First Course on Partial Differential Equations - II
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Lecture – 04
Hamilton Jacobi Equation

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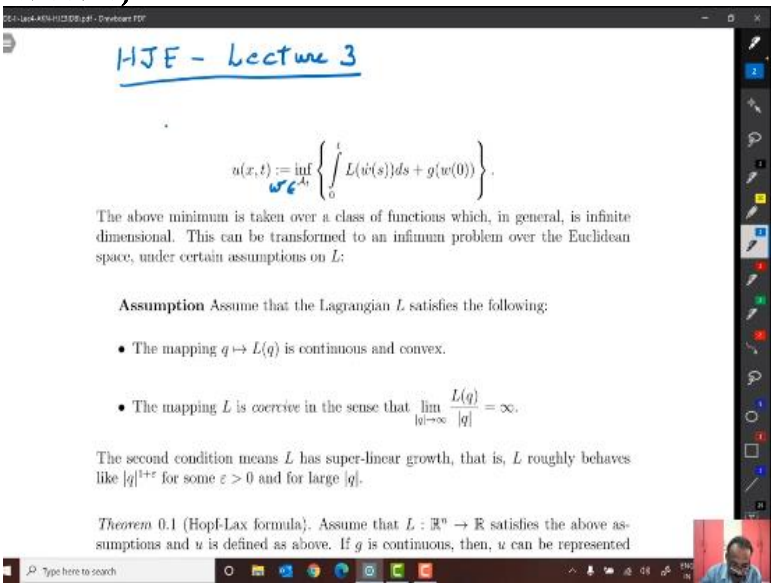

Good morning and welcome back to the lectures on Hamilton Jacobi Equation, this is the third lecture in the section Hamilton Jacobi Equation, let me briefly recall what we have done in the previous class, we have basically derived the equation the Hopf-Lax formula for a minimization problem. Since, we need to do some computations I will have a kind of a typed material here, because doing the computations will take a lot of time.

So, what we have done in the last class, we have see introduced the minimization problem and the u x, t is the minimal value of this functional integral 0 to t L w dot of s w ds w varies overall so w varies in A t. A t we have defined all the C 2 trajectories taking values from 0 and x t = 0 it is y and t = time t it is x. So, you are minimizing and what we have done 2 important assumptions we have made one is about the Lagrangian L. L is called the Lagrangian.

You will see more types of Lagrangian here and we have seen 2 conditions, one is the mapping at least a continuous and convex function, convexity is an important assumption of course, one can consider non convex problems etcetera. So, for this particular thing where you are derived Hopf-Lax formula, we have used this one. The second part is a coercive condition and that means L has a super linear growth, something like a growthlike this. Need not be in this form, but you need a growth which is a bigger than you are mod q.

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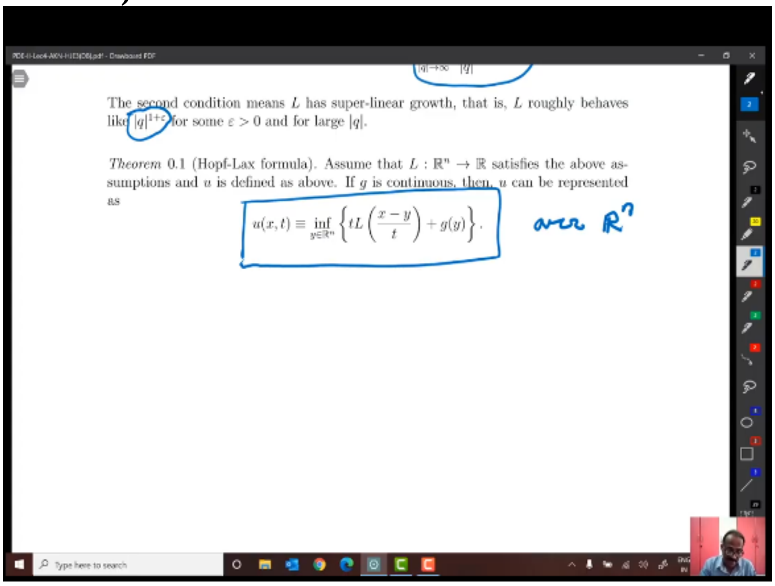

And then we are derived a Hopf-Lax formula that if L is satisfies the above condition, and g is continuous then u satisfies this minimization problem. The difference between this minimization problem, this minimization over trajectories, on the other hand and this minimization over R n. So, that is a different so, we are going to use this Hopf-Lax formula to derive some properties of this thing. Eventually want to show that you satisfy some Hamilton Jacobi equations.

What we are using that we have started with a minimization problem and derive the converted that minimization our trajectories into a minimization over what we call it an Euclidean minimization, that is a kind of finite dimentional minimization. The over trajectories is an infinite dimensional minimization. So, we are going to use this formula in deriving some important properties of you and that is what we are going to do in this lecture.

