First Course on Partial Differential Equations - II
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Lecture - 04
Weak Solution 4: Distributions

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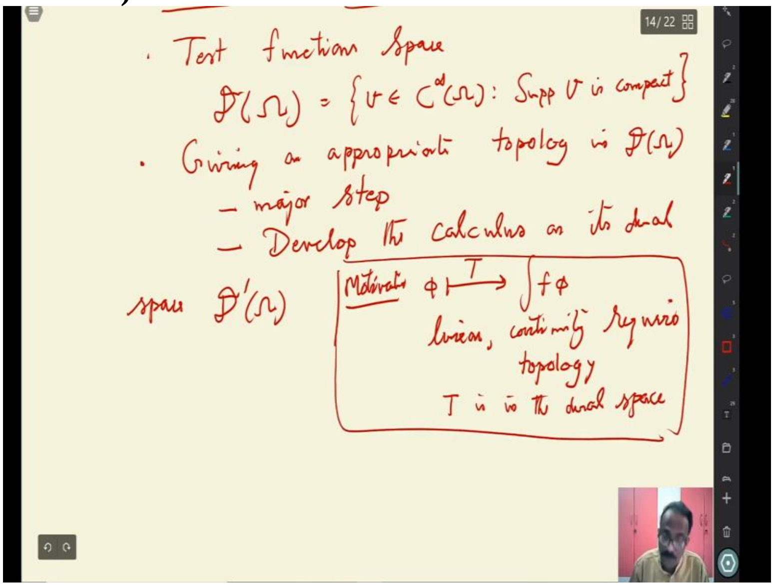

So, we will continue our lectures on distribution theory so let me recall our test functions first. So, test function space D omega is equal to set of all C infinity function D of omega with the support of v is compact that is the whole thing. So, the important development by shorts is giving an appropriate topology that is a major thing giving an appropriate topology in D omega and this is the major step.

And then develop the calculus on it so you need to correct locally convex type topology which is what we are going to it develop calculus on its dual space that is what if you look at the motivation dual space D prime of omega which should be quite bigger. So, if you look at phi going to all these things integral of a f phi the motivation which you have seen this is linear continuity requires topology so that is what we are looking at it.

And a correct topology which we can do it and so this association T actually is in the dual space so we have to understand that calculus on the dual space and that is what we are going to do it here. So, I will give you the topology the more easiest way.

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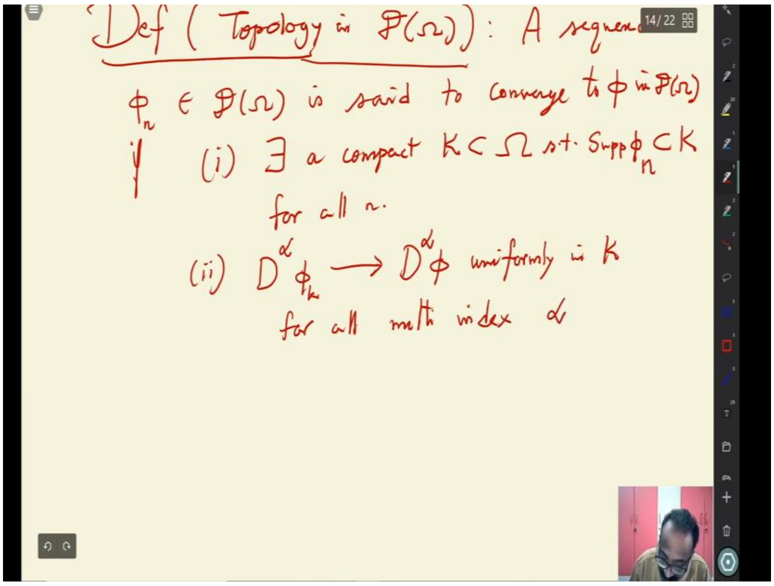

But you can define why are the which I will tell you so let me give you the definition of topology in terms of topology I told you this is the most important thing to begin with topology in D omega because you have to take care of that boundary condition also a sequence I will give in terms of the convergence but you can also give a neighborhood systems and other things inductively topology.

So, there are it is a bit more hard work to exactly understand in terms of the locally convex topology but giving this is enough for your regular working a sequence phi n belongs to D omega is set to converse. So, you have to understand the very clearly this topology set to converge to phi in D omega if there are 2 conditions that is 1 there exists a compact set compact K contained in omega such that support of all the functions is in the same compact set.

Support of phi n is contained in K for all n that is a very it should be a you already know that each phi n has a compact set but that may vary from n to depending on the function. But we want a common compact set so that support of phi n contained in for all n and 2 all its derivative D power alpha it is a multi index. So, D power alpha phi n D power alpha of phi K converges to D power alpha of phi uniformly.

