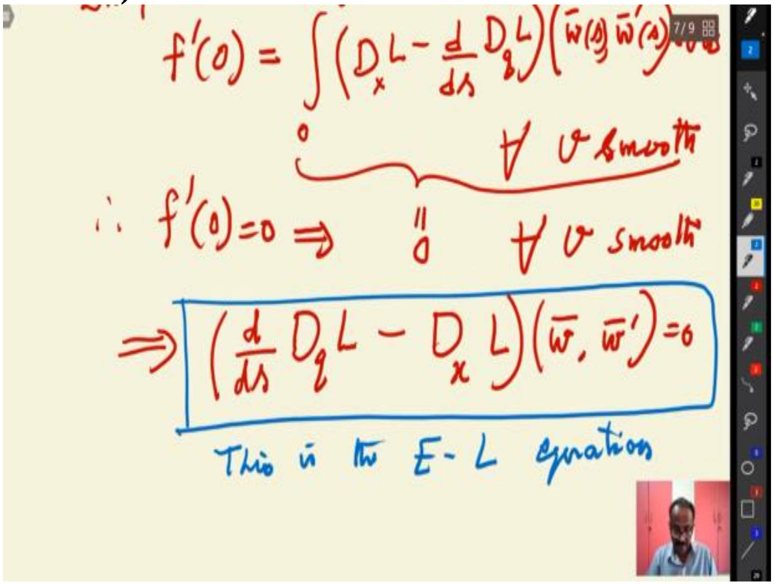

So in particular, let me use the same colour. So in particular, I can compute my f prime of 0, because it is an argument in tau = 0, f prime of this comes arguments are w bar + tau V. So if you want to compete at 0, you put the tau = 0, so, you get w bar, w bar prime everything in s. So you will get 0 to t, in D x of L - d by ds of D q of L evaluated at w bar of s, w bar prime of s that is our given dot product V with the ds, and this is true for all V = 0 for all V smooth.

Since it is true for all V smooth therefore, since there is some integral action on V 0 for f prime of 0 = 0 implies this term the equal to 0 for all V smooth with 0 boundary conditions. And that implies the integral is 0 so, that implies you can change the sign if you want and let me write it in a standard format changing this D q of L - D x of L is of course, these are all evaluated, so this is an evaluated at the optimal solution. So w bar w bar prime this is equal to 0 and this is the Euler Lagrange equation. So, as I said that, so, if the w bar is an optimal solution, then it satisfy.

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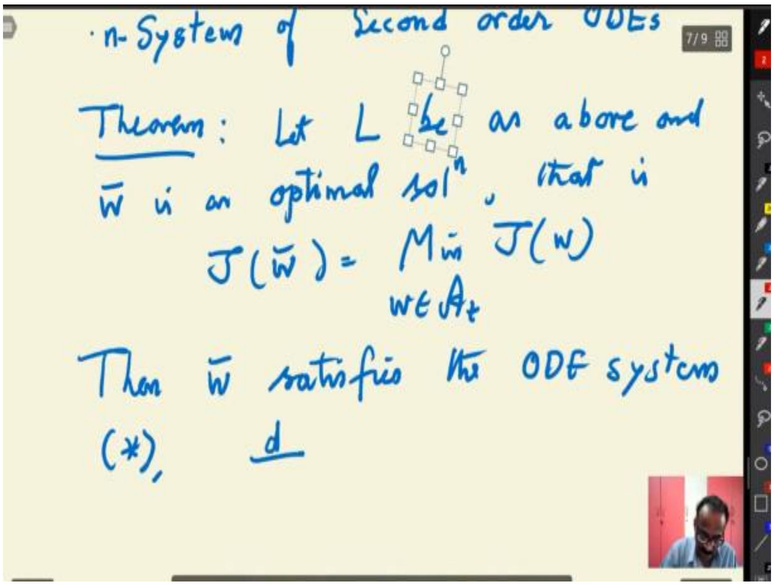

This is a system of n system n equations are there system of second order equations second order ODEs. So, we will eventually write this system n second order equations into a 2n first order system later using what is called the Hamiltonian. So, defining Hamiltonian involves some work which we may not do it today, but we will before going further. So, what is the theorem basically, so, let me complete the theorem format theorem what we have done.

So, let L be as above and w bar is the optimal solution minimising solution when I write optimal solution that is J of w bar is equal to minimum of J of w, w in A t then the w bar satisfies the ODE system star. What is star? This is your star systems star that is d by ds, maybe we will put it whatever way you like it, otherwise do not get confused. If you both are same minus here plus here both are same because it is an equality equal to 0.