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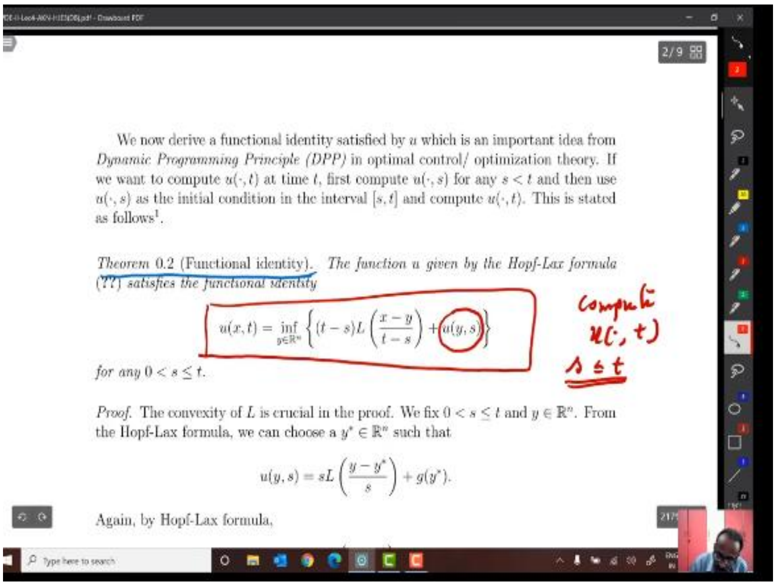

So, we are going to derive a what is called a functional identity, this functional identity that means, u is the solution given by the Hopf-Lax formula as you see here is a formula given by this formula which is the solution to that one, the advantage of this is that because you are taking an infimum over R and then you know that compute its derivative and you will be able to get the explicit formula for your solution.

So, what we are going to do we want to understand certain properties, showing that it satisfies certain Hamilton Jacobi Equation, first we need to introduce what is the Hamiltonian which we will not do it today and maybe a next lecture or a lecture after that. So, what we have seen is that there is a functional identity relation, this is a very, very important formula u x t is equal to, so look at it, so, you are want to compute, so, you basically want to compute u at time t.

So, you take any time less than or equal to t, you can take any s and then compute your minimum value at s t = s with u y s and y can also s. So, you fix y in R n and then you compute u y s. So, accordingly you will have a formula here so, this is x t. This formula is valid for all x and t. so, you compute u y s. So, what you are doing is that you compute, so that is what you are you compute u y s for any s less than t and then use u dot of s as your initial value over the interval s to t.

So, you are looking at is you have the initial value here y so, starting with an initial value at t = 0 and then you are compute this thing, but on the other hand, you compute the minimum value required and add to s time s and then take this as your initial condition and in the interval s to t and then compute u dot t by a minimization. So, for that, so, you understand that cause to from on that interval is precisely if you can connect these 2 in the any dynamical system.

If you want the initial value and if you want to compute something at some other point, you can take any other initial value and then compute from here. And then you have this portion, and that is what he exactly tells. So, this is an important thing of a functional identity, because it is an identity between you about the function at the functional level. And this is very good, this is also called what is called dynamic programming principle in optimal control and optimization theory.

So, this is basically up to and actually an infinitesimal version of this. So, this is a functional identity an infinitesimal fraction of functional identity namely dynamic programming principle will lead to Hamilton Jacobi equations, when the solution is smooth, when the solution is not smooth, how do you interpret that way? That is the thing.

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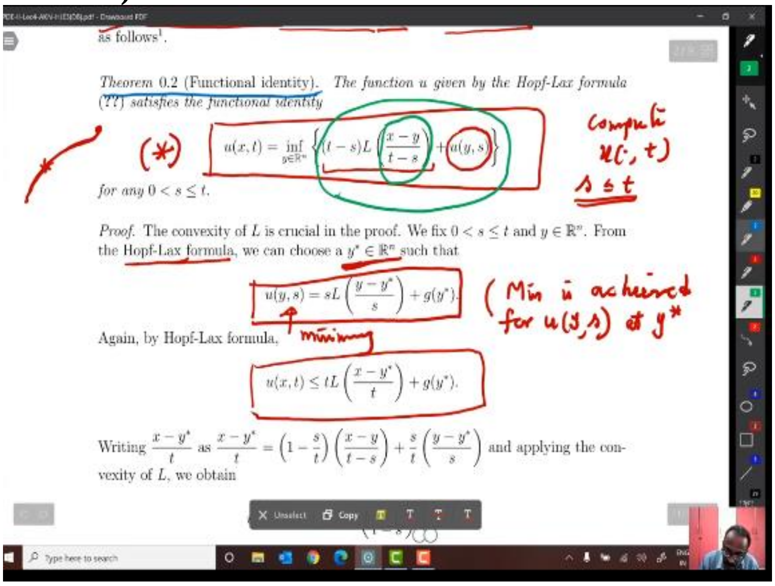

So, let me break the proof is a bit of a technical thing but basically cleverly using the Hopf-Lax formula. So you see so, I have to prove this equality. So I call this equality to be star. So, you want to prove that equalities star. So, I look at it u y s so, I take u y s, u y s also you cannot apply Hopf-Lax formula. So, this is given by a minimum u y s is a given by a minimum that is this y star. So, because u y s is a minimum, you can write here u y s, you use another parameter when you write y s here.

