First Course on Partial Differential Equations – II

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Lecture – 05
Weak Solution 5: Distributions and Sobolev Spaces

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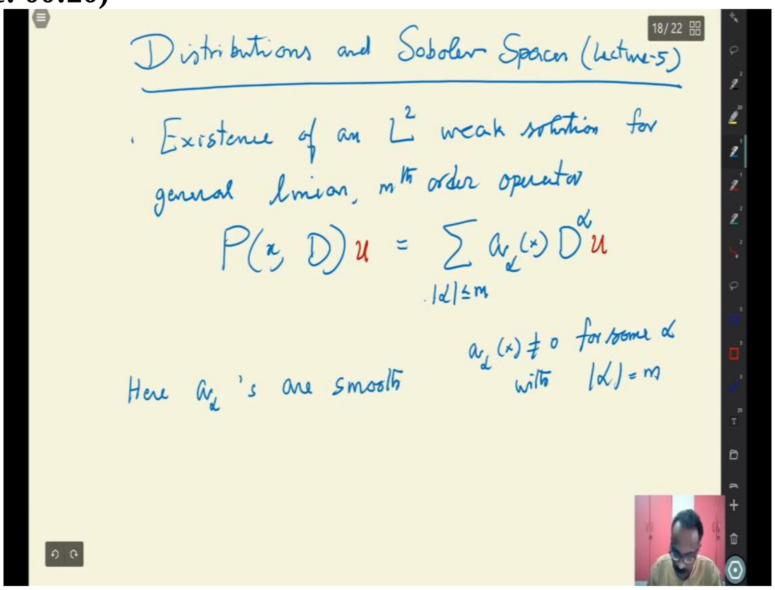

Good morning. So, this is the last lecture to a very short introduction to distribution theory and Soboler spaces. As I said it is only a glimpse of this theory but those who are interested in studying should get into that and spend some substantial amount of time to really get the understanding of distribution theory and of course Sobolev spaces we do not do much because it involves a lot of technicalities.

So, we will just say introduce as Sobolev space which is relevant to our weak formulation which were introduced 2 to 3 lectures back. So, before going to that so distribution theory we already exists explained it i is a D primer of dual of a D omega and then we have seen some examples and all that and before going to Sobolev spaces this part of this basically consists of 2 parts one I will give the existence of a weak solution and the second part of the talk will introduce quickly the Sobolev spaces.

So, this is one thing existence of an L 2 weak I will not give all the proof but how you see the existence of an L 2 weak solution for a general linear, mth order operator because what is your operator may look like this. So, you have your P x, D this is your operator so your operator is something linear operator should look like D power alpha. So, this mod alpha is less than or equal to m.

And it to be a genuine mth operator at least one of the coefficient of the largest term should be not 0 of such that a alpha is not equal to 0 for some alpha with the mod alpha = m. So, that precisely to make it a secondly thing that means so if it operates on u that is it. So that is what you have an operator here you assume an alpha are smooth. So, not deal you can deal with other cases a alpha's are smooth so and before we will do.

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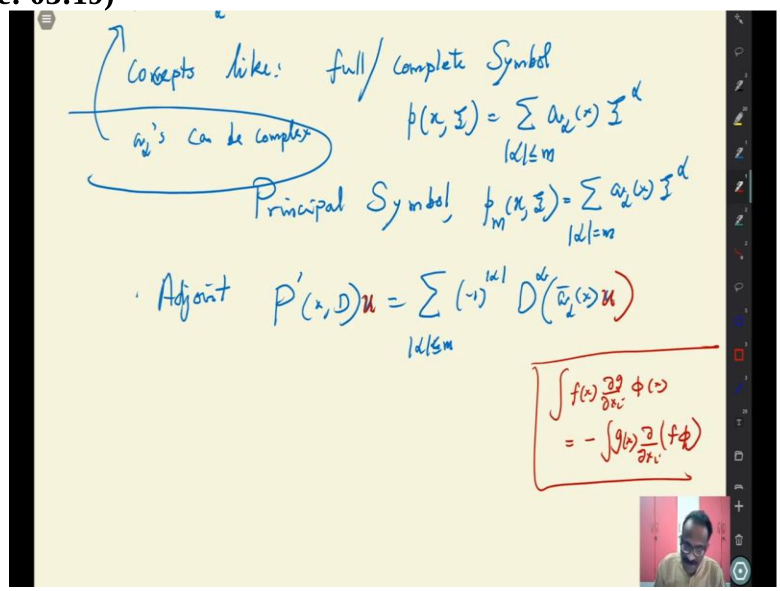

So, there are some concepts like which we do not need it here concepts like simple concepts like full or complete simple because when you read other books and learn more you will say what is called the full symbol which is a polynomial of degree m that is equal to sigma a alpha of x it is a multi index psi power alpha but mod alpha for less than equal to m and there is also a notion like principal symbol.

That is p m of x I these are all useful may not be in this class anyway this is kai of a by x set of lectures alpha with only the highest thing you know that even in second order you see how you use a second order to define electricity and all that corresponding to this one. So, we will define what is an adjoint operator a alpha can be complex can have that also something here. So, adjoint you denoted by P prime is not prime P dash of x, D is the same thing is adjoint comes from the integration by parts.

If you do that - 1 power mod alpha mod alpha less than or equal to m D power of so you get this alpha bar if it is a complex the a bar of x and then if you apply to u so this will be inside apply to u. So, you will have u here this one that is natural when you do an integration by parts for example when you have something like say f of x d by dx i pf g some other phi then it will go like this. If it is valid g we want to do an integration by parts you will have something like d / dx i of f phi you will see that is why so you write down your P x, D and that but P x, D is everything needs given in this general setup. So, you are looking for a solution.