Whether it is you say that uniformly in K or omega does not matter because of the first condition because outside K anyway D power alpha of phi K = 0. Because there is no value it is 0 outside K uniformly. So, whether you mentioned in the book on K or omega because of the condition 1 if that is not there it is that is where the top of the research standard topology in C infinity and that is not the topology which we are taking it.

And is the C infinity of omega topologies given by a family of semi non linear spaces and this is not that topology. But that is used to get the system send everything uniformly in K. And this should happen for all multi index alpha you can construct example.

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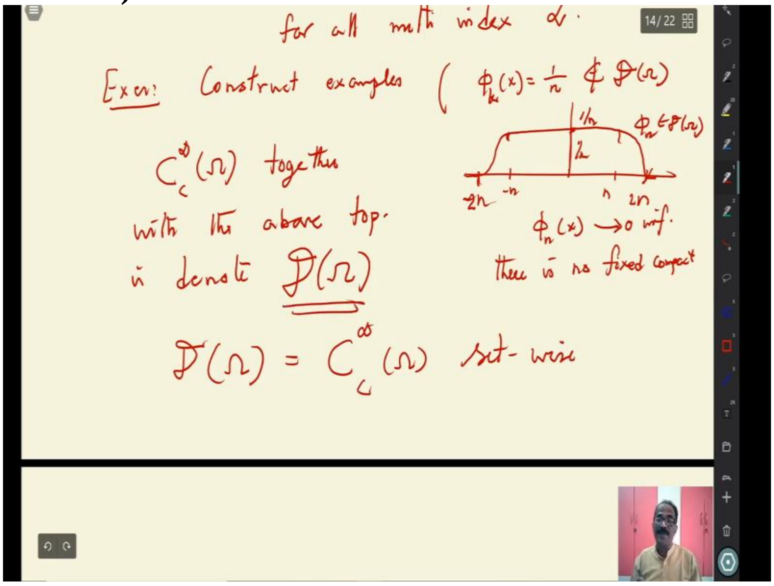

So, as I said I do not have time this is only a kind of took a glimpse of that one. So, your exercise we will study construct examples of course we can do plenty of example but we do not want to do it see if you look at it my phi T of x = 1 over n this is not in D omega constant functions are not there. So, you so not get constant functions are in D omega function. On the other hand if I define something like say minus n to the n.

And if I take functions like say it is going like this and it goes to 1 over n then you can see that but these this sequence such sequences. This is 1 over n this height this phi n belongs to D omega. Because it is vanishes outside but then and this phi n of x converges to 0 and all its 0 phi n of x converges to 0 uniformly you can see but does not have a compact set there is no common compact set there is no fixed compact set.

So, you can get say for example this is up to maybe you can very precisely contract up to minus 2 n this is up to minus n and here you can construct such functions n this is 2 n. So, it will come down that function sequence will converges very nicely because pi n of x and even if you take the derivatives all will can do that one it will nicely come down. But then you do not have a so you make a sequences which converges to C infinity uniformly.

So, the convergence here is not a C infinity convergence. So, this convergence is so that is why the construct examples more and more. So, C c infinity function this is C c infinity C c infinity so whenever together with the above topology is denoted by D omega. So, when we use D omega it is a point is set wise D omega and C c infinities D omega is the same as C c infinity of omega set wise. So, we use this notation when the above topology is given. So, D omega is a very specific space it is not just a set it is a space in this topology.

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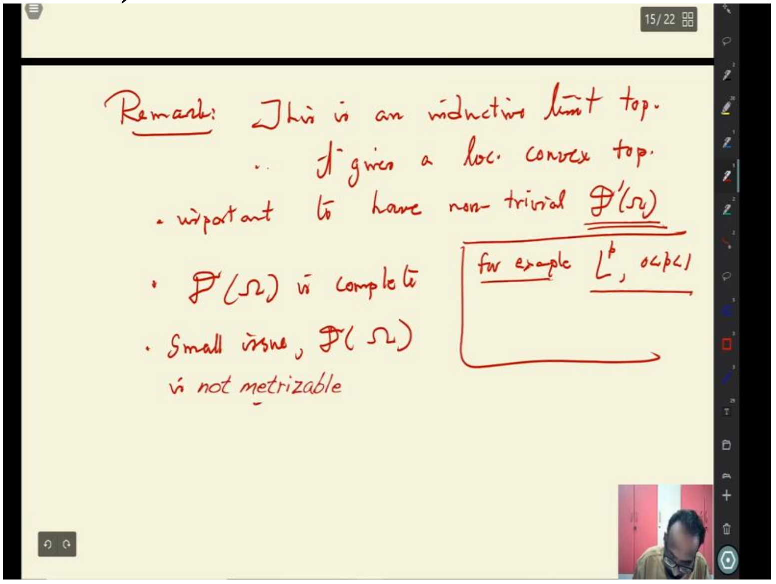

So, some comments or remarks so I will not say a word but if you want you can read this is an inductive limit topology. In C infinity you will get a semi normally topology and thought this topology is also here locally this simple it also gives a locally convex topology and this is important to have a non trivial D omega this is a very if you do not have a locally convex topology is this will be maybe empty.