And then the infimum is achieved when you have L is a quadratic growth and all that you can see that that is where use the assumptions, you can see that a minimum is achieved at some point y star and which were going to use it repeatedly. So, you will be able to choose y star in R n were the minimum is achieved, so, your minimum is achieved for u y s. If you write down the Hopf-Lax formula for u y s use the parameter properly and at y star so, the minimum is achieved at y star. So, that is why you get u y s = g y star.

So, so, you have using the precise information where the minimum is attained. So, hence, you apply again the Hopf-Lax formula for u x t, u x t is again is a minimum, you see and since it is an infimum. This is true for any y u x t will be less than or equal to this bracketed quantity for any y in particular u x t will be less than or equal to y star at that point. So, you are here you are only using the inequality, but in the minimal thing here you have an equality.

So, you have used this equality by attained the minimum here, this result will be true for any y in particular for y star. So, you have again applying the Hopf-Lax formula here. Now, you have trick these that trick you are used. So now you want to represent this is the term you want to get it. So you see this is the term you want to get it x - y / t - s. So, I am going to write L is a convex function.

Now I am going to apply the property of convex functions, so you write your x - y star / t and this term you want to come inside. And that is what you are going to you want to prove that one way in equality first to prove it one. So you have to get this term on the right hand side. So, I want this term to come into picture, so I will do that.

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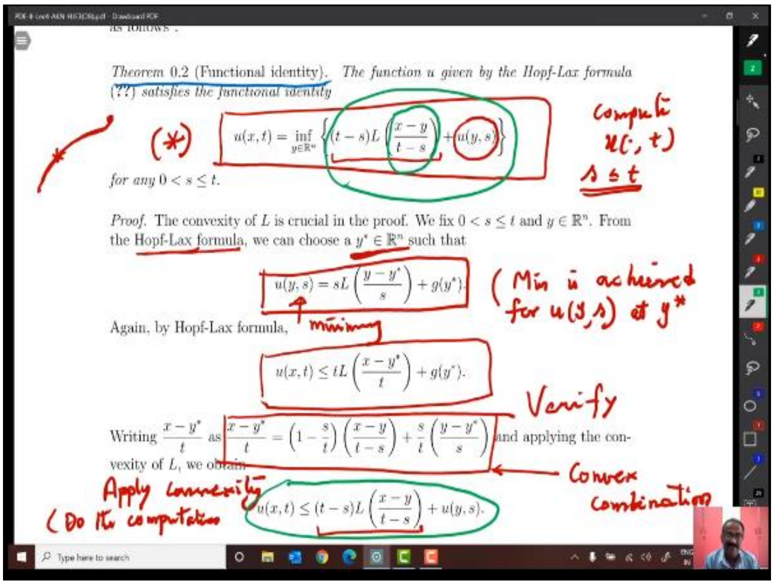

So I will write my x - y star / t. So you can verify. So you verify that I am writing x - y star / t = 1 - s t into x - y / t - s that is easy to check. Because if you take t is here, t - s / t, t - s and t cancel, the s s cancel, you will see it is exactly x – y you can. Now you will see this is a convex combination means 1 - s / t + s / t is 1 these numbers are less than 1, s / t is less than 1. Because it is less than or equal to t, this is also less than or equal to 1.

So it is a convex common, not just a linear combination, this is a convex combination. So you have your convex combination, so I can apply convexity. Apply convexity, what you will get it? You will get a L of x - y star / t is less than equal to 1 - s / t into L of this one plus s / t L of that. So, you can apply convexity and use this one, and you will have a term and then use this term together.

So, I had left some computations here, do the computations. It is an exercise, do the computation. Some computations I have left, which you have to learn it is simple, because I applied L of x -y star / t is less than or equal to 1 - s / t into L of x - y / t. And that is this term, and then s / t into L of that, but s / t and 1 t will cancel in your u x t formula, this t will get cancelled, and you get exactly this one and that is u x t.

So you take the minimum or that, so you get this equality now, do the computations and you get the just to one line computation, applying L on this one and use these 2 formulas, which we described above, and then you get this one. So you have this formula. And then this is true for any y, and you have one way the equality. So you go to one way inequality, now you have to get that reverse inequality.

So to get the other way, the quality you use the took which we have used, what is the trick we have used 2 steps we are used to here. One we have a exactly computed equality where the minimum is achieved for u y s, and then just in infimum definition is used to for u x t, to get the other way the quality, you use the minimum for u x t. So that is what you are doing it.

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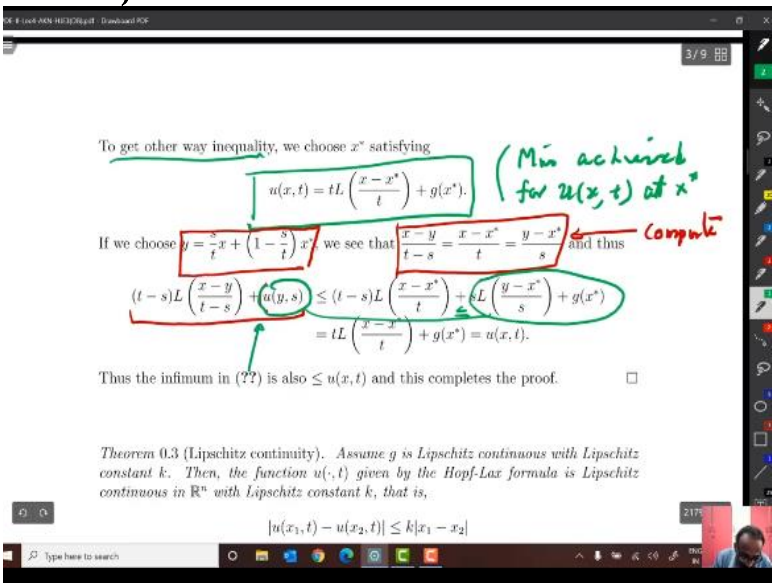