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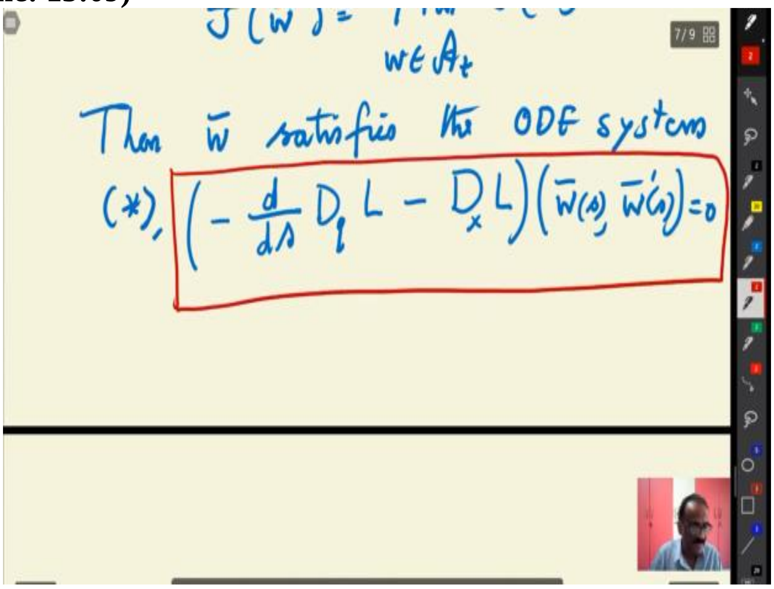

So, let me write it minus d by ds of D q of L - D x this means you can solve it D x of L. Of course, you evaluate at the optimal solution w bar of s, w bar of prime of s = 0. But Euler Lagrange equation, how you will divide? So, let me make a remark so I am going to give some examples.

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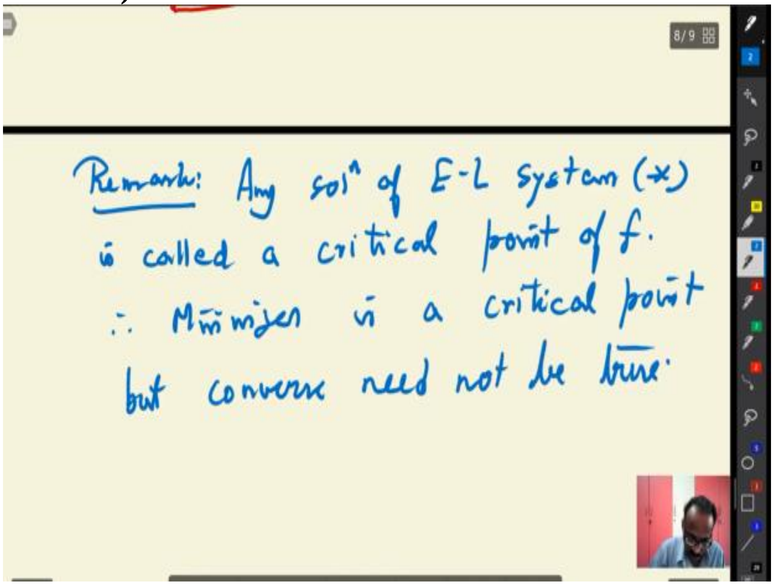

So, let me make a remark before going to examples, which is exactly like 1 dimension. For example, in 1 dimension x 0 is a minimising f of x in an interval, and then you know that if f is differentiable f prime of x not equal to 0, but f prime of x not equal to 0 need not be a minimiser. So, similarly, if w bar is a minimiser, it satisfies the second order ODE system, but that does not be the second order ODE system can have solutions and but need not be a minimiser.

So, that is the why any solution of Euler Lagrange system star is called a critical point is called an exactly like called a critical point of f. So that minimise the 4 minimiser is a critical point, but the converse is not converse need not be true. So, this is similar, and then we call it so basically, you have derived something your f prime of tau here. So you see your, derived of f prime of tau here, and so, we can compute in particular, you calculate f prime of 0 this one, that is what they and that is true for a V d p.

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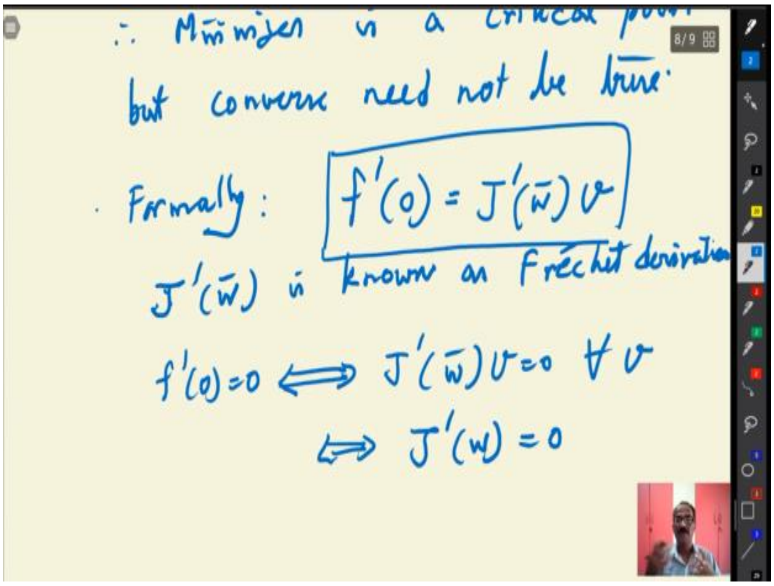

So, informally, we can write though those who have study so formally, you can write f prime of 0 is nothing but J prime of the w bar because the bar evaluated at V and this J prime of w is known for Frechet derivative. So you say you have a concept of derivative known as Frechet derivative, see this is an infinite dimension, so, you are basically looking at this thing by fixing at V.