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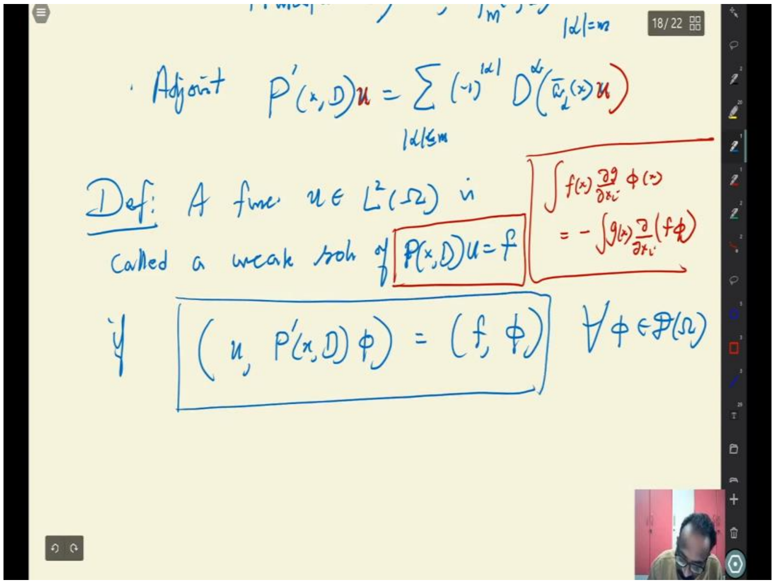

So, you can study these things in advance other PDE courses if you look into many classical PDE you will understand all these things so. So, definition A function u in L 2 of omega is called a weak solution of there is no boundary condition or anything I am defining here solution of u solution of P of x, D of u = f if it is a solution of this equation if u is a basically P prime of x, D of phi = f, phi.

So, only the solution in the sense of basically in the sense of L 2 you are defining this is true for all phi in so basically you are understanding it as a distribution solution. So, for all this thing you have a solution is defined in the sense of distribution that is how you define you can interpret this in the sense of distribution. So, when you integrate by parts you get this one so this is what your definition.

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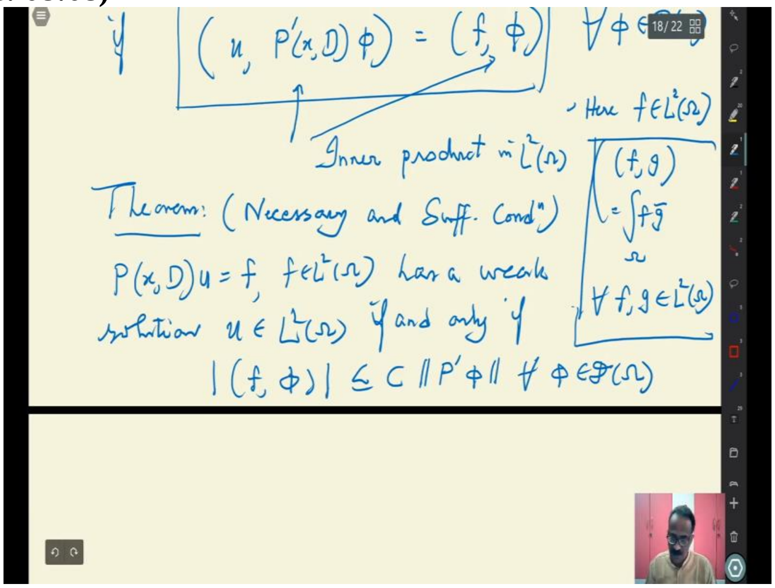

And this is the inner product because this u is in L 2 so this is an inner product this is also inner product in L 2. So, you understand that here f is also mean L 2 omega so everything in the setup of for L 2 you are defining. So, this is inner product in L 2 of omega what is the inner product in L 2 omega f g = integral of f g bar if we are working with the complex thing for all f g in L 2 of omega. So, this is the inner product in L 2 of omega so if you have this one.

So, you have a theorem I will not give you the complete proof thing. So, let me give you a theorem here this proof I will give a necessary and sufficient condition P x, D of u = f of course you start with f in L 2 of omega as a weak solution u in L 2 omega if and only if this is a necessary only if there is an estimate modulus of a f phi is less than equal to some constant in do nom of P prime of x, D and L 2 of omega this is true for all f phi in D omega and C is a constant this is what they are. So, there is a necessary and sufficient condition for the existence of a L 2 weak solution.

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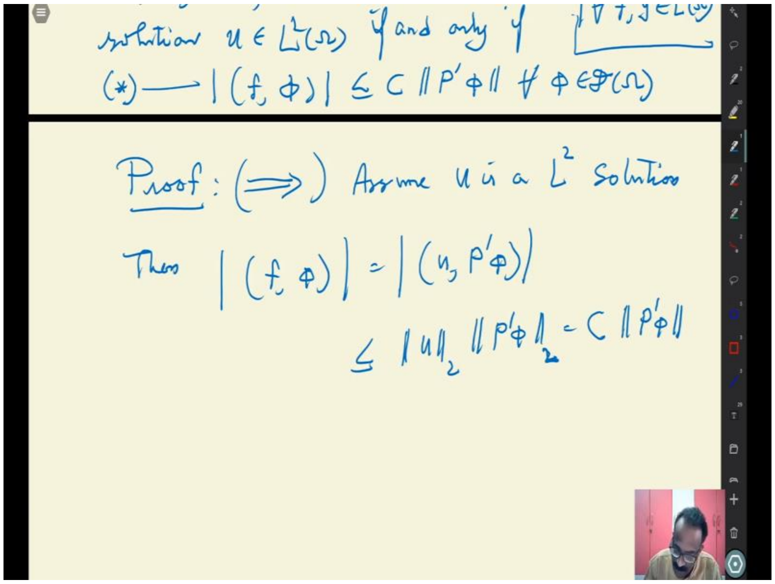