For example you have you can define L p with the p less than 1 you never you will always study with the L p with p greater than or equal to 1 the reason is that the dual you so not get a good dual. So, you look into such kind of thing so if you so not have proper local convex neighborhoods that dual space maybe empty so and you are looking at your dual space it is a good topology so you have a D prime of omega and D prime is also complete.

So that is also is a complete space the only small issue we will not face a little careful issue D omega is not metrizable. So, this is an important point you should be noting down so these are all some remarks.

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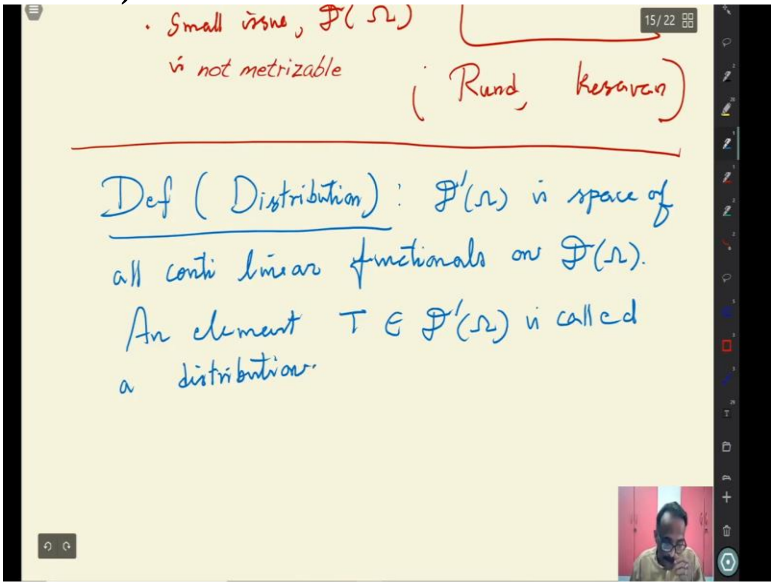

So, if you want to know more about this topology you can look into the book of Rund may be a Kesavan of and it is all given in our reference book. So, you can see of course this topologies are not mentioned in the book because the PDE book is not about the advanced topology. So, this is a small inconvenience or small issue because we cannot take to do this one. So that is a remark with that we are going to define this we have almost defined definition distribution.

D Prime is set of all for continuous linear functionals so let me recall D prime of omega is the space of all continuous linear functionals on D omega and an element T in D prime is called a distribution. So, D prime is a set of all called a distribution so it is very clear about the distribution. So, what is continuity? Continuity you have to understand properly.

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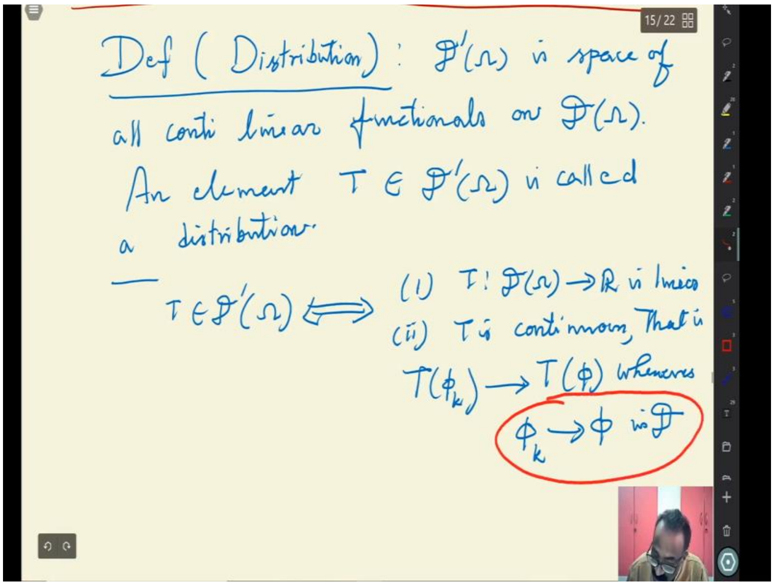

So, there are 2 things 1 is linearity so there are 2 things T belongs to D prime of omega so let me write down the continuity also if and only if 1 T from D omega to R is linear 2 T is continuous that is T of phi K this is what you have to verify all the time T of phi K converges to T phi whenever phi K converges to phi in D. So, you have to see this so the according to that is meeting whenever something converges in D omega.