So, f prime of 0 is basically is equal to J prime of w, and then f prime of V is 0, it because f is depends on V. And then f prime of 0 = 0 = J prime of w bar equal to therefore, f prime of 0 = 0, you see in this f is depends on V, please keep it in mind. So, that should be a dependence on V, that the V coming here in their prime of w equivalent to J prime of w bar V = 0 for all that is we also write it as the Frechet derivative is 0.

That is the meaning of a function and that Frechet derivative is going to be a function. So, with this we will connect as I said the system of 2n equations, n equations in second order, we will convert it into a first order system, and that first order system using the Hamiltonian and we that inverse little more work. So, before that, let me go to some examples.

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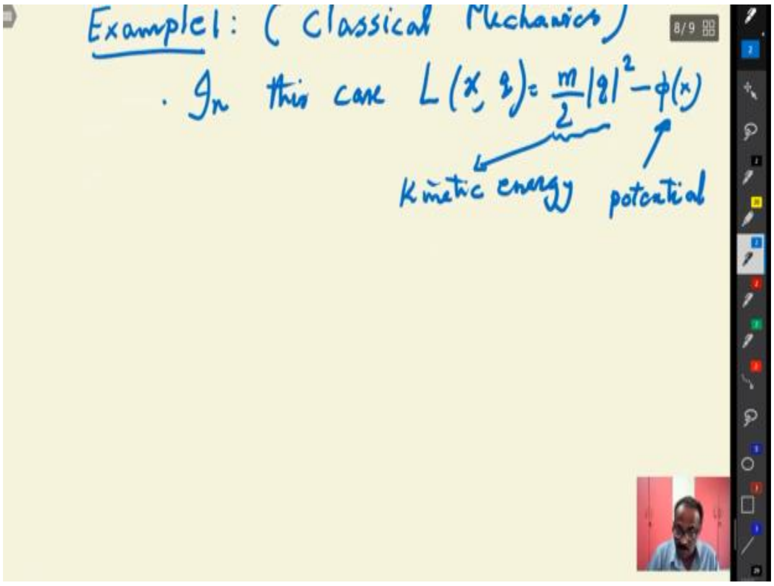

So, we will start with some interesting examples and in this class some of the details how to work it out. So, I will not be giving the first example one is from classical mechanics, this Newtonian mechanics. So, what we are doing is something more than the Newtons law of motion, you have to understand that. So, all the things available for Newtonian mechanics, may not be available.

So, the Newtonian theory of about the world view is a specific choice of L, so, let me write it in this case, let me I will tell you the problem which you know, in this case, your L x, q is basically m by 2, m is a constant, mod q square - V x or let me write it as phi x. So, this is, in this case, this will become the kinetic energy. You see, I told you in this case, q has to be x prime eventually and that is nothing but when velocity so this is the kinetic energy and this is your potential.

So, it is a difference in kinetic and potential energy. And they are what Newtons law of motion basically tends to that there is a least action principle, that is what you do least action principle. So, the Newtonian mechanics follows this so, if you want you should read some classical mechanics.

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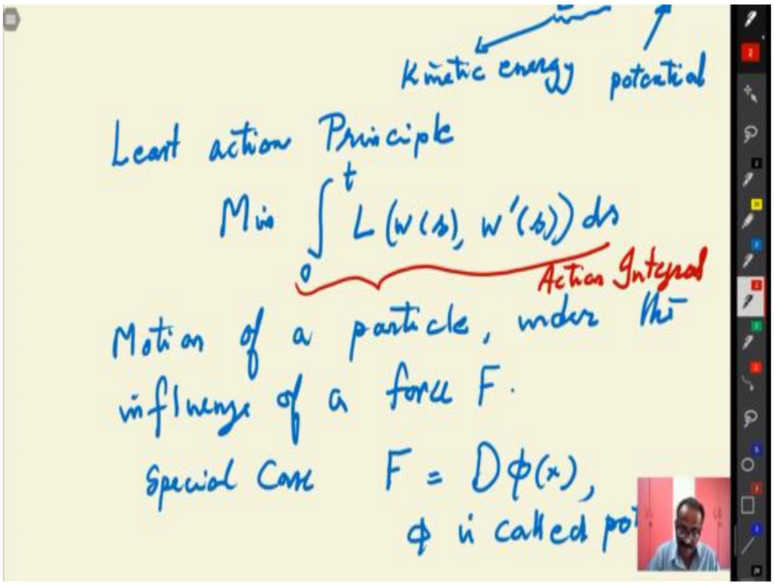

Newton least action principle, you minimise the action integral. So you have your m by 0 to t L of x of s, x prime of s. So since I am using w, let me use w only. So I will use w prime of s, you can use any notation you would like so, L x, q is that one. So, you see, and how do you get to Newtons mechanics. So, Newtons basically want to study the motion of a particle under the influence of a force.

And in special cases, in concert with forces F and under like conservative forces, this is given in a special system or conservation system and a special case is a conservative system where F is given by a potential F is equal to some D phi x and phi is called the potential that is what is physics basically in the Newtonian phi is called potential and that is the thing we have used. So, you see, so, you eventually, you want to find the least action and this is called what is it is called action integral?