So, let me give this necessary and sufficient condition so to prove this existence so you look for operators for which this condition verifying is enough. So, let me give proof of this and then we will go to Soboler space and then just after making a comment. So, assume one way so we will prove that assume u is L 2 solution then what will we get modules of f phi is equal to modulus of u P prime this is an L 2 inner product.

So, you can apply Cauchy Schwarz that will give you norm u at L 2 into norm P prime phi at delta so this is given to you so this is your constant this is your constant in the P prime phi. So, you get this constant norm u into less then equal to C into this one if there is a solution this is true for all phi any phi.

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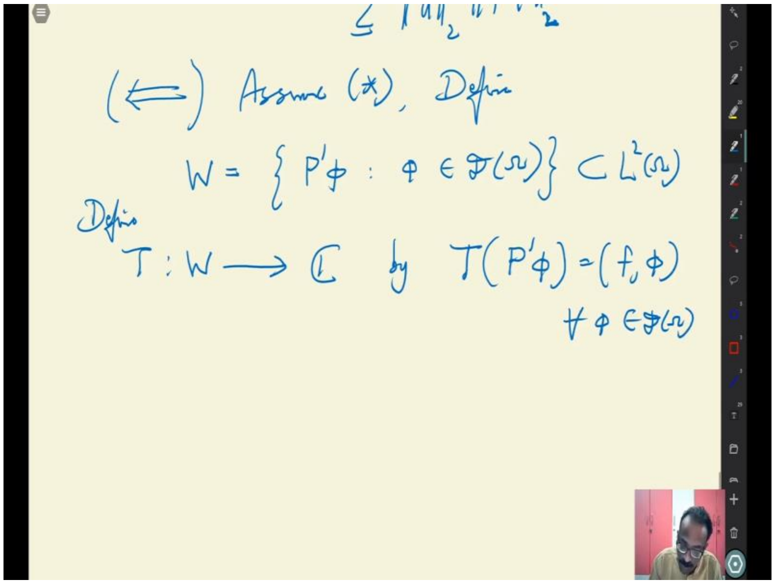

To conversely assume star and then you define a space defined W equal to a space set of all P dash of phi your phi is in D omega and which is a subspace of course this is a easy to check that this is a subspace of L 2 check that because phi is the smooth functions a alpha all are smooth. So, these are all they so you can define now T define a linear operator T from W to C by is a every element of W looks like a P prime. So, it is enough to define T prime of phi or T phi prime by f phi for all phi in D omega T P dash of phi is in W.

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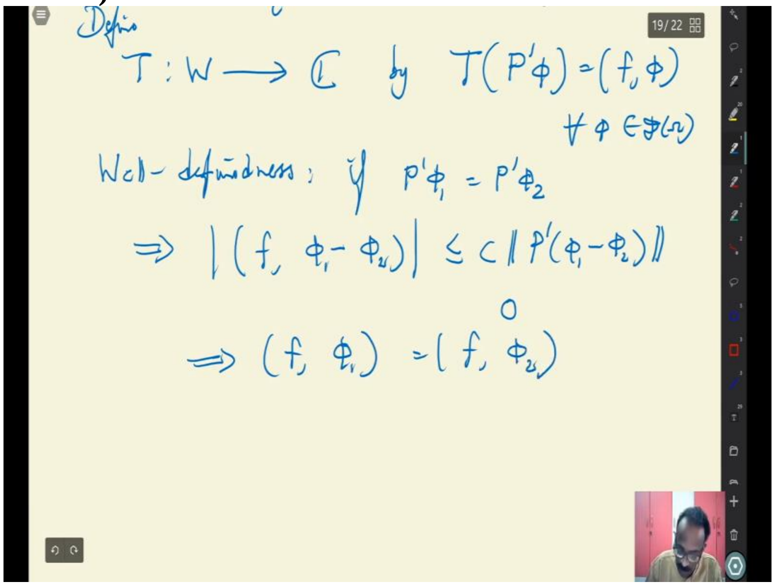

Of course you want to check the well define us in the sense that suppose P prime of phi 1 you can suppose well define us because suppose to have the 2 different phi 1 has the same thing you do not know suppose phi 1 and phi 2 gives the same P prime of phi 1 and P prime of phi 2 you have to show that f phi1 equal to f of phi 2 it should not have a difference. So, well-definedness that is easy that well-definedness by the condition if P dash of phi 1 = P dash phi 2.

Then you can immediately that implies you can see modulus of f phi 1 - phi 2 apply the inequality given the star inequality given. So, you have this star inequality so we can apply the less than or equal to constantly do norm P dash of phi 1 - phi 2 but P dash is linear. So, this is equal to 0 it is linear so you can apply that to one so this 0 implies f phi 1 = f phi 2.