So you applying the Hopf-Lax formula by minimum achieved, so minimum achieved, so here, you are using it for achieved for u x t at x star. So that is what applying so, you have your equality correctly. Again we use similar trick, I want to get other terms, I use my y little more cleverly here, I will write y as a convex combination in this form, I write a choose your y as a convex combination of things, once you use it, you can get this equality formula compute it can see that all these 3 are equal.

And this is what you want to computed because this has to come to the left hand side. So, you see, if you look at here, this is right hand side so you have to come to the left hand side. So you come back to the left hand side and you have your L of x - y / t – s look at here, here. And I can right here and then x - y / t is equal to I can apply this here t - s into L of x - y / t is same as. So I replace correct so, there is equality here.

So, there is no I am not applying the convexity property here x - y / t is same as x - x star / t and you write u y s now, your u y s already there u y s is less than or equal to this 1. So, for u y s this quantity this again by infimum this quantity is less than or equal to this quantity, this quantity is less than because you apply the minimum at x star. So, exactly earlier so basically you are reversing the role of u y s and u x t and writing down that properly.

So, once you do that one, you will have y - x star / s is the same as x - x star / t, so s and this cancel, you get t into L of x - x star / t and you are g x t and that is nothing but your u x t here. So, you get so you have this formula less than or equal to this one, u x t, now you take infimum over all y, that is what you want to show it. So, you want this to be on that side, this is true for any y. So, you take infimum over y in here, infimum in this case.

So infimum here that is the question infimum here you will get is also less than or equal to u x t and this completes the proof of your important functional identity which you are getting it. Please understand it physically. For any s, you look at it, minimal value and start that so, you have a trajectory basically. So look at the minimum value at time t = s basically and take that as your initial value and proceed your compute your cost basically, and take over all such possibilities of y.

And then you will get back your u x t some sort of a comparison between what you will see in the ODE's that is what I have marked compare these with the Semi Group property enjoyed by a ODE system, this property is part of any abstract dynamical system. So, you proved the Hopf-Lax formula that u the minimal value given by the Hopf-Lax formula satisfies a dynamic property I mean property.

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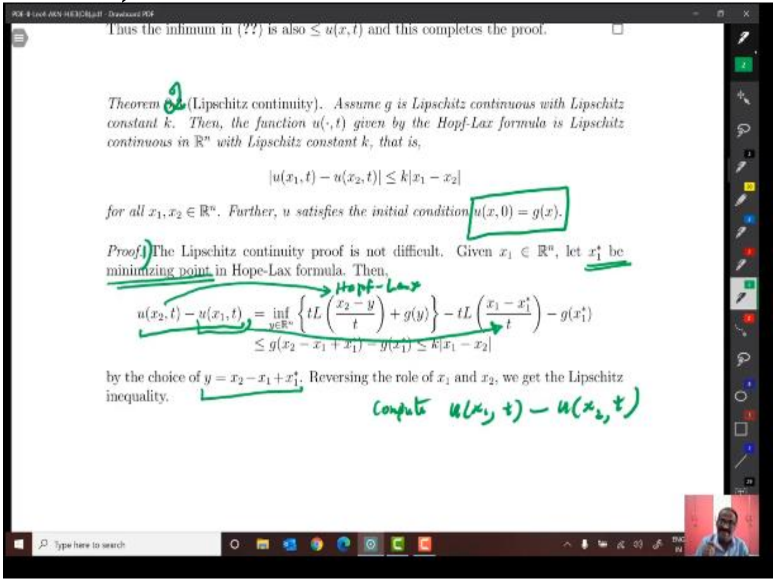

So, we are now going to derive so, this is one thing, so, you have your theorem to basically what is that is about the Lipschitz continuity, but that is what we are looking for either Lipschitz continuity and then not the differentiability. So, the solution given by you eventually happened to be the solution need not be differentiable, but it will be more than continuity, in fact it is informally continuous in the variable x.

So, you are having you are assuming if g is Lipschitz continuity with the Lipschitz constant k I hope all of you know what the meaning of Lipschitz continuity.Tthen otherwise, you verify our look into our earlier lectures or any books you can refer to for that matter. Lipschitz continuous g is continuous with Lipschitz constant k, then the function u dot of t given by the formula is Lipschitz continuous in R with constant k in that case independent of t you see. So, u x 1 - u x 2 t is less than or equal to k this is true for all x 1 and x 2.

And further you also satisfy this, we have not described any PDE problem. But it is satisfies an initial condition and this is going to be part of your PDE later, when you introduce your Hamilton Jacobi Equations. So, you see so, there are 2 steps. So, the Lipschitz continuity proof is not difficult, but this proof of this initial condition deriving that u x 0 = g of x inverse little extra work the Lipschitz will do it. So, to prove this again is not a difficult so, you look at it this is what you want to compute you both you have to compete u x t – u x 1 t because you want to consider the modular.