So, we will use the same terminologies action integral, and we are going to use the same terminology. So this integral and so, Newtonian mechanics is nothing but a very, very special case of a Lagrangian. So, what we are doing is the present study is more general Lagrangian as I said, everything will not be that easy, especially construction of a Hamiltonian because there is a in the classical mechanics.

There is a Hamiltonian which turned out to be the total energy that means sum of kinetic and potential energy and that you can go from one system to Lagrangian formalism to Hamiltonian formulation which I will litigate later in the next class.

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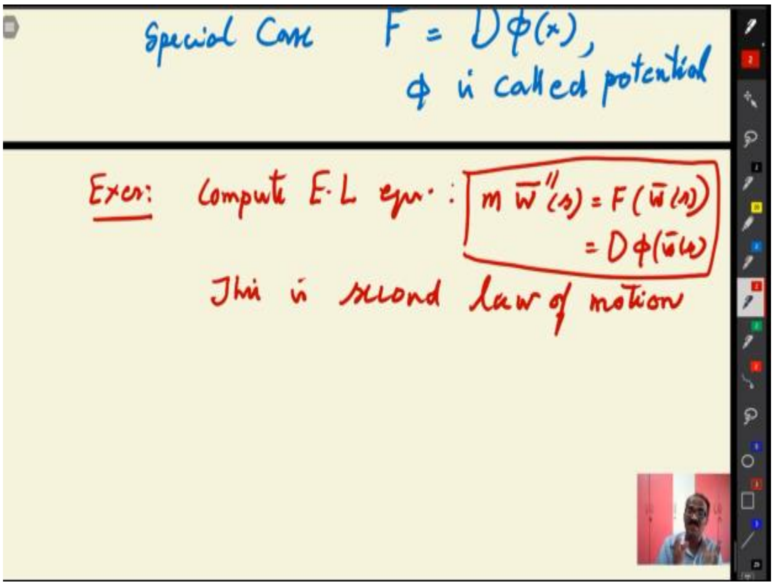

But in this case, so, you have a simple exercise you can prove it. It is not an exercise, but you can easily do it while you compute an Euler Lagrange equation which you already know that Euler Lagrange equation and this turned out to be what do you get it is m into w bar double prime of s and is equal to of w bar of s and that is nothing but D of w bar s you see this is nothing but your second law of motion Newtons second law of motion.

You see, so, what we are doing is a kind of generalisation of; this, you just write down your Euler Lagrange equation, you will get it second law. So, second law of motion is a necessary condition for your action integral to have a minimum that is what the study.

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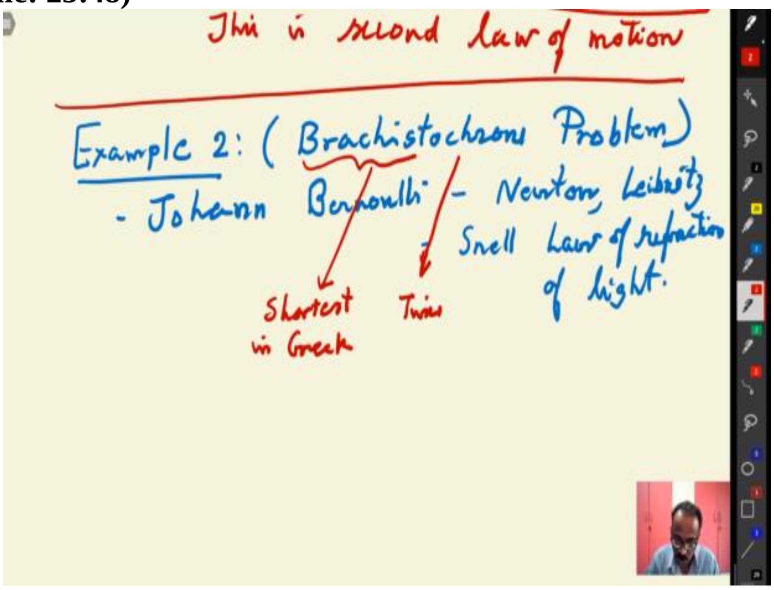

So let me go to another example so, you want to give 2 more examples before finishing the example 2. There, I will not write down the equation so you it is your job to write down the Euler Lagrange equations, I will just leave that one. This is what is called a Brachistochrome problem which you probably many of you are already familiar Brachistochrome this problem was introduced by Johann Bernoulli in the 1600s or something like that.

I do not know exactly when it is introduced maybe anyway, but the material for the time being Johnn Bernoulli and then many people like Newton Leibnitz are all work to solve these problems. So, you got to understand that the early days 1500s and 1600s the era normally people proposes problems to the scientific community and you look for solutions and later people tried to give a general theory making mathematics this little bit of an abstract subject.

And even the Daniel Bernoulli and other people worked on these and the more interesting thing is that this users the initial proof users, Snell law of refraction to see how physics is content. So, it is not so we are what we are given is a much more general theory what then what do you know it. So, it uses that kind of Snell law of refraction of light so, you have this thing.