Then corresponding T phi K converges phi in D then D phi K converges to T phi and that is the natural continuity the convergence in the domain should give you the convergence there. So that is what D so the thing is that you want to be carefully, take this is the convergence D so that is a distribution.

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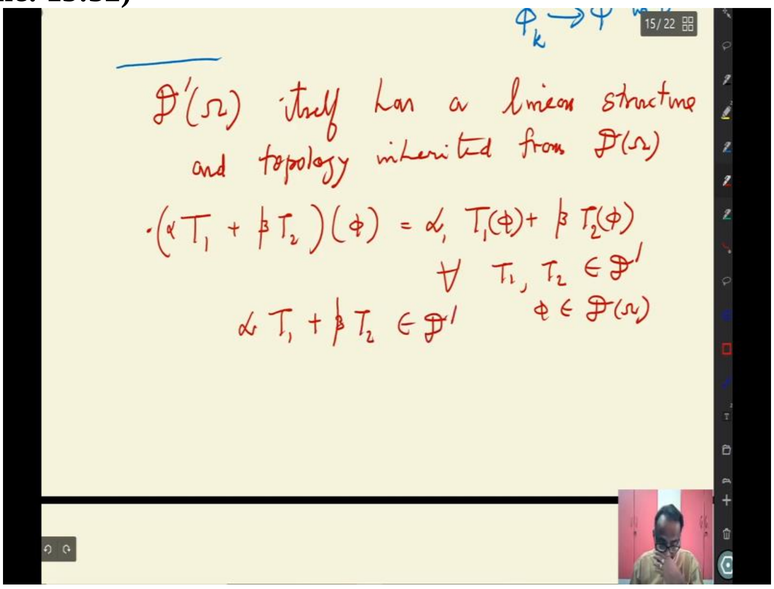

The thing is that D prime itself is a topological space D prime itself has a linear structure and the topology itself has a linear structure and topology is inherited from D. So, whenever you have a dual space you can define a topology using the topologies from D. So, what is the linear structure? That is T if you have 2 elements T 1 so you can define alpha T 1 + beta T 2 acting a phi = alpha T 1 phi + beta T 2 phi this is the definition.

For all T 1, T 2 in D prime anything and of course phi listen D omega always so we can actually show that there is a simple exercise if you want you can show that alpha T 1 + T 2 beta T 2 is D prime that is a linear structure.

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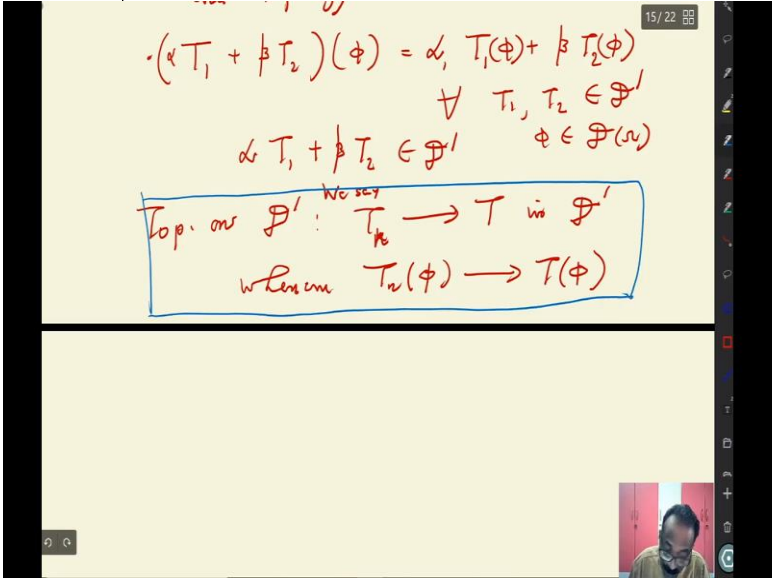

So, what is the topology? Topology on D prime you need all these because for the analysis because now you have to develop the calculus on D prime. T n converges to T in D prime because when you are dealing with PDE you will see no solutions as distributions which are more objects T n converges to T whenever so we say T n converges to T whenever T n of phi that is what it is inherited from phi T n of phi converges to T phi These are real converters because these are all real numbers. So, you have a topology so T prime is also called as a topological space with a convergence like that.