So, you have to compute u x 2 t – u x 1 t and the other way also u x 1 t – u x 2 t so, you see you look at here u x 2 is given by this Hopf-Lax formula. So I am just writing infimum but, you can write infimum here, but I will not be because of the minus infimum cannot be estimated, infimum of something is less than or equal to for a (())(19:59). But minus of infimum, only it will reverses thing, but here for u x 1 t.

I computed the minimal value which achieved, so they corresponding to x 1 there will be an x 1 star t for which u x 1 t is achieved. So, x 1 star t it is already mentioned here, so, I do not have to mention so, x 1 star is the minimizing point in the Hopf-Lax formula. So, this is exactly equal is not that I am removing the infimum and looking for. And for this point, now, I will cleverly choose my y I want this I do not want my L here. So, to L here, I choose my y like this.

If I choose my y here, I will get excitedly x 2 will get cancelled here I will get x 1 - x 1 star. So, t and this term and this term will get cancelled. So, I will get this will be less than or equal to so, I this is infimum this is true for any y because now we can use because this is a positive side. So, I can choose for the infimum anything that because of these minus in the second term I have a problem. So, I choose y = x 2 - x 1 star - gx 1 and g is Lipschitz continuous.

So, there will be less than or equal to k into this minus this and x 1 star and x 2 star cancel so will get so, u x 2 - u x 1 and getting. Now reverse the role of x 1 and x 2 and the reverse role of means that means you compute u x 1 t - u x 2 t. And in this case, you choose u x 1, you write it as an infimum and for u x 2 you choose the achieving point that you look for the next 2 star so that u x 2 t is a exactly achieved at that point. So that gives you your Lipschitz continuity and I want to check my initial condition.

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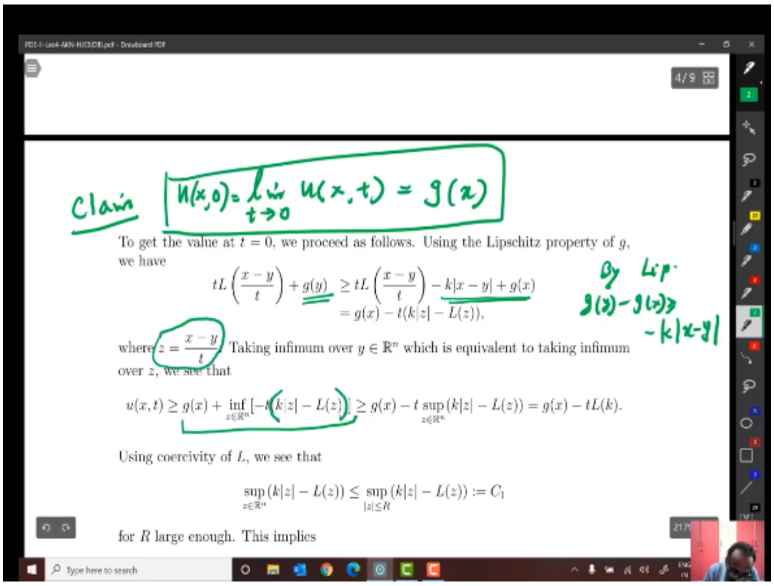

So, I will go to the initial condition, what is the meaning of initial condition, I want to u x 0 is interpreted as a limit t tends to 0 u x t. I want to prove this is equal to g of x this is what I want to prove it claim this the claim I want it, so you use the Lipchitz property. So it is not difficult, but it is a bit of a technical thing. So you look at here so this term is this one. So I use this one and this one, you apply Lipchitz contiguity by Lipchitz g y - gx.

I know that modulus of g y minus gx is less than or equal to k into mod x - y but then, if I want a lower equality, you put a side, this is true mod x - y then greater than equal to minus k of mod x - y that is true you see, so g y - gx and that is what I put it here. So g y put it here, gx n - k, I have taken here. So this is just an application of the Lipchitz continuity of that one. So that is gx so I take t and put z = x - y / t, so that is a notation given here.

Now, you want to understand, I want to minimize this with respect to y, but x and t are fixed so the minimizing LHS is equivalent to minimizing z because x and t are not changing. So when you want to minimize something, whether it is if you are not convinced just verified that the minimum over this is minimums minimum over z, so it is enough to minimize over z. So once you do a minimizing over z. so, minimizing our z or y does it matter equal to minimizing.

So if I minimize here with respect to y, I get my gx and if I write the minimization, I get this term. So this is greater than or equal to u x t greater than or equal to gx into I minimizing whether you minimize our y or minimize our z is same because x and t are fixed, you know, I am not varying that. So minimizing this you get it this one, and this is gx less than or equal to when I take I want to take this minus outside, so, when I take a minus outside, so, there is a big bracket here careful here. When I take minus outside it will be coming supremum, this is wrong, it is nothing here.