So, what does this mean actually, Brachisto means I will tell you, Brachisto means shortest this is a Greek word shortest in Greek and chrome means time so, the shortest time when I explained you will understand the meaning of the shortest. So, you want to study so, let me explain the problem first.

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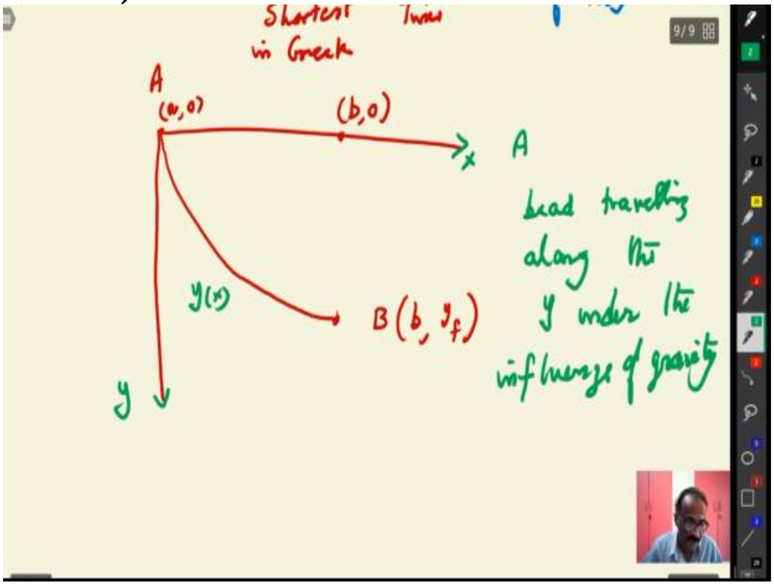

So, you have a 2 points you can think it as a vertical plane, I will put the coordinate later A and this is a point B. So, you want there is a bead, so, there is a curve here and bead this travelling if I put the coordinate system in this way. So, this is the point a, 0 and this is the point corresponding to b, 0. So, you are travelling from this point so, a, be 0. So, this will be some b some other number B, you can put some endpoint in the y coordinate y f final coordinate.

So, a bead is travelling so, this curve we call it y of x. So, this is your x and this is your y so, let me quickly write down a bead is travelling along this curve y and bead travelling along the curve y under the only influence of gravity no other forces acting. So, only it is moving towards.

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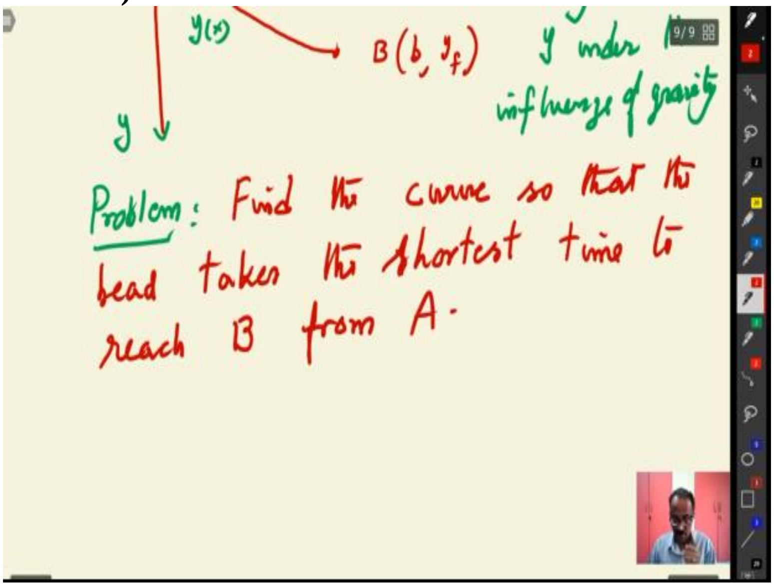

So, what is the problem? Find the curve y, by the curve y so that the bead takes the shortest time not shortest length the shortest length is the straight line, shortest time to reach B from A so, that is a problem.

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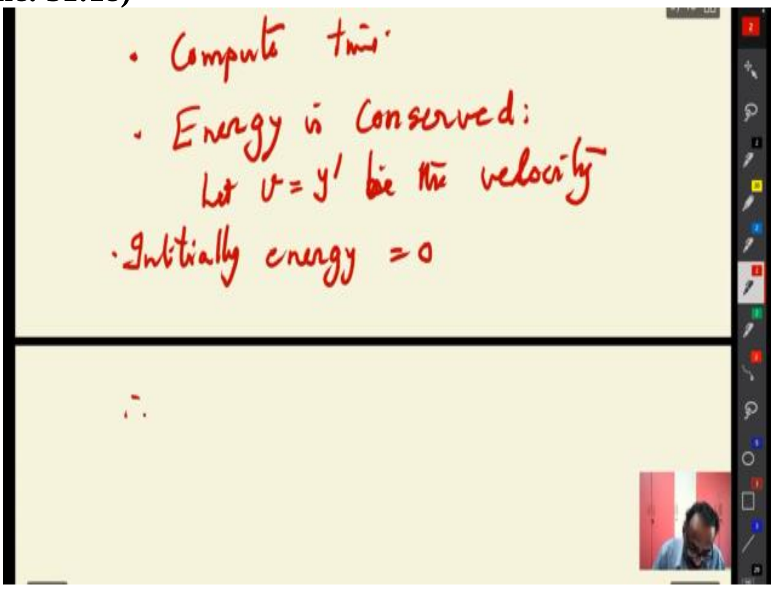

So, you want to find the problem is to compute the time, so, we want to understand. What is the computation of the time? So, what is physics? The physics is the energy is conserved. So, you just compute your energy is conserved. So, you write down the energy, so, let the V so you compute the position so, you compute any point here. So, for any point x here, so, this is your y of x the distance and let the velocity be V.