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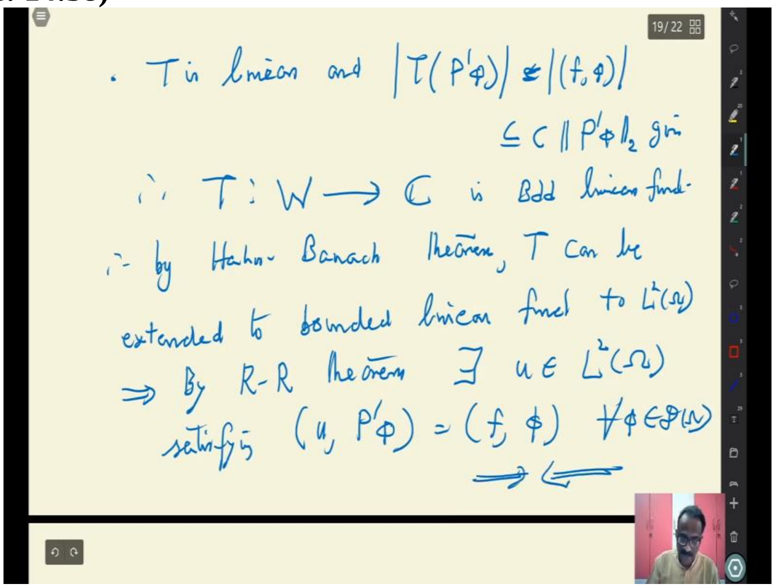

So that is a well defined so that is one point. The second point is T is linear immediately and bounded in that space because modulus of T of P dash of phi equal to modules of f phi is less then equal to constant into this is given to you by given condition this is given you see therefore T from W to C is a bounded linear functional this is with respect to L 2 know linear functional. Therefore by Hahn Banach theorem T can be extended.

To a continuous linear function extended to a continuous or bounded both are same bounded linear function to continuous linear functional to L 2 of omega then apply that implies by Riesz representation theorem there exists u in L 2 of omega because you can identify it with a satisfying u p dash of phi = f phi for phi in D omega that is how you prove existence. So, that sounds so you have proven if and only if conditions that is P x, D of u = f the if and only if u = 0 so I will not prove there.

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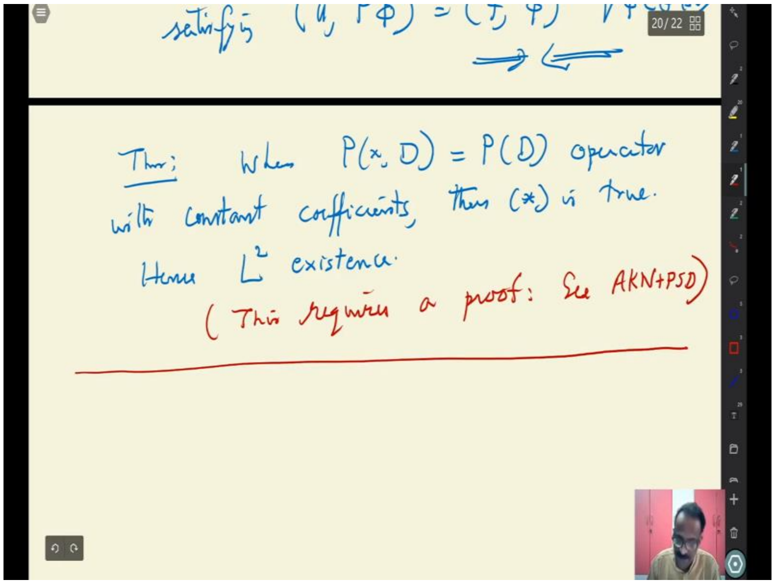

So, there is a theorem which I will not state here when P x, D is equal to P of D that means operator with the constant coefficients then star is true hence there is a this is one example star is true and hence L 2 existence that is all. So, this requires a proof referrer book see AKN plus PAD. So, we will not do that one and we will stop here on this thing.

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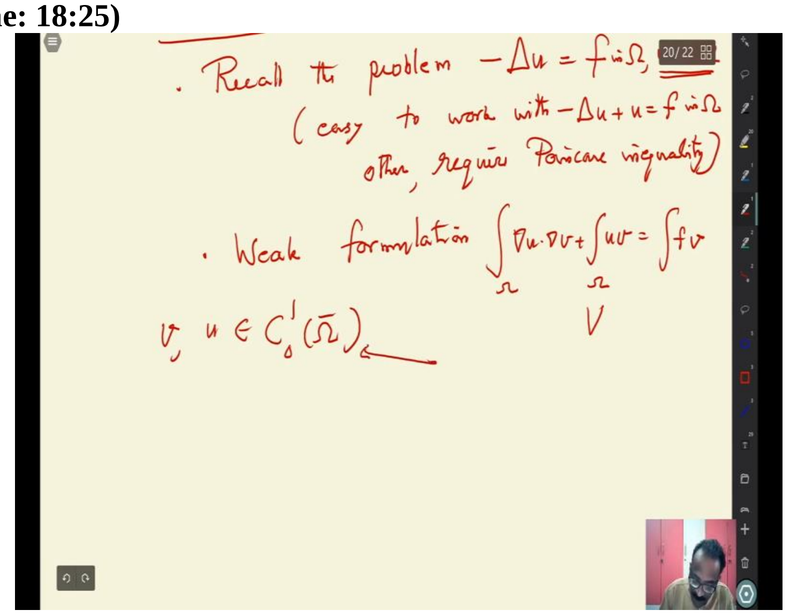

Quickly another 10 minutes or 15 minutes let me go to Sobolev spaces in context of our early problem spaces and corresponding we do not mention so what are so recall the problem -Laplacian of u = f. So maybe little easy to work with -Laplacian of u in omega u = 0 on D omega Laplacian of u + u = f. Otherwise what we require is the Poincare inequality mentioned earlier. So, we will not so you can use the same thing because of 0 boundary conditions.