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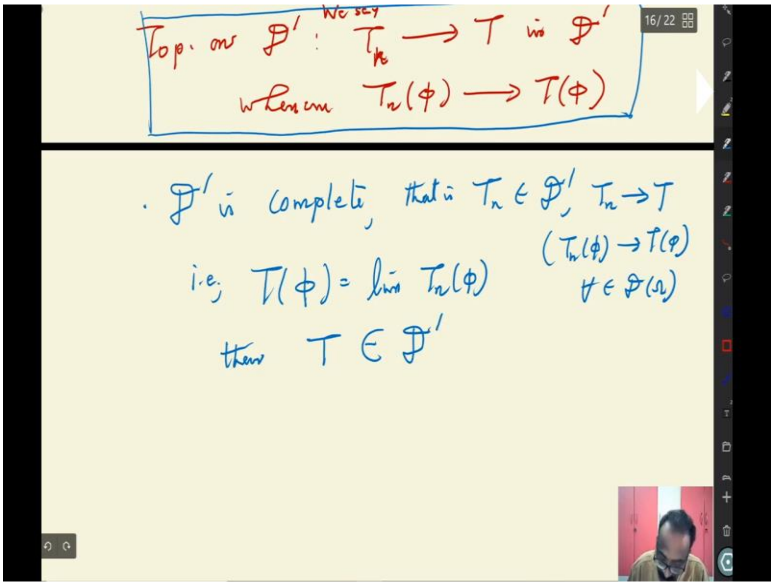

And you can also see that D prime is complete these are all some exercises of calculus that is T n is in D prime T n converges to T that means the T n of phi converges to T phi for all phi in D omega or more generally T n of phi converges to that is what we are going to do is that we are defining actually T phi suppose T n is a sequence suppose in the sense that limit of T n of phi that is the meaning of this in the sense I am talking. Then T is also the limit of a sequence is D prime so that sense I am telling in some sense it is a complete space in this T is distinct. So, you have all these very nice properties here which you can use it.

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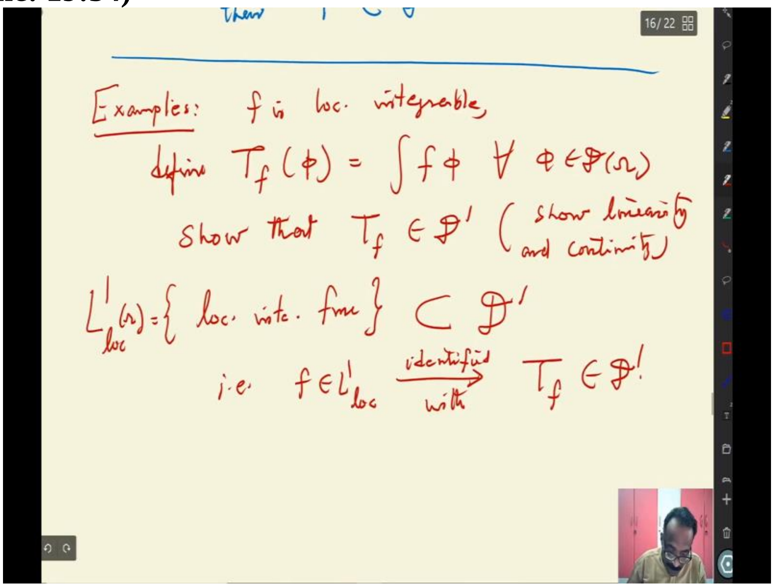

So, let me recall the examples what are these examples recall the previous example. You have your f given by equal to mod x and then what is your the correspondingly you have your may not be example may be little later after this thing. So, let me do the derivative of a distribution so we will do that some more examples f is locally integrable define T f = phi = integral f phi.

So, these are all exercises also f phi for all the in D omega so these examples even if it is easy please work out if you want to get a good understanding of that show that T f is in D prime so you have to show linearity and continuity. So that implies every locally integrable function. So, the classes of all locally integrable functions are the class that may be L 1 log if you want a certain set of all locally integrable functions is contained in can be identified with an element.

In that sense that this is a distribution that is f here in L 1 log identified with. So, always an identification identified with the T of in D prime and later once you are comfortable with that we will not use a separate notation here f. So, when f is an integrable you also treat it f as a distribution it means that for every phi integral of f phi, phi going to integral of f is defined and then it is continuous and linear.