So, can let V = y prime be the velocity. So, the energy initially the total energy is so and y is in the negative direction. So to understand initially total energy 0 because there is no kinetic energy and it is moving under the influence of the gravity is at the point A, there is no energy initially energy is equal to 0.

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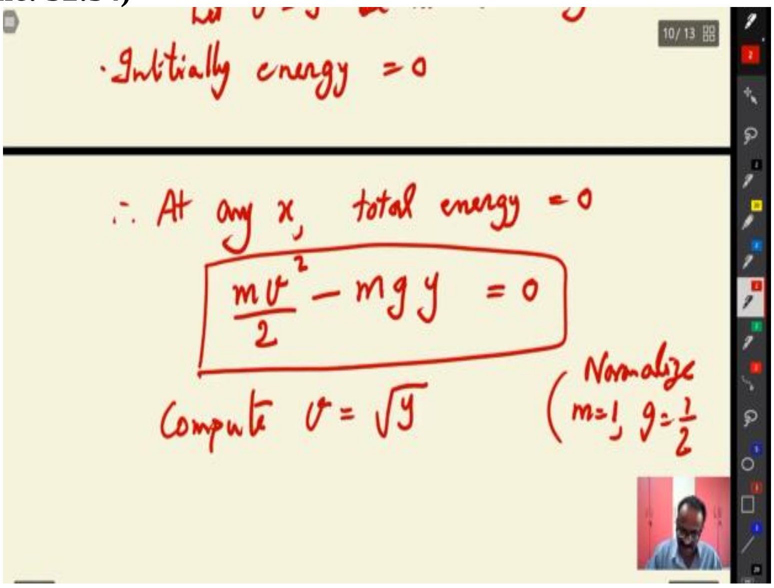

Therefore, at any time any x any x the total energy is 0. So you just write down the total energy because once it moves, it picks up the energy is equal to 0. So you have your kinetic energy, kinetic energy is m by 2, and then you write down the potential energy m g y this is equal to 0. Because the minus sign comes because it is moving down what we have done that coordinate in that direction that is the reason that.

So, from here you can compute V compute a relation between V you get V = 2. So, you normalise it is not compulsory, but you can normalise because for your computations, it is easy, you take mass 1 and g = half this is not necessarily I am telling you and this no but otherwise, you have to keep tracking of this one and then you will get your y = V = square root of y in that case, otherwise there will be a parameter seen here. And then what is the function? And so, you compute the length of the so if you want to understand the time taken.

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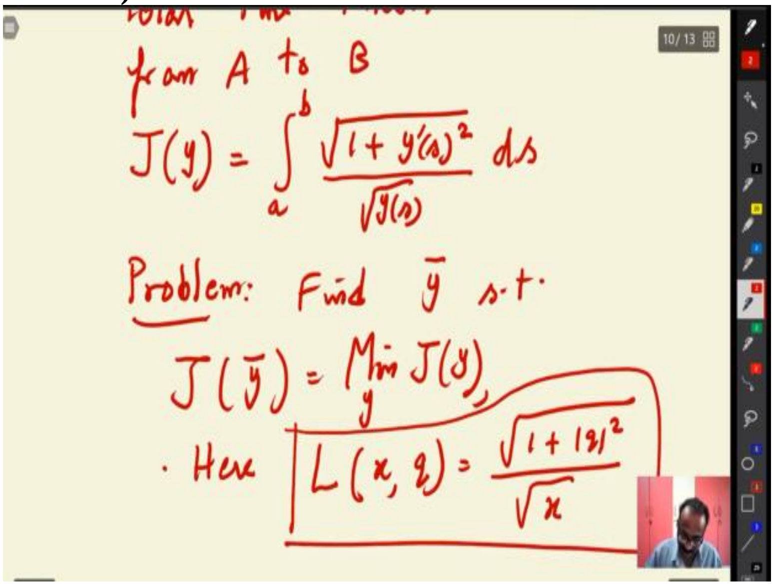

So, the total time taken to move from A to B you compute the length of the curve y prime of s square and then divide by the velocity basically s. So this is your J of I used the y here. So this is your so, what do you see a problem so they are your problem. Find this is the time taken so the problem, find y bar so, you see y bar such that your J of y bar = minimum of over y or that the smooth curves of J over y and what do you see here? What is your L x, q? Your L x, q is nothing but square root of 1 + mod q square by it is 1 dimension thing and you have square root of mod x that distance y is taken as to be positive so you have your x.