So, if it is not a 0 boundary condition the Poincare inequality may not be true. So, when you deal with these problems with the knowing mind and other condition you have to be a little more careful. So, studying any of these 2 problems are fine with due to Poincare inequality so what is the corresponding weak formulation? So, let me please recall what we have done it in this thing weak formulation is grade u grade v plus integral of u, v if we add that term.

So work out even without this so since we do not have time we do not want to make further remarks on the and this should be so where are we looking at initially we started with u in C 1 naught of omega bar and you are v is also there but you will seen this is not complete.

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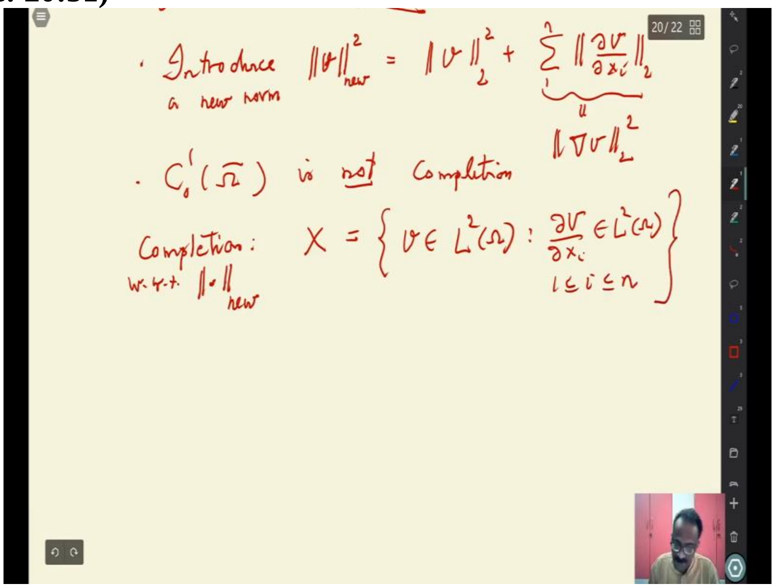

So, you have to introduce the norm now introduce this is what we are now we can be justify whatever they have done in the first 2 lectures of this series introduce a norm of u for these are norm of v. Let me write it in the square form so norm of phi in the space this is L 2 norm this norm is different so maybe I will put one that is not necessarily so this norm is not an L 2 norm I mean introduce a new norm.

So, maybe I will introduce new here later I have change it new norm this is the L 2 norm and then your gradient norm that is nothing but your this is well defined norm dv / dx i square this is already done this is nothing but 1 to n this is nothing but norm of grade v square this L 2 but what we have seen is that C 1 naught omega n bar is not complete and that is why you want to take now completion is exactly what you are doing now completion.

This is done earlier completion with respect to the above norm so what norms you are taking is very important completion with respect to this norm then you can actually get this is what gives you the first set of Soboler space set up on v in L 2 omega with dv / dx i no boundary condition is used yet L 2 of omega with 1 less than equal to i less than equal to n.

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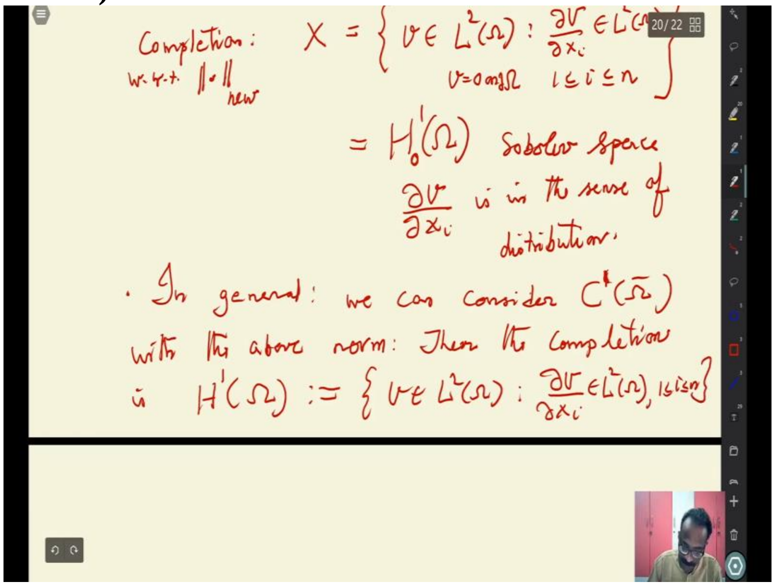

And this is called the and that is a notation H 1 of omega and then this is the Soboler space and dv by dx i is in the sense of distribution now in this because it says 2 functions can define all LP functions or distribution in the sense of distribution. So that is the point so that is a space we are looking. In general not the complete C 1 naught of omega but in general we can define so this is denoted by H 1 naught of omega here v is also equal to 0 on D omega.

In general we can define we can consider without the boundary condition C 1 of omega bar with the above norm and then you will have then the completion is given by which will become a Hilbert space and which we did completion is some H 1 of omega is defined to be set of all v in L 2 to of omega it such that dv / dx i is in L 2 omega without boundary condition. So, do not pick up any boundary condition.