So, we will not use in any distribution theory book or PDE book you will not be using you will not see that T f is written again and again that is only the beginning f is identifying with that. So, whenever something is saying that any element here in a unique fashion identified with an element. Not that 2 same thing will identify and that is only thing always you must you do this identification you need to wait so that is example 1 which you will see now let us go to the second example.

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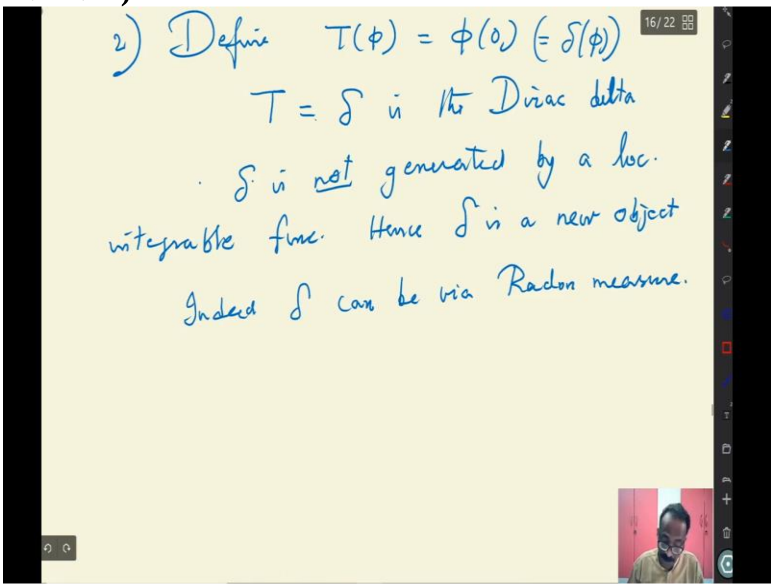

Second example also this example you have seen define T f everything you verify T f of phi = phi 0 which you are got is as a derivative of thing we got this as with a minus sign probably so, T f = phi 0 that is a Dirac delta basically. So, this is a Dirac delta function T f is the Dirac delta function and T f = delta is the Dirac delta. So, you see now a very clear mathematical definition of Dirac delta which was doing.

Because it is a Dirac delta because there many people get confused it is a function that is not a function and you have seen that this cannot be this is T f is locally it is not sorry there is no so, define T phi = phi 0 this is a special notation T = Dirac delta is the Dirac delta. So, this is by definition this is delta phi whenever this is there we use this notation. So, we have seen earlier by exercise delta is not generated by a locally integrable function.

This is a new object by a locally elected hence delta is a new object delta is a new object recovered from here but then there are other ways of understanding Dirac delta indeed delta can be defined via radon measure a measure which is finite on compact sets which you may in a measure theory you would have studied that.

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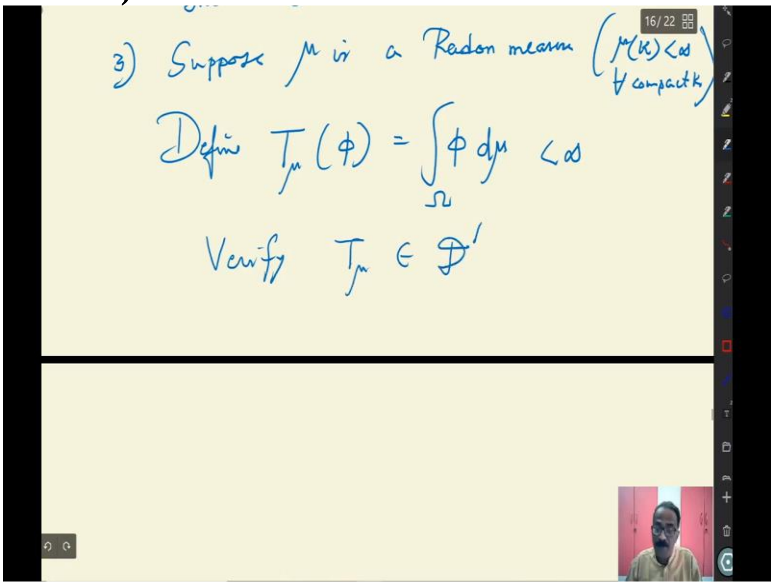

In fact you can define more distribution these are all examples suppose mu is radon measure this is important because radon measure important because mu of K is finite for all compact K in general measures need not be finite of all compact K generally when you define a measure of course layback is a radon measure because it is finite on things but in general the measures need not be finite on complex.

So, set of all Redon measure defined phi can define a distribution and measure is a bigger class than functions for using locally integrable or integrable functions you can define measures. So, but this is a new object every measure so the distribution theory consumes everything. So, it has all kinds of objects which you are familiar with so define your T mu acting at phi you integrate phi d mu is not multiplied mu.