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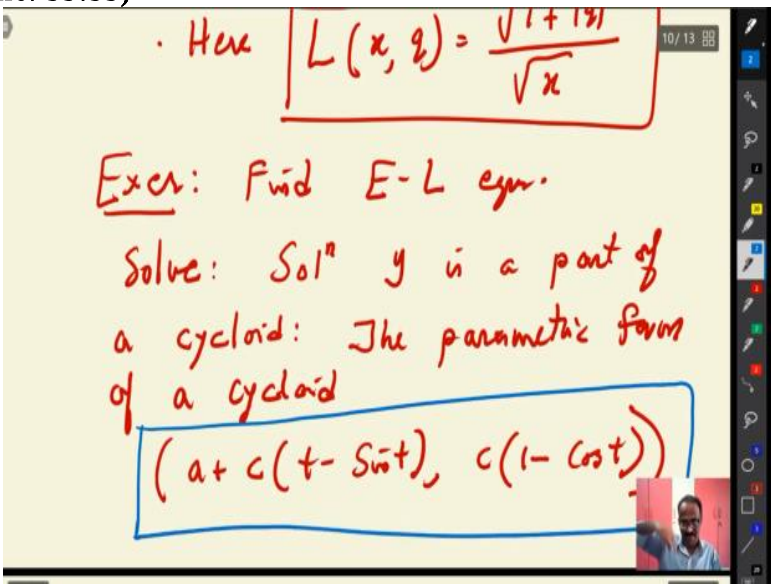

So, you will see so, the exercise I will leave it as an exercise I will just tell one more problem before I exercise find Euler Lagrange equations and once you solve this one I will not do all this work you can refer some books or something like that that is not the main aim because we have other thing to solve the solution, y is a part of a cycloid so, you understand what is a cycloid.

And all that the parametric form of a cycloid is which are the parametric form of a cycloid form writing a parametric form is that a + into t - sin t and c into 1 minus you can do all these are all exercises. So, maybe we will provide you exercises later. So, first write down the second order equation solve it when you solve you get this equation a solution that is a part of a cycloid, cycloid is obtained by rolling a wheel it will form a curve so, it will be part of a cycloid.

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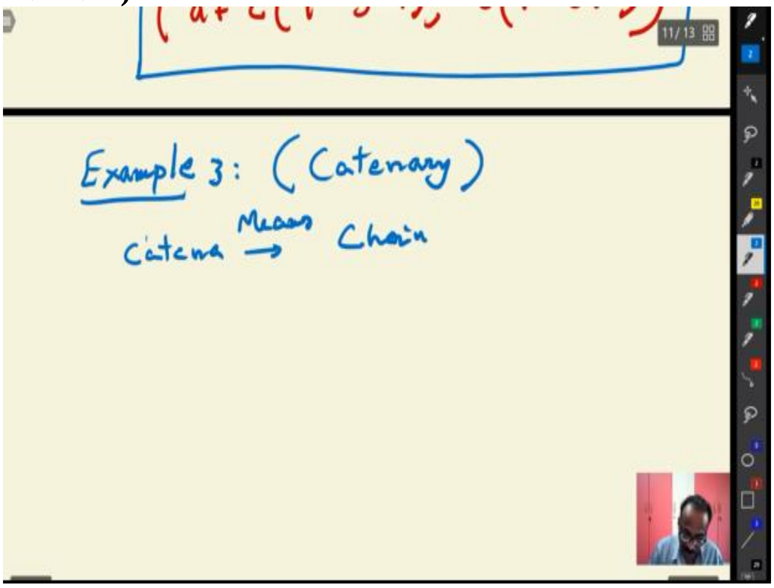

So, let me present you one more example. Before I conclude example, again I will just write down the example, but you do this further things if the time is up. So, we will example 3 this is what is called like catenary. So, let me again this is introduced by Bernoulli and then really many other people worked on this one. So, typically you have a chain catena means chain so, let me describe here a problem.

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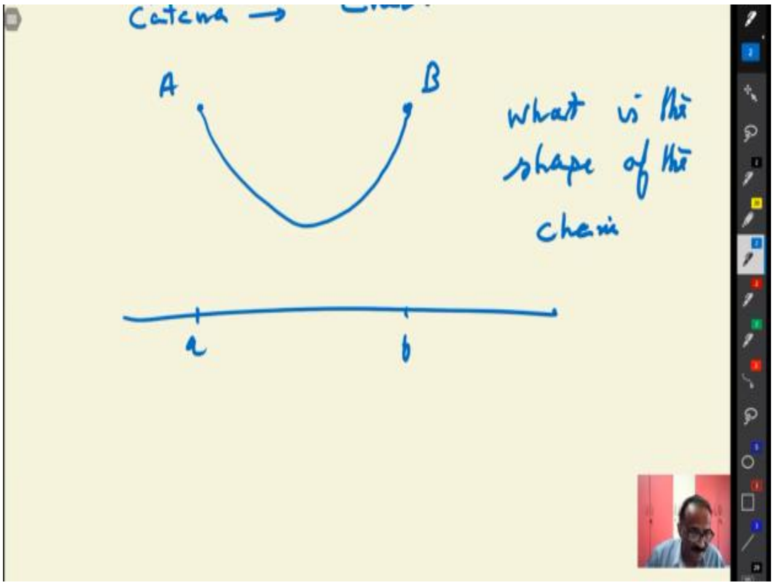

So, you have let me also introduce the coordinate here. So you have 2 points and you have A and B and you have a chain at if you want it, and chain is hanging here. So what is the shape of the chain? The problem is what is the shape of the chain? Of course, chain is too big, it is possible that chain will come and stay here. So we are avoiding that also you can solve it, but we want a looks like a chain. So typically people will think that it will be a part of a parabola and things like that, but that is not true. Unfortunately, that is not the case here.