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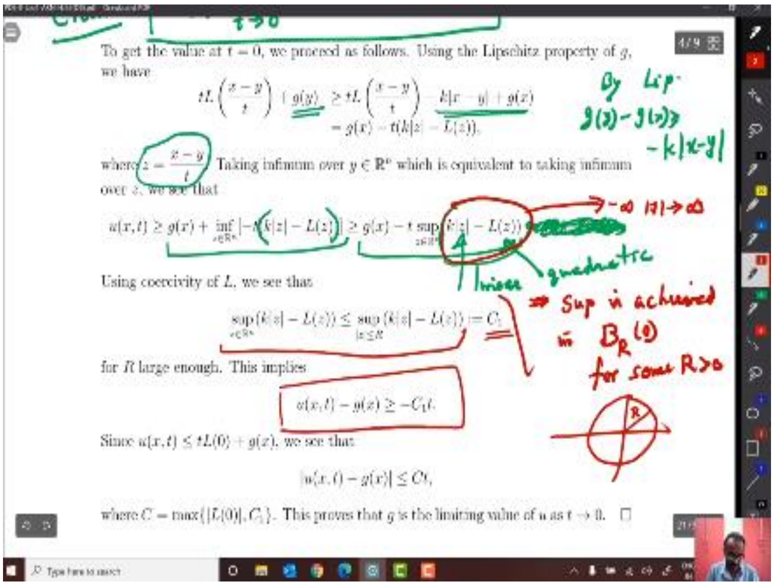

Now here is some more trick which you want to understand, you have to understand the coercivity of L, what does the coercivity I will tell you, so, this is linear. So, you see the linear and this will quadratic, quadratic. So, as mod z is large, this will go to basically minus infinity. So, you have to understand that little bit of an analysis here. So, this is a guy going to because of this quadratic, this is go to infinity.

So, minus that, that will be going to actually minus infinity as mod z tends to infinity, what is the implication of that the supremum cannot be achieved when mod z is larger. So, that means that supremum is achieved this implication is that supremum is achieved in a ball of radius R, this implies that this one achieved in B R 0 for some R positive. So, if you are trying to look for the supremum maybe in a ball of radius R and this R is fixed now.

So, what I am saying is that this supremum over R n is less than or equal to supremum because after R it will be very, very small so the supremum and I call now the supremum is one closed to set. So, the supremum is achieved by call this to be C 1. Therefore, you will see there is a minus sign here the supremum of this one is less than or equal to a positive constant. So, therefore, u x t - gx will be greater than equal to minus C 1 t because, it is supremum is less than or equal to since the minus sign is there, so, you get this inequality.

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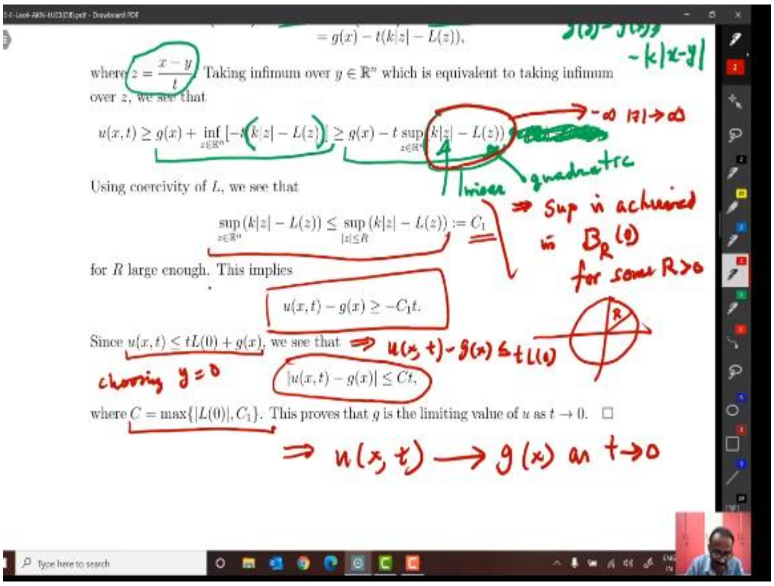

And, of course u x t is always this is true for any number, this is from the infimum. In particular, when x = y you get L0, we go t into infimum of t into L of x - y / t once you choose y you get L 0 here by choosing y = 0 that means, you get this implies you get u x t - gx will be less than or equal to t into L0, which is another number t into L0. So, you have u x t - gx is greater than or equal to minus u t and then u x t – gx is less than or equal to t into L0.

So, if I choose my t is equal to this one and you will see that your mod x is this one because u x t – gx lies between these 2 numbers. Minus C 1 t and t L0 which is another number. So, if you choose C you get 1 this proves that as t tends to 0 u x t. So, this implies u x t tends to gx as t tends to 0. So, you see, so basically what do we have, so, before concluding this part and continuing little more and we will introduce something more general. So, you have we proved the functional identity, we proved Lipschitz continuity and we proved the u x t.