And this is finite because mu is phi has a compact support and hence this integration integral d phi is actually on a compact set because I would say the compact set phi is 0 so these are the things you have to verify and then this is finite mu is finite. So, verify these are all the examples we have to verify T mu is in D prime. So, you see so you have plenty of objects. Now let me give you a newer property other than that minus 1.

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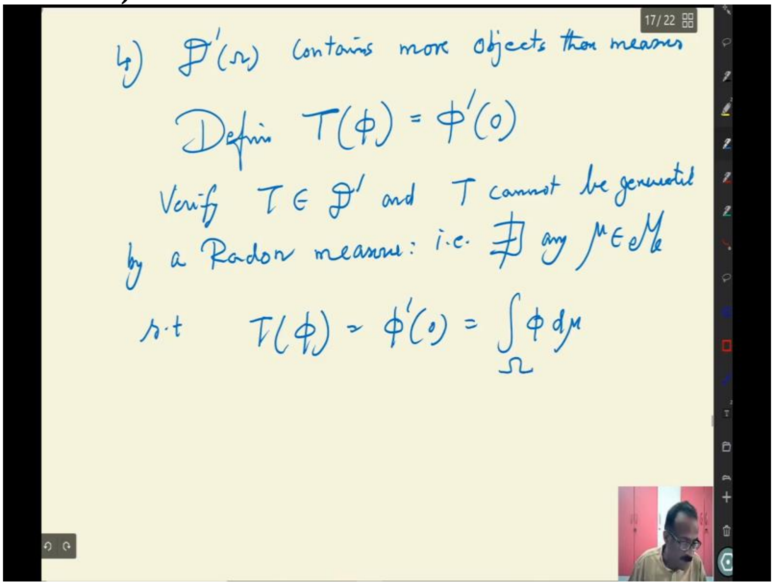

So, actually for D prime contains so every radon measure so you can see that the class of radon measures contained in D prime and D prime omega contains more objects than measures contains more objects then measures. So, here is an another example defined T acting at a phi is not earlier if defined for Dirac delta phi. Now I am defining acting at phi prime of 0 and verify that is easy verify T is in D prime. And the important example T cannot be generated from a radon measure that is that does not exist any mu in M such that you are T phi = phi prime of 0 = integral of phi d mu you cannot appear a term in a measure.

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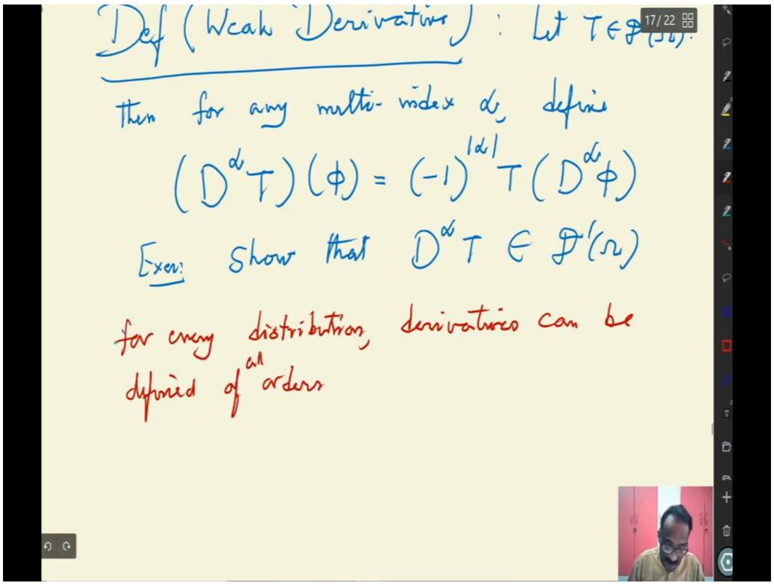

So, you have all kinds of objects and you also produce an object which is something new, new falling this measure theory but these are all objects used by the Dirac. In fact the phi prime is associated with the definition of a derivative. Now we are in before completing these lectures we define what is called a weak derivative and then we will stop here and we will complete this with one more lecture initially we thought of completing in this lecture the whole thing.

But I require one more lecture to do it so, for every distribution. Let T be called a distribution now an element that D prime of omega then you can derivative so you see now I do not assume any smoothing or whatever it is for a very distribution I am going to define a distribution derivative then for any multi index alpha defined the D alpha of T this is by definition D power alpha of T phi = - 1 power mod alpha mod alpha you know that T acting at D.

So, you see how the smoothness of phi used them so your exercise show that D power alpha of T is also we have distribution. So, you see for every distribution so therefore the command is that for every distribution any amount of derivative can be defined is this is an derivatives can be defined for all orders for that is very, very important thing.