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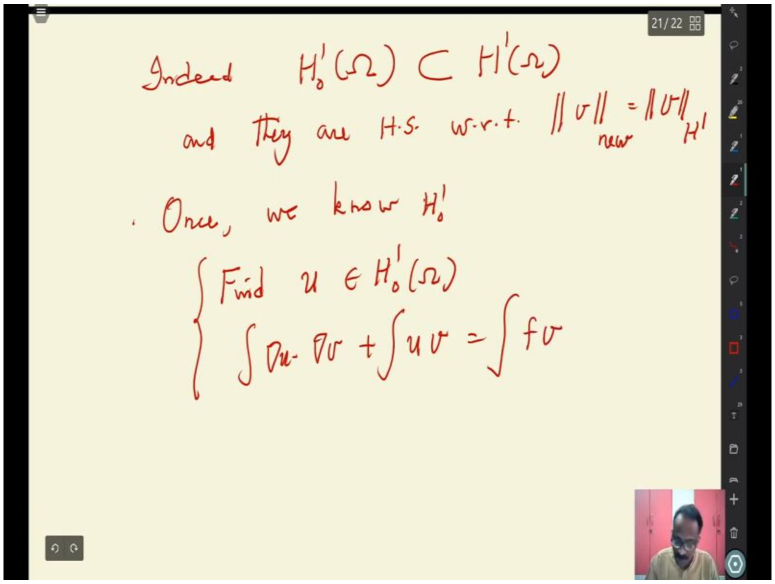

And it of course indeed H 1 naught of omega this is to take care of the boundary contained in H 1 of omega and they are Hilbert spaces with respect to the and you can extend the same norm and they are with Hilbert spaces with respect to the norm which we have defined norm of v new. So, this is normally write it has norm H 1 so both these spaces are Hilbert spaces H 1 naught of omega and H 1 of omega both are kind of some Hilbert spaces which you will be.

So, this is the space setting which you do it and hence you can study and then once we know H 1 naught so now you are weak formulation you find u in H 1 naught. So, you have a formulation now very mathematically stable gradient of u gradient of v + the integral of u, v is equal to integral of f v you see and you can get a solution to this problem.

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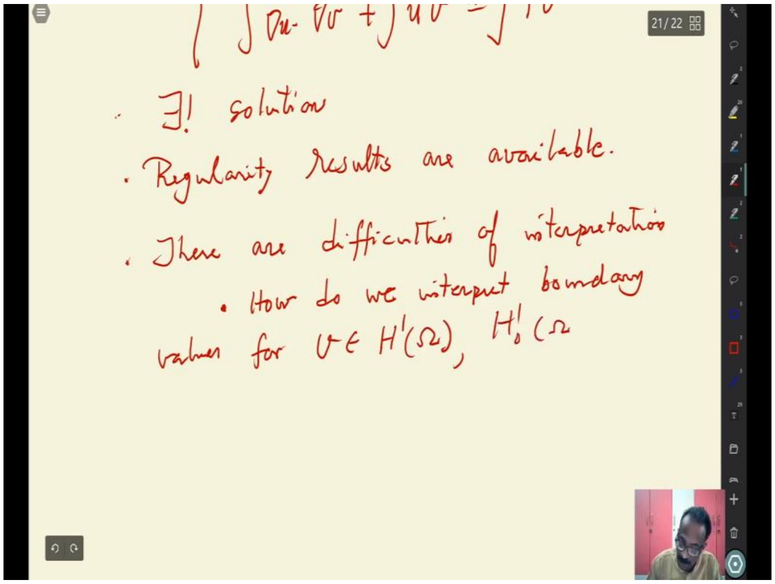

So, we already seen that so there exists for these there exists a unique solution and there are regularity results if you need some regularity of the domain there are regularity results available. If you have more and more smooth in H f you get more and more regularity and if you have a f is very nice you get back your solutions you cannot do all these things regularity results are available there exists a unique solution.

And you have this and this how Hilbert spaces with respect to that you know what we have done this but then it looks everything is easy there are difficulties of interpretation still there are many difficulties one of the difficulties are many things how do we interpret boundary value? This is important in a boundary value problem boundary values for v in though we have written there, there is a problem there is an issue or H 1 naught of omega how do we interpret.

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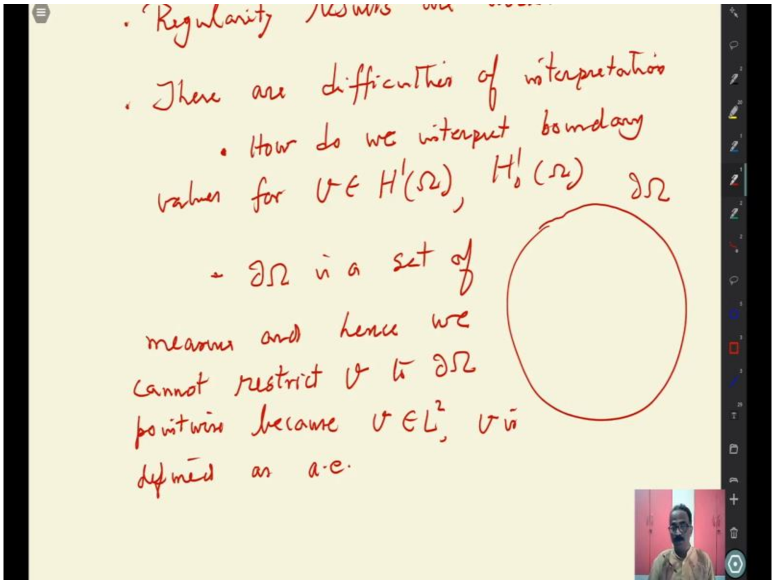

Because these are all integrable functions and the boundary the boundary D omega reason for this is boundary D omega is a set of measures 0 and hence we cannot restrict v to D omega point wise because v is only L 2 or H 1 v is defined and all most defined as all most everywhere and so that is no meaning to the boundary values. So, as such right now so if you start with the function v in L 2 or L 1 the boundary is a set of measure 0. So, whenever so it is a set with that almost anywhere since so if you have continuity up to the boundary you can restrict when there is no continuity you cannot restrict to that. So, how do we interpret the boundary value?