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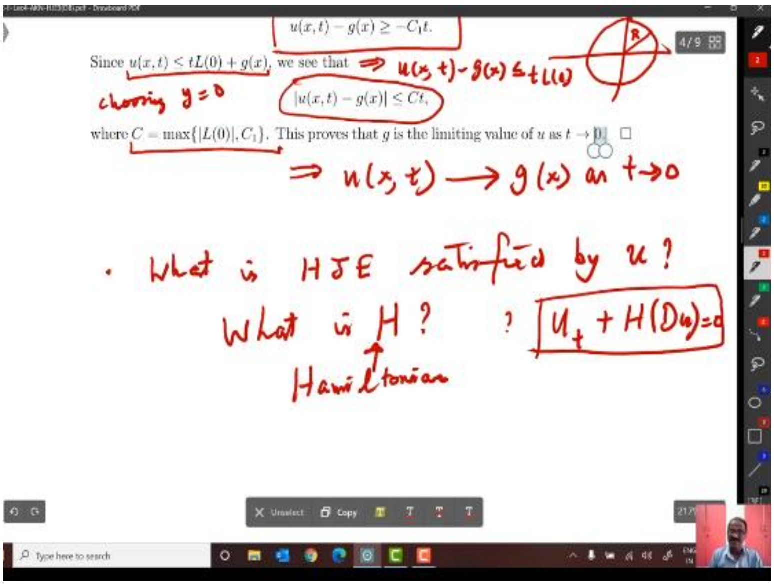

So, the next we actually want to see that what is the HJE Hamilton Jacobi equation satisfied by u, for that, what is it H? First of all what is H? So, what is this Hamiltonian is called Hamiltonian? We will come back to this later, maybe the notes and couple of I will do something more here today. So, and more examples of this kind of minimization problem before coming to H and H is called basically the Hamiltonian and you eventually prove that this is what you we prove? You will prove u t + H of Du =0.

And these we will do it next lecture or a couple of lectures later because we need to introduce first of all, what is H. So, before completing today, what I am going to want to actually complete one more section, but that is not possible.

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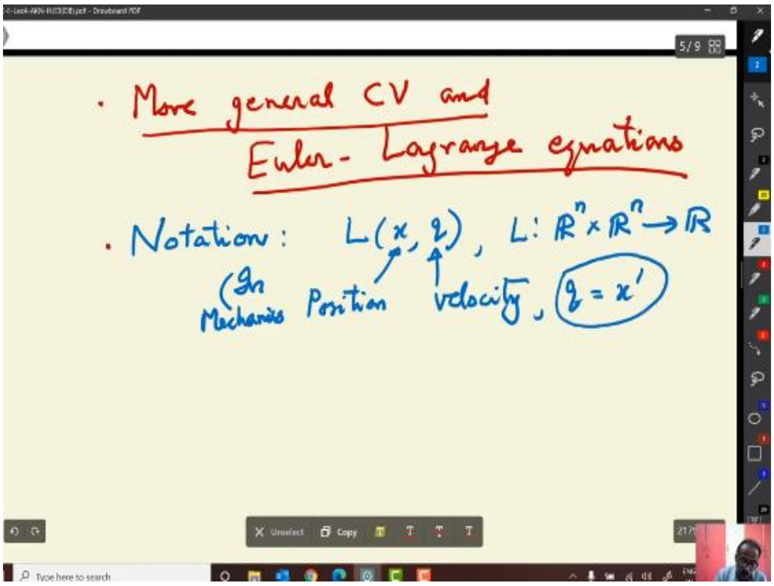

I am going to study a little more general so, what I am going to do the more general, we have studied one calculation problem. So, we are going to study more general CV and Euler Lagrange equations. So, I will do the setup today like Lagrange equation and then we will give you some examples and after that we will come back to Hamilton Jacobi Equation late. So, we are going to consider a little more general calculus of variation problem. So, let me have my notations first here.

So, let me set up the notation today and probably we will do tomorrow the other thing, so, notation is we are going to consider variable L x q that means, L is a mapping I used this is R n x is in R n to R n. So, to see that in these are all more general than the classical mechanics and Newton's law of motion which you will see here. So, in that terminology x is always see the position which you will see need not be in general in mechanics, it going to be like that will give the example precisely. And more general a position and this always represent the velocity. So, eventually q is going to be something like x prime.

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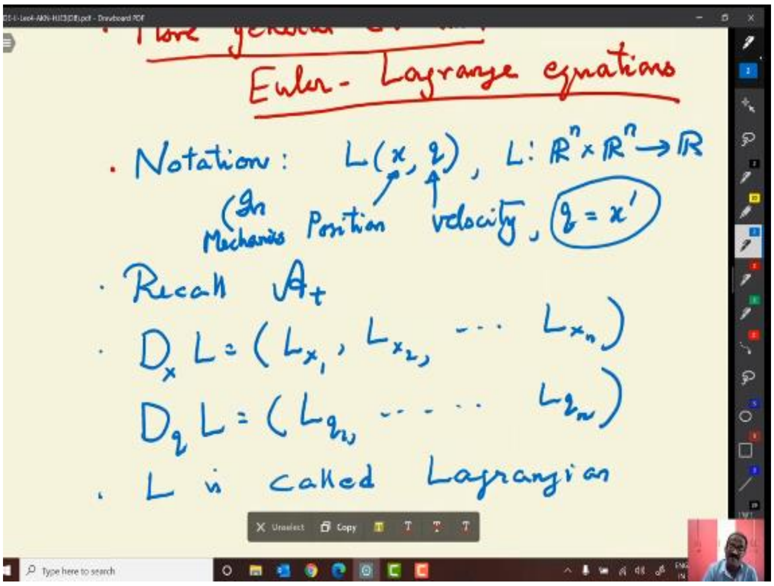

But right now it is Ellis general function which will have an application and recall A t already defined the set of all admissible class A t which you will set off all t is continuously differentiable functions satisfying at t = 0 y and t it is x set of all trajectories here. So, I will also put a notation when I have a this is a n vector So, I will have my D x of L if they do all the derivative with respect to x of L and the derivative with respect to L x 2, so, this is a vector L x n.