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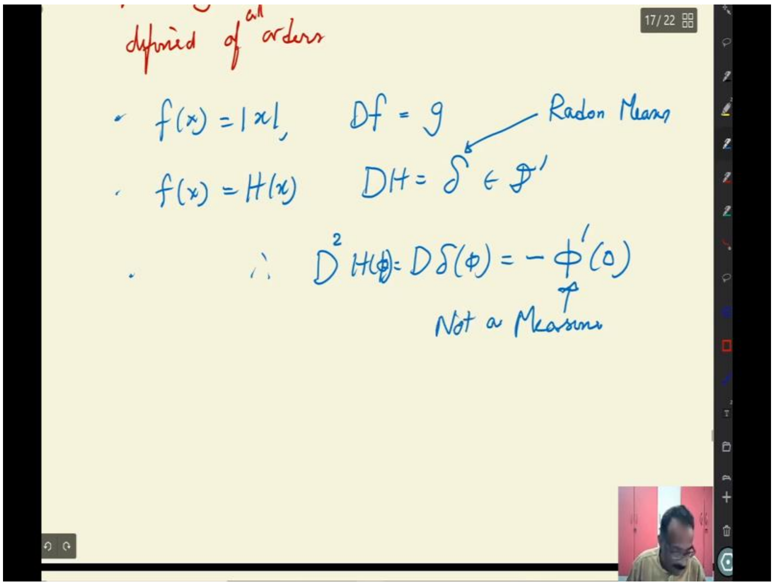

So, with another command let me stop here with whatever you see. So, if f = mod x you already established that your D f is g the same data at the same thing matches with that and when your f x = H x the heavy side function then you are D H = Dirac delta which we are defined and you can also define with the delta is also a distribution. Now so therefore the next is excise is that you can define D square of H that is equal to D of delta.

So, D of delta acting at a phi D H curve H acting at phi D delta you can actually see that it is phi prime of 0 you see so this is actually a radon measure but this is not even a measure these are all called some dipole distribution used to be not a measure.

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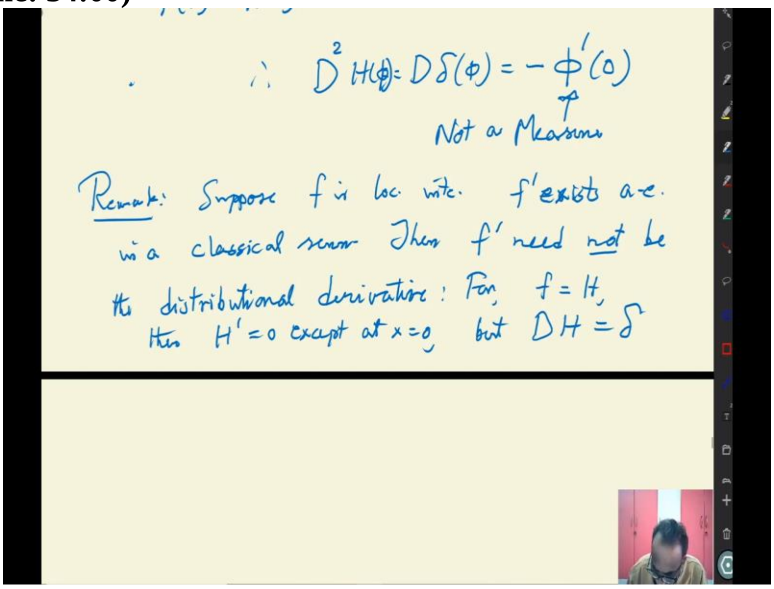

And then there is one more remark which I want to make it suppose f is with this I will stop f is locally integrable and f prime x is almost everywhere in a classical sense. So, be careful when you are defined in a classical sense and all most anywhere sense all most classical sense then f prime need not be your distributional derivative you need additional conditions generally that we do have seen already in a example.

For example take f = H then H prime = 0 except at the origin is almost except at x = 0. But you are distributional derivative D H = delta so you see it is picking up that information. S, the almost a everywhere thing may not pick up your physics. And whenever you have a discontinuity like H and that discontinuity you have something derivative that is what you are picking up the picking up really the distribution derivative.

So, I will stop here. And then in the last class of this thing we will give a for a general PDE a concept of a L 2 we have a proven existence when necessary and sufficient condition for the general mth order equation and L 2 to solution quickly without all the proof and then maybe briefly introduce sobolev spaces to understand our weak formulation we have introduced in the first class of these weak solutions and how to prove that one. Thank you.