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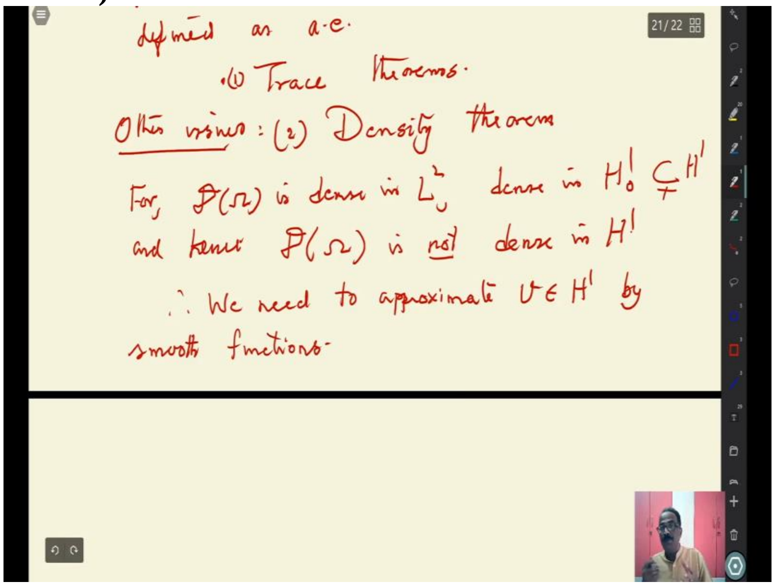

And these are already very serious issues what are called Trace theorem so to understand Trace theorem the other issues which you have to understand other issues other this is one Trace theorem the other issue is density theorem see most of the time even though we work with the non smooth spaces but then you look for that non smooth functions are in approximations for example D omega is dense in L 2 is also dense in with the corresponding norms H 1 naught.

But naturally this is already H 1 which is strictly less than H 1 so and hence D omega is not dense in H 1. So, you need results to approximate so therefore we need to approximate because these are all part of analysis because quite often you are to come to the classical thing to do then you need to approximate v in H 1 by smooth functions not necessarily with the compact support with the compact support you cannot do it. Because you already got all the elements H 1 naught you have a H 1 so need to approximate by smooth functions. So, these are all it is not very easy but you need to have these are these is the calculus you have to develop.

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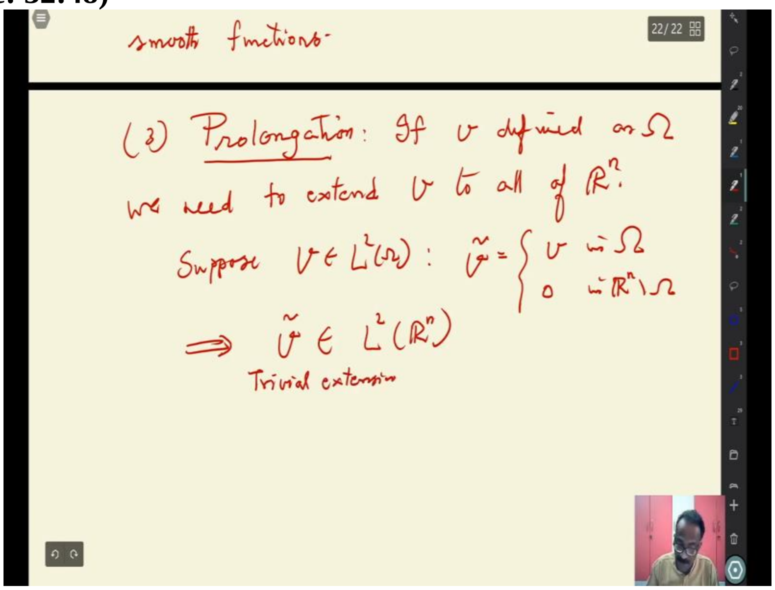

Another serious issue what is called for prolongation result? So, what you have to expect prolongation quite often we need if f is given if v is a defined on omega we need to extend v to all of R n because you in full space you will be able to do use other machinery like Fourier transform etcetera. So, doing analysis in a bounded domain in a domain with the boundary will be difficult.

So, you need to extend the domain to without boundary to full domain R n and then do the analysis and come back. So, you see suppose v is in L 2 of omega and then you define v tilde a trivial extension v in omega and 0 in R n - omega then by the additive property you immediately get the v tilde is in L 2 of R n. So, you have a trivial extension this is called a trivial extension.