Similarly, I will have D q of L is the next n vectors q1 etcetera L q n. So, you have these notations to be used to be a continuously use this notation and L is called Lagrangian. In the earlier problem which we have described is more special than these in which the dependence of x was not there, there was only dependence of q. So, what is your minimization problem?

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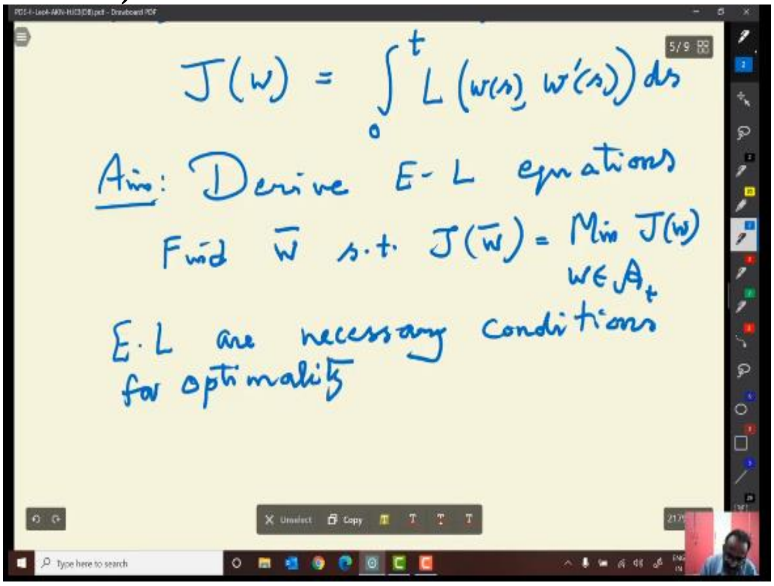

So, you have your minimization Jw is equal to integral o to t L of w of s w prime of s I told you this is a thing like the velocity into ds. So, in this generality, I do not write the g function, you can include that cost also terminal cost, the initial cost or terminal cost, but let me concentrate only on this part. And that is what more important. The other part you can add in there is nothing wrong in adding that one.

So, as you know probably we want to derive the aim is to derive Euler Lagrange equations. So, what is the problem? So, the problem is find w bar such that j of w bar is equal to minimum of J of w w over A t. So, we are going to do that minimum thing. So, these Euler Lagrange equations are necessary conditions for optimality.

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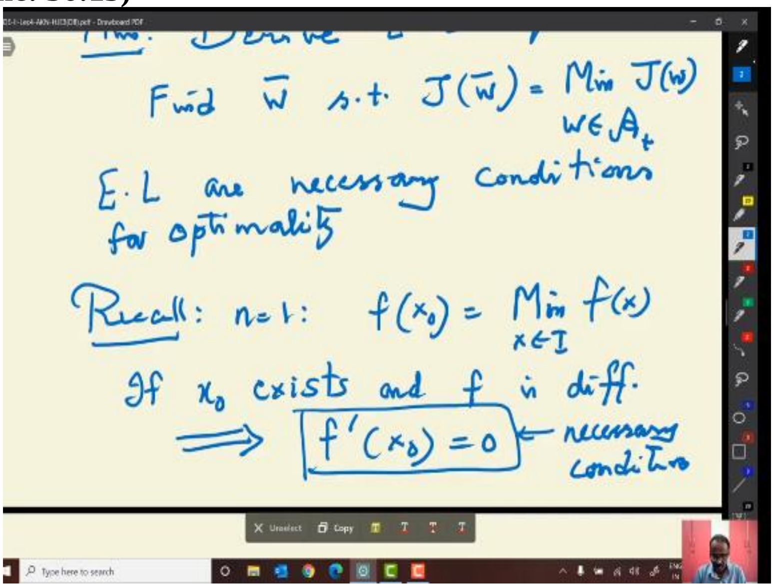

So, let me recall suppose, you want to have a one dimensional situation n = 1 I want to find f of x naught say minimum of f of x, x in some set in something like that, then you that if x naught is exist and f is differentiable that implies f prime of x naught = 0. So, you see, so, this is a necessary condition is not a sufficient condition you know that necessary condition. Now what we are doing is that we are doing a minimization problem not on finite dimension domains, we are doing a minimization problem in an infinite dimensional setup.

And then what do you mean by? Yes we already got a w bar, but then you are to understand the differentiability of J and there are concepts of modal concepts like differentiability what are called fresher derivatives and total derivative in infinite dimension setup. And we are going to derive a necessary condition if w bar is optimal, which minimizes thing we are going to derive what we call it the Euler Lagrange equations, which is going to be a system of second order ODE’s. So, we will stop here and continue in the next class and then we will give some set of examples. Thank you.