First Course on Partial Differential Equations - II
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Lecture - 39
Weak Solutions 3: Weak Distributions and Distribution Theory

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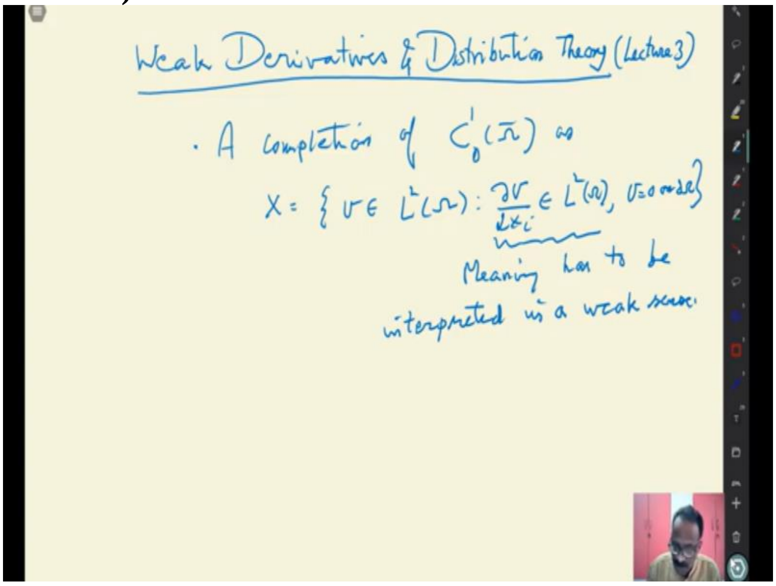

Good morning, in the last couple of lectures, we have tried to see the solutions in a very generalized set up in a Hilbert space thing and we also seen examples of minimization problem and leading to a formulation weaker than the Laplace equation. So, Laplace equation in a classical sense we have studied throughout this course, but, quite often the physical solutions are not captured by the PDE rather another formulation which is called a weak formulation.

In the process of explaining that what we have seen we have introduced a completion of C 1 0 of omega bar is equal to completion as x is equal to a leading to a v in L 2 of omega such that your dv / dx i is also in L 2 of omega with the v = 0 on d omega. So, the issue which we have not understood this space properly, what is the meaning of this? Meaning has to be interpreted in a weaker sense. So, as I said earlier this was the beginning or end of the 19th century and the beginning of the 20th century.

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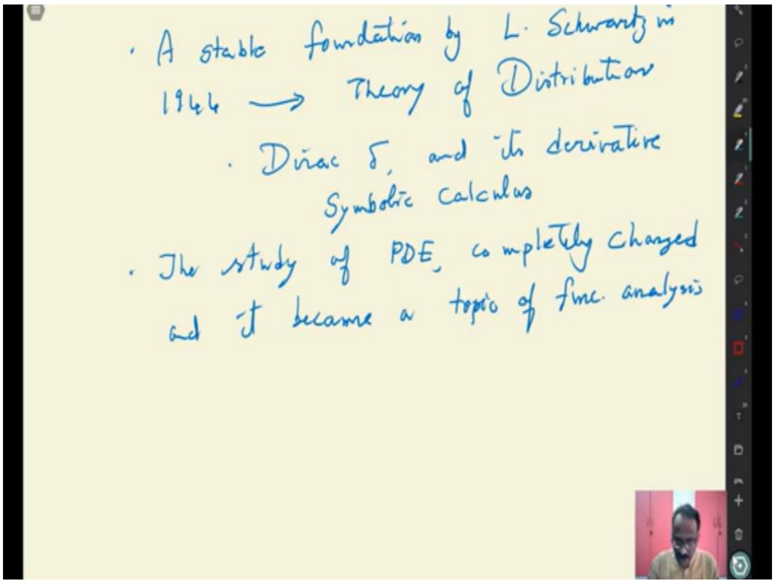

Finally a stable foundation was given, mathematically by Lauren Schwartz in 1940s, maybe in 1944 and introduced theory of distributions. So, the earlier prior to this, the distribution not only helped in understanding PDEs in a weaker sense, it also helped in understanding other physics, like the Dirac delta and its derivative. Dirac introduced in the 1920s and 30s. Because these are all some objects and Dirac was using the symbolic calculus.

So, its unified the math, physics and other things doing in a very symbolic way without having a proper mathematical foundation was justified by the theory of distributions. And after 1940s in the last 70 years or so, the theory of the PDE has completed the study of PDE has completely changed and it became a topic of functional analysis. So, together with the abstract development of functional analysis helped to study PDE together with this theory.