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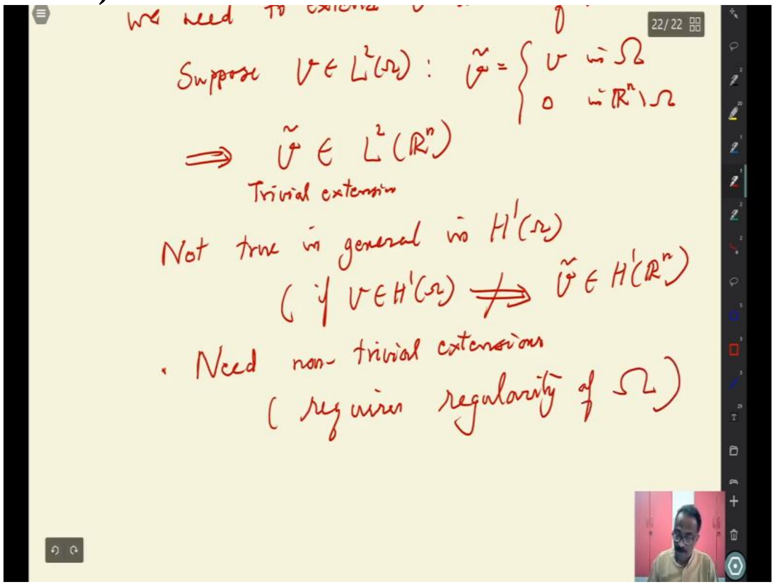

The issue is that this is not true in H 1 not true in generally in H 1 if you have a trivial extension that is but I am saying that if v in H 1 of omega need not imply v tilde is in H 1 of R n. So, you need non trivial extensions does it exists this is not always true which requires regularity in some other form not necessarily the regularity what I defined earlier the need requires regularity of omega in some sense.

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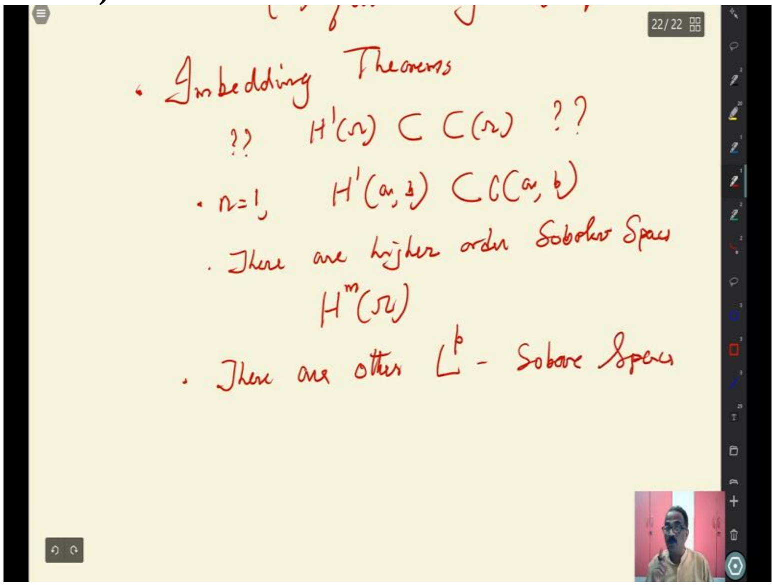

And that is the last point what we call it Imbedding theorems which I do not want to tell much here because you need to introduce Imbedding theorems because you want to prove whether a weak solution is a classical solution. So, the questions what do we are asking? The question is the can H 1 contained in some continuous functions? This is a question. Actually what can prove that higher dimension is true not true?

But when n = 1 you can see that for H 1 of interval a, b is contained in C of a, b so you do not get to C you see a H 1 means you have that function and it is a distribution or derivatives are L 2 in 1 dimension that will imply that function is continuous. So, you have C of a containing C of a you get that such thing need and there are a order of spaces there are higher order Soboler spaces H m of omega there are other L p Sobolev spaces not necessarily with L 2 L p Soboler spaces they are not you know that only L 2 Hilbert space other L p spaces are not Hilbert.

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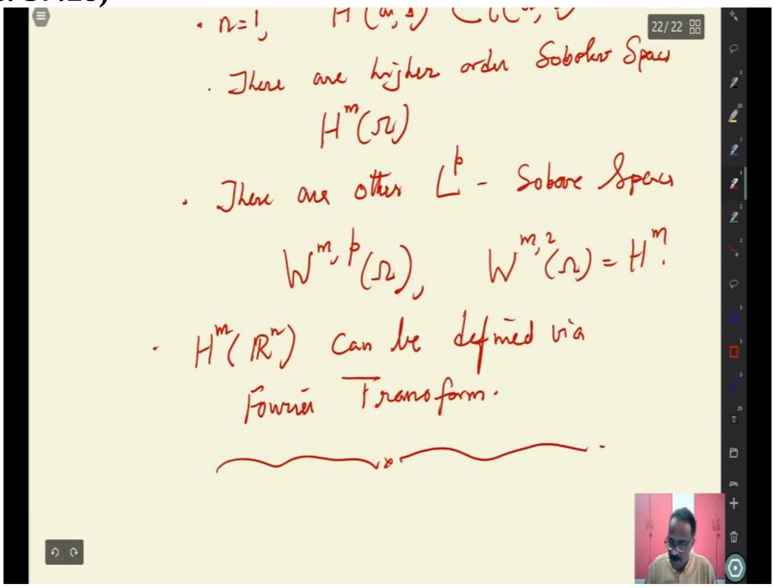

So, you can define higher order space with respect to p for such these are all Sobolev spaces when p = 2 so you have your that is your H 1 and as a final thing it H m mean R n can be defined via Fourier transform. So there are lots of literatures a lot of analysis Fourier. So, I will stop here and thank you very much and this completes this set of lectures. Thank you.