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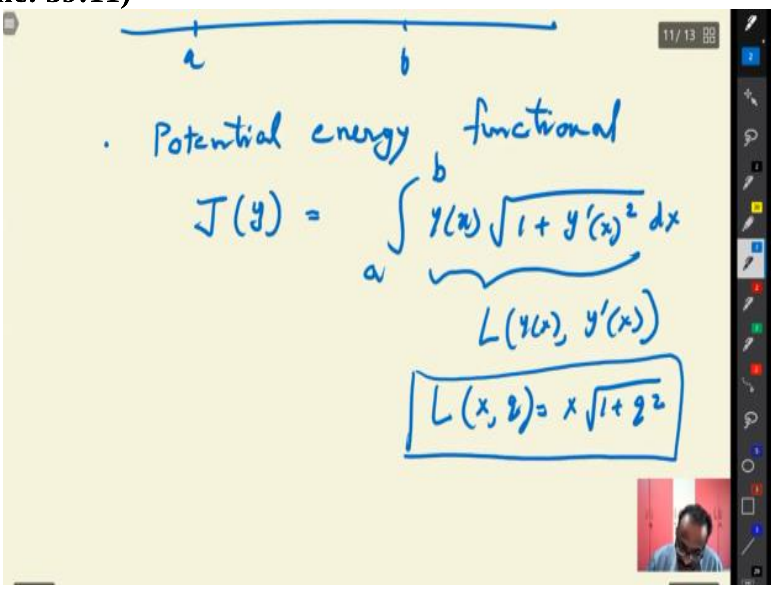

So what is the function here to be minimised? So, what is the minimise thing when a chain is hanging the only physics or only force acting on it is potential energy. So, you just write down the potential energy function that you can compute any point to put the coordinates appropriately and the potential energy is happened to be m g h at you can normalise it if you want. The potential energy is integral a to b and y of x square root of 1 + y prime of x square.

So you for a small place you compute it and then you write all this. So this is your L of y x prime og x I just change it at the motivation that is all L of y x y prime of x. And so, what is your L x q? Basically, if you want this notation, L x q = x into square root of 1 + q square I told you the q comes as a velocity most of the time.

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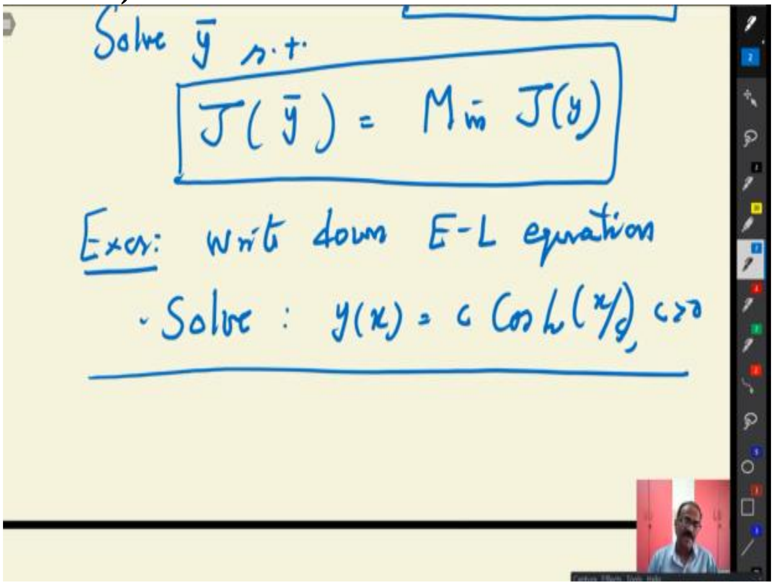

So, you want to find an answer so, you want to solve y bar such that J of y bar = minimum of J of y over a smooth curve. So the exercise is again write down Euler Lagrange equation then solve it and then solving you get your y x to be cos hyperbolic constant. So it is not cos hyperbolic cos hyperbolic of x over c where c is positive I leave it these are all exercise. So I will stop here, you see only 3 special cases one is Newtonian mechanics.

Then you have seen a problem from Brachistochrome and then the third problem from the Catenary and you can actually of course initially the solutions given by their methods individual methods. But now you have a very general theory you see more Lagrange equations and then if you can solve it that second order equation you get the solution to this problem. So, I will stop here.

And we will continue a little more about deriving, converting into a system of 2n equations. So, we start introducing the Hamilton ODE and eventually Hamilton ODE and what you call it, how do you introduce the Hamiltonian, Lagrange transformations, so let me stop here.