So, what we are going to see in a couple of lectures or something, a glimpse of distribution theory, not that because that needs if you want to understand it properly, you need at least a one semester full course and that is what I said it will be an advanced PDE if there is a new course on that one, but we try to give these processes.

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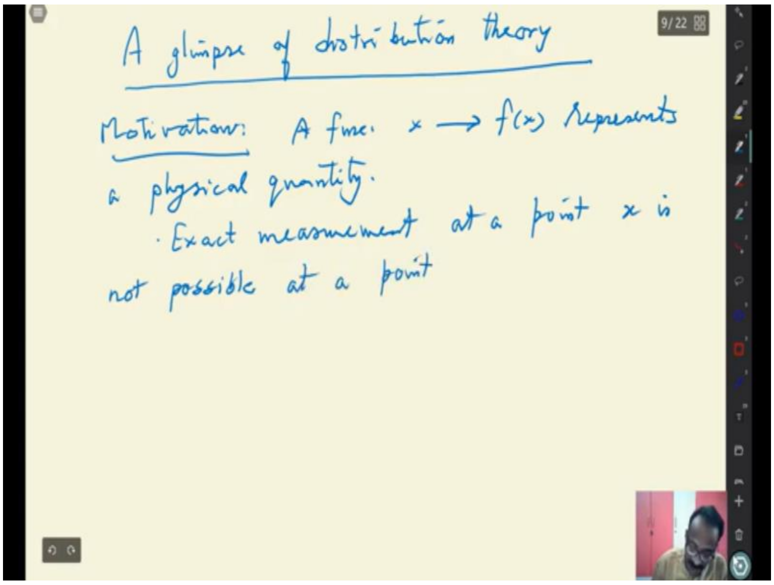

So, we want to give a glimpse of distribution theory so, we want to try to understand. So, let me motivate you first, quite often a function x going to f of x represent a physical quantity. What is interesting is that you cannot measure x is a mathematical concept. A point is a mathematical concept and no physical quantity can be exactly measured. Exact measurement of effects is not possible at a point x is not possible. So, what we get basically is some averages around a point.

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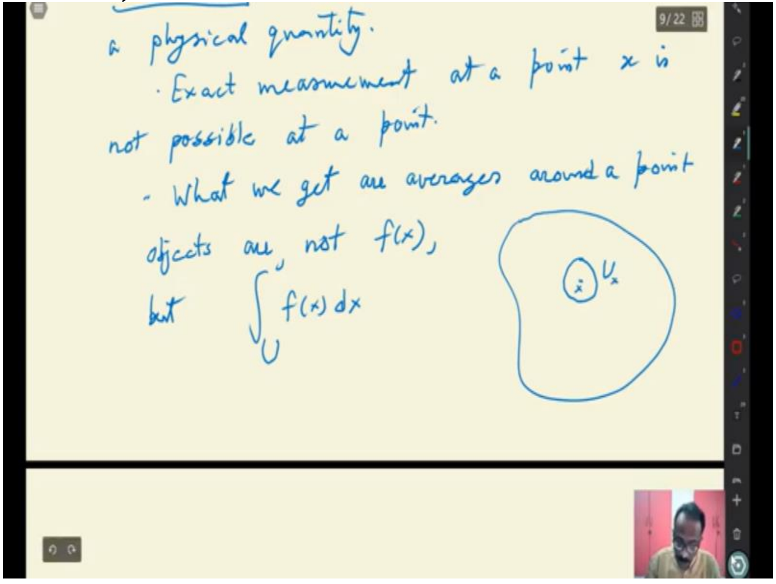

That is all you can measure say suppose you are measuring the temperature at a point with an instrument you will not be measuring exactly at the point you basically get an average around that. So, if you have a domain here, so basically if you have a point here, so if you look at the neighbourhood U this is x. So, the objects available are basically objects are not f of x actual physically, but integral of f of x dx over the inter neighbourhood.

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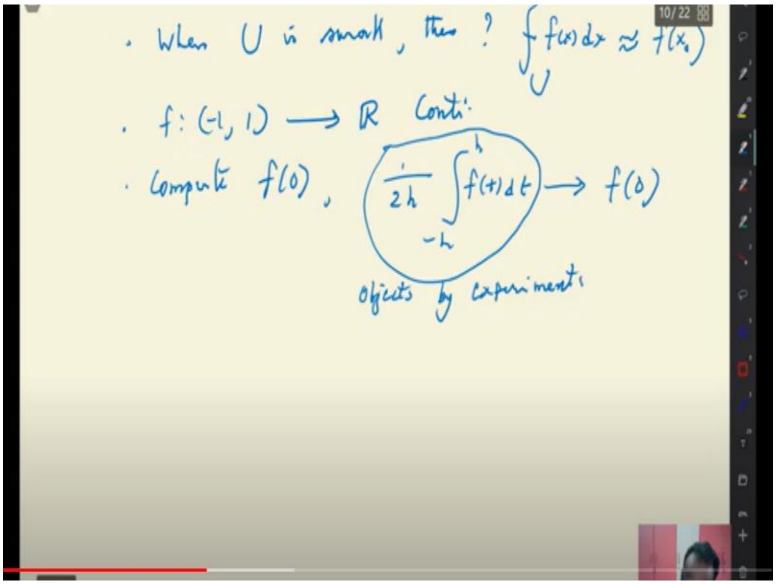

But then when the U is close enough, so most of the time when U is small. Then the question is that is this average, you take an average if you want is close to you are averaging about? So, let me fix a point index so, you have a point index. So, you are fixing is this close to f of x 0. So, let us look at an example suppose f is from a, b to R continuous and maybe I will take very precisely example from minus 1 to 1 and we want to compute f of 0.

So, instead of f of 0 what may be available is a neighbourhood minus h to h f of t dt. If f is continuous this will converges to f of 0. So, these are the experimental objects available by experiments basically, this is one way of motivating that.

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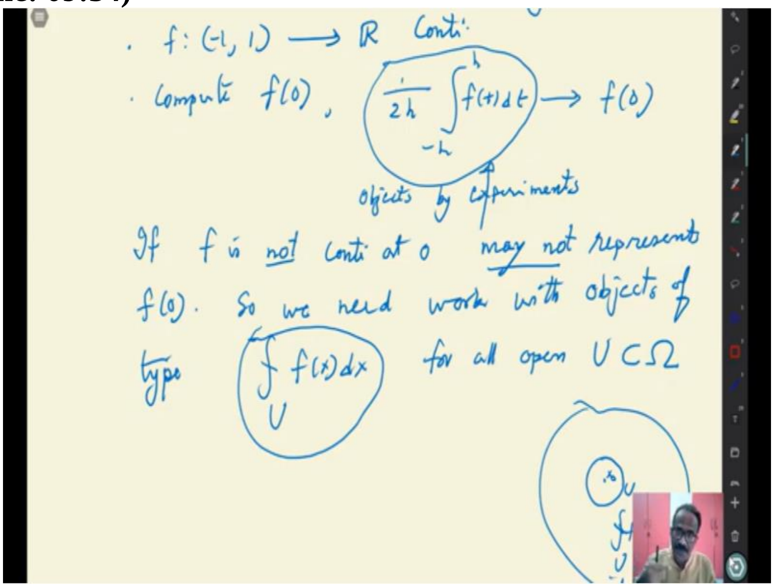

But if f is not continuous, at 0 this may not represent f of 0. So, we need to work with objects of type integral U f of x for all open U in omega open or close does not matter. So, these are the objects to work with because the averages if you consider here may not represent that this is true even in higher dimension if f is a continuous function so, if you take a point and if you take a neighbourhood U. So, if you calculate x 0 here, if f of x over neighbourhood averages this will go to f of x 0 if f is only continuous.

So, you need smoothness if there is no smoothness then your physical quantities may not be smooth all the time. Your physical things can exhibit singularities. So, on a singular function if you try to work with f of x 0 point wise may not yield the correct result or may not produce the correct solutions. So, you have to work with these objects frequently.

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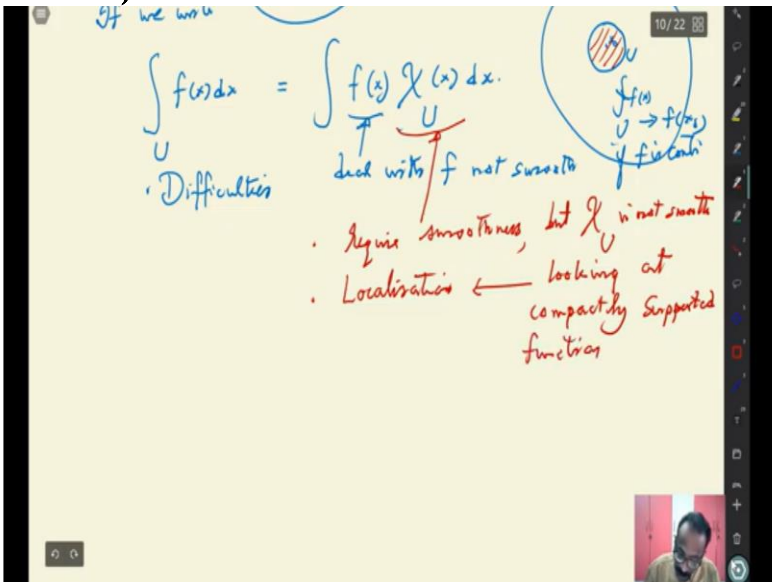

If you write U f of x dx I can write this by taking it as a characteristic function f of x characteristic function of U of x dx. So, there are 2 difficulties if you work with these sets. One difficulty you want to deal with f not smooth therefore, when you interact something, this indicates you need smoothness here. Smoothness for that function require otherwise both are bad these objects will be very bad require smoothness; the best smoothness possible see infinity but chi U is also not smooth the characteristic function.

So, that indicates and second thing you want a localization because we want to find the value approximately here. So, you want to integrate over smaller and smaller neighbourhoods. This is called typically a localization problem. In mathematically this is handled by looking at compactly supported functions, you will see most precisely so, instead of looking for all neighbourhood U.

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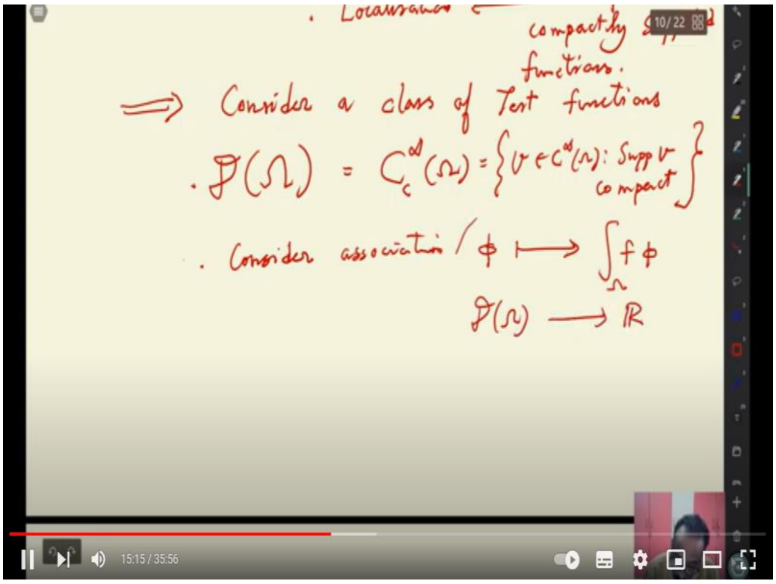

So, this force to consider a class of test functions which we denote by D omega now is for a notation because d omega is nothing but C c infinity. Right now, it is C infinity of omega later we put an appropriate topology the set of all v, c infinity function with the support v compact and these are all intuitive motivations. So, consider the expressions of the form, or association of the form. Consider association phi go into integral of f phi for a given function. So basically, we are getting out of the class of functions and making it bigger class of functions where phi for also this is from phi omega to R so, these are new objects.

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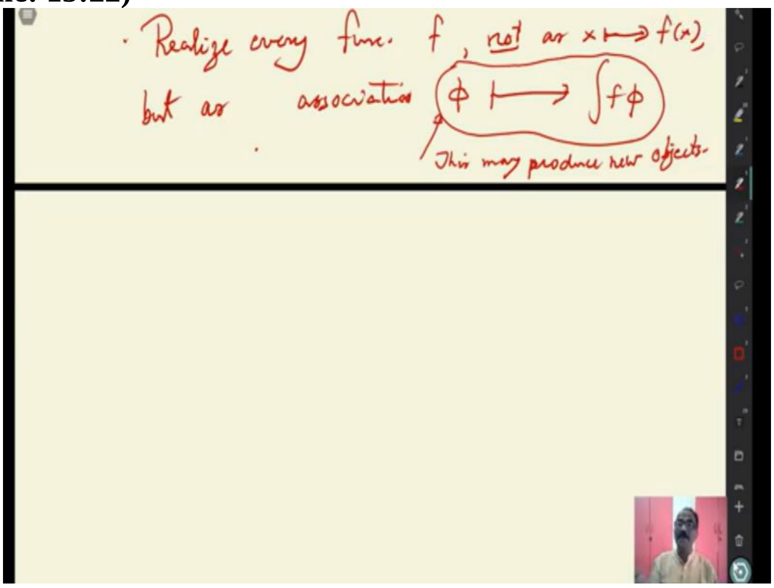

So, every function, we will state an immediate reset. So, we want to realize every function f is some integrable function not as a point wise association not as x going to f of x but as an association phi going there are difficulties, that is where the old theory you are to understand but you have seen that. The thing is that this may produce new objects. So, every function has a point wise association and you are associating with an objective.

This is not a function this rather a function which you will see soon but this may produce new objects which need not be coming from a function also. So, let us try to first understand that more with a theorem we will understand that every function can be do this way. Unique it can be done because if you create one function is associated with something end of it with that new object not uniquely identify with this one. Suppose it identify 2 functions then that may not be a good association. So, here that is what is called the basically the localization.

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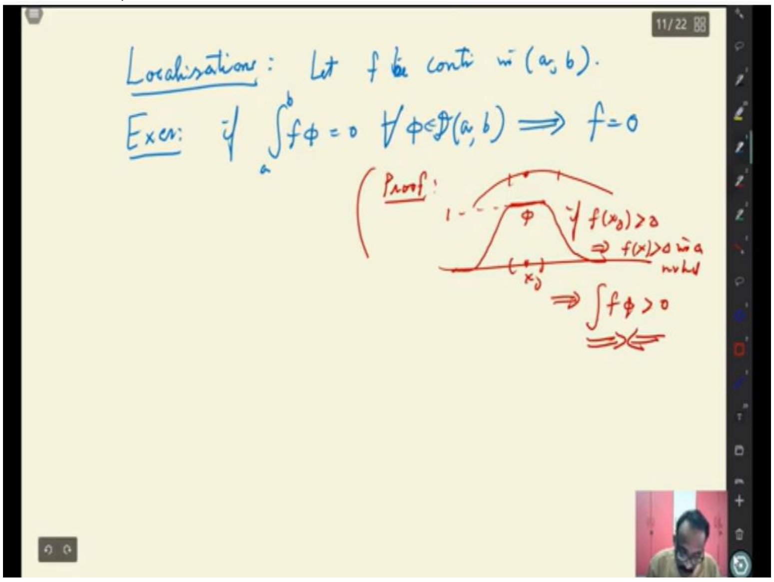

So, let me do this localization now so, again these are all initially told you I am not going to develop theory and prove results rather we want to understand things. So, later let f be continuous on it. Suppose f is continuously maybe is continuous on it. So, we will always see at continuity derive the property and then try to remember the small thing if it is conflictive f be continuous in a b then this is a small exercise you should work out.

If integral of f phi = 0 from a to b for all phi in d that one C infinity you do not need C infinity function, but it is enough for continuous functions. So, but the compact support is important d of a b that implies f = 0, that means you can identify uniquely for a phase continuous thesis I will tell you a little more about the proof is easy what do you do? You just do this exercise this is an important factor with this one. So, a function can be completely recovered from this thing.

So, I will give geometrically suppose you have a function like this and you have a point here x 0. So, if f x 0 is positive, you can find the neighbourhood in such a way that on that neighbourhood you take phi to be 1 and such functions you can construct so, this is at 1. So, you take phi to be 1 here something like that in that neighbourhood. And this is 1 you do not need 1, but what I am saying that you can easily prove that.

So, this will imply integral because there exists a neighbourhood for which if f x 0 = 0 and if f is continuous, this will imply f x will be positive in a neighbourhood and you are choosing phi also to be positive. So, that will imply integral of f phi will be positive which will be a contradiction so, make things clear so, you leave it as an exercise.

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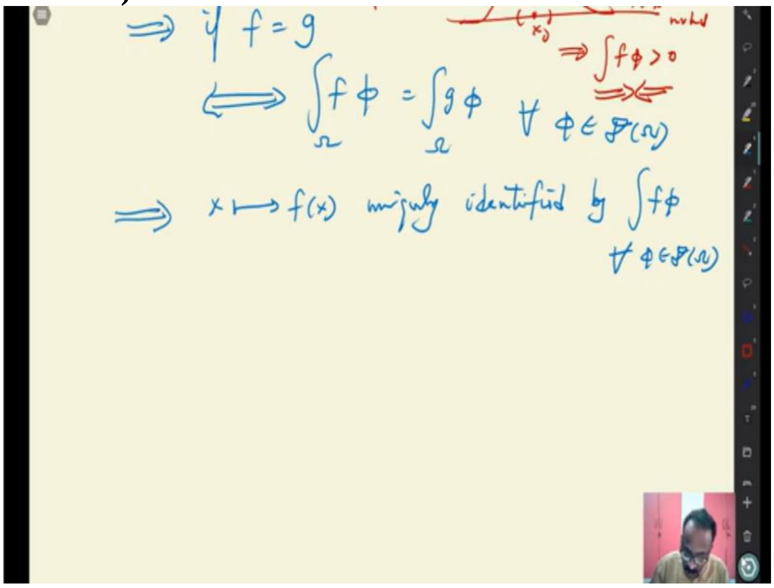

So, what does this immediately imply? This implies, if f = g if and only if integral of f phi = integral of f g phi for all phi in d omega. So that means integral of f phi uniquely determines f that is what it shows. So, this is in every function so this basically implies that f is uniquely identified by integral of f phi. That means you can recover your function if you know this integral of f phi for all phi in d omega then you can get back your f. So, as long as this is different but what I am saying that this association which we are going to do and if f phi is not continuous or f phi is locally integrable.

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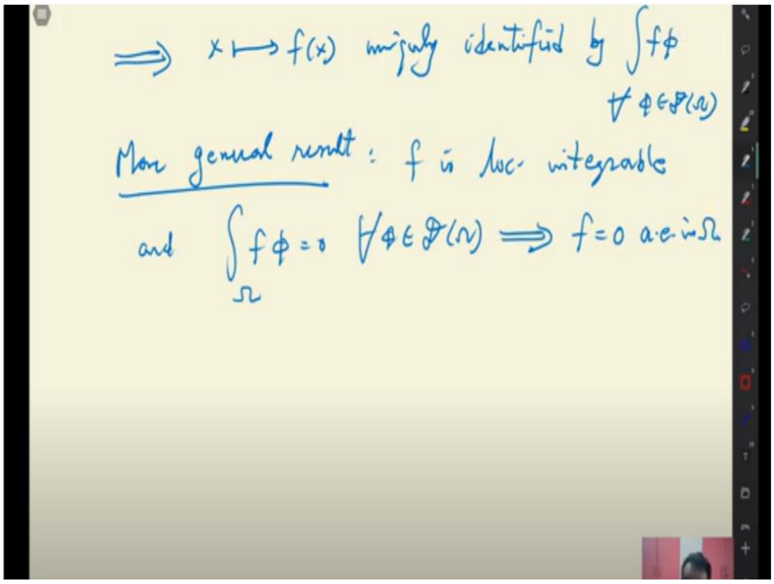

So, there are more general results so, we can develop you do not need because you want to get out of that. The result integral of f is locally integrable not continuous locally integrable and integral of these are all in measure theory you would have learned it equal to zero for all phi in d omega then that will imply f = 0 almost anywhere and that is enough for these things.

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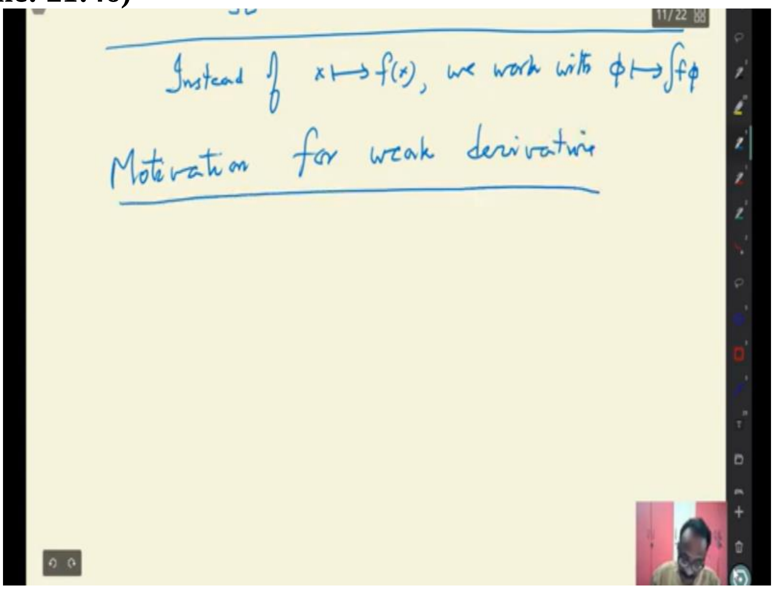

So, what we are going to see now from now onward instead of x going to f of x whenever this is possible that is continuous it is fine. Otherwise, we work with and that is the basic motivation. Phi going to integral of f phi and with this let us get a motivation for weak derivative. So let us see how things will work.

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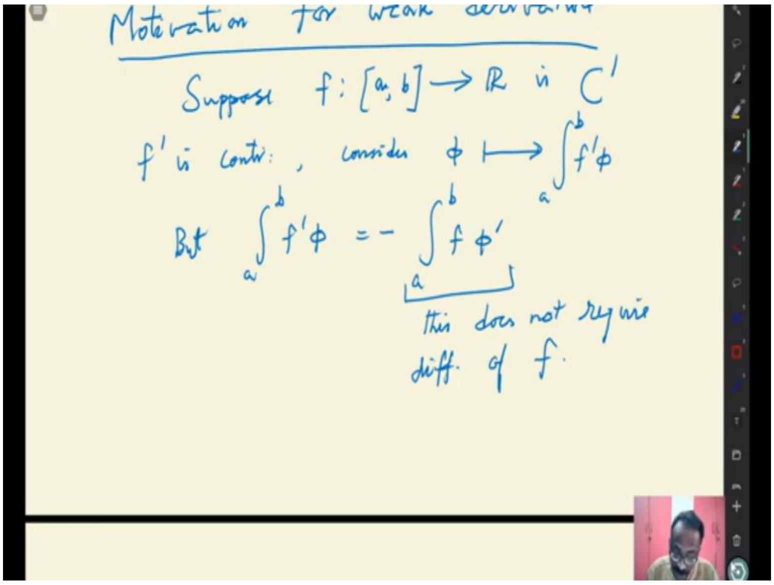

Suppose we will start with the f is in suppose f front a b to R is smooth, you will see R is C 1. So that mean, a phi is continuous and a prime is continuous, we will do that maybe you can take from closed interval also does not matter so, you can do that one. So, C 1, in that case f is differentiable, f is continuous, f prime is continuous, and then f prime is continuous. So let us try to understand in this notion.

So, instead of f prime as a point by thing, that means consider, then, so you want to consider not x going to f prime of x, I want to treat this as in phi going to integral of f prime of v that is what we try to do. Now do by in because f prime is different continuous, therefore we do an analysis, but integral of a to b, f prime of phi = - integral a to b, f phi prime. And look at this one. So, these 2 quantities are equal, but this does not require differentiability of f. So, you can do this one so even if f is locally integrable, you can define the derivative of f

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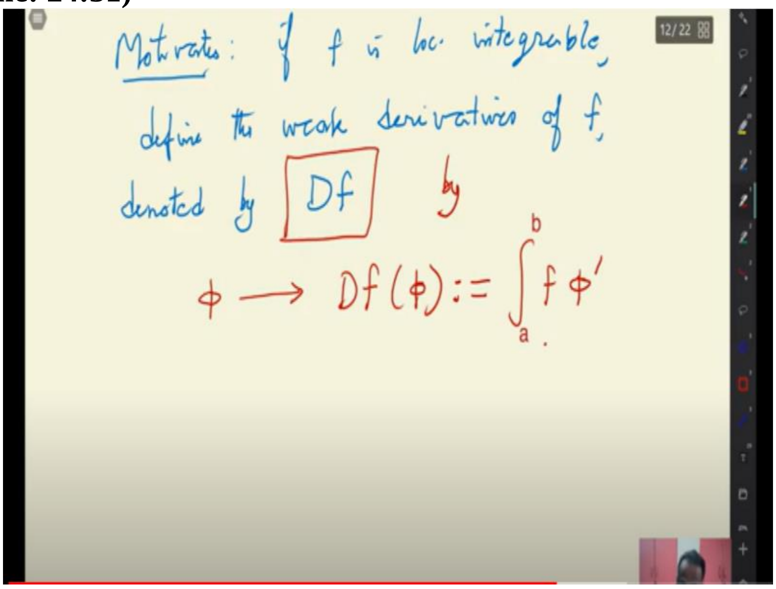

So this motivates if f is locally integrable define the weak derivative of f denoted by the Df. f prime I look for the classical thing and then the denoted by so Df is might be weak derivative notion not as a point wise now everything is defined, I am doing it by these things phi going to Df acting at phi and by definition this one integral of f phi prime.

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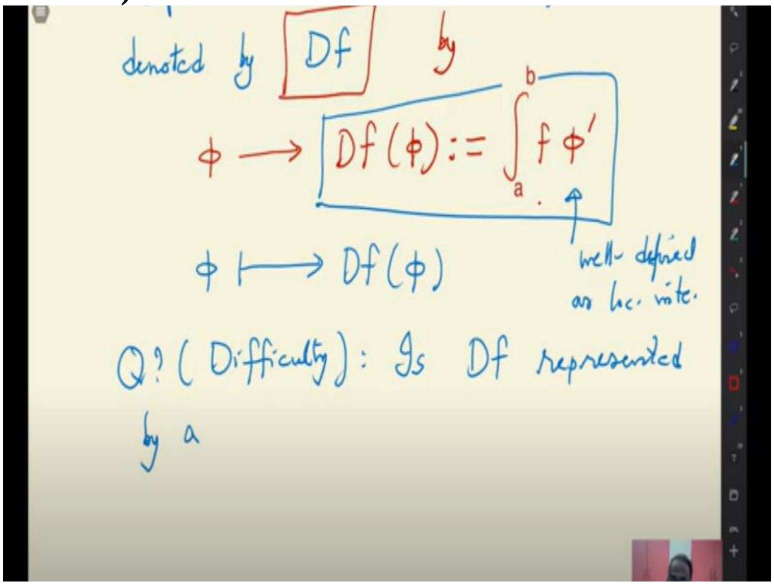

So, one of the questions you can emit, you have the definition of the derivative. This is your definition of weak derivative. So, you are continuing making an association phi going to Df acting at a phi. In this way and this is well defined as F is locally integrable same as locally integrable. The major question may come or the difficulty may come is Df represented by a function that is important, otherwise Df may be only an association.

So, what we have seen that every locally integrated function produces an association uniquely but the Arabian association which may not be coming from a locally integrable function. You will immediately see an example now is Df represented in by a function.

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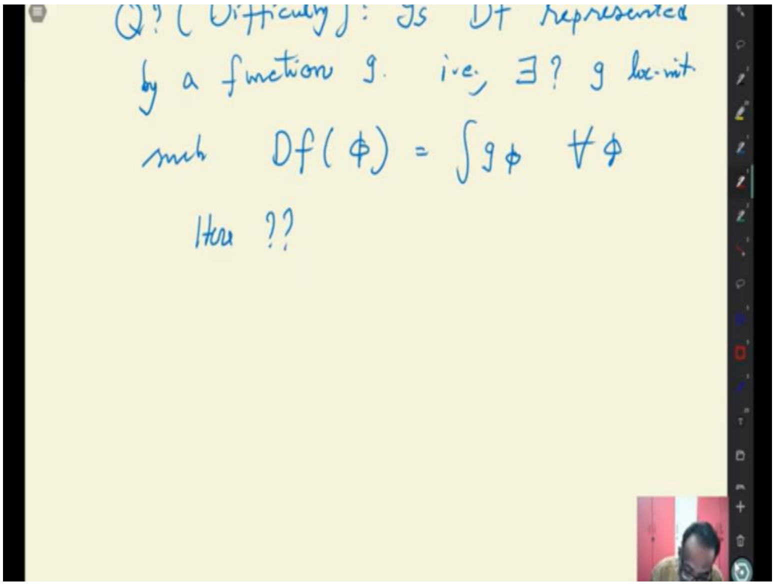

So, we will give a couple of examples before going. Some function g that is you want to identify the Df of e is does there exist g locally integrable such that Df acting at phi = integral g phi for all phi that is the question here. The question is that does that exist here is the thing Df at v is equal to I have to put a minus sorry I forgot to put a minus sign. So, there is a question here this double integral is does there exist?

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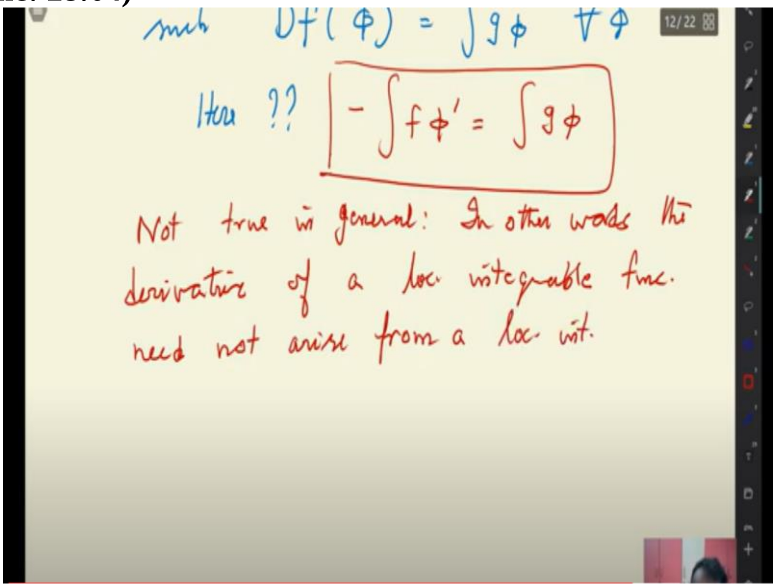

So, that means minus integral f phi prime = integral of g phi that is the question. So, unfortunately this is not true. In general, you will see some simple examples. In other words, the derivative of a locally integrable function need not arise from a locally integrable function. So let us see a couple of examples and stop quickly so you can do it examples or exercises also. So, you do the computations examples. This you can also view it as exercises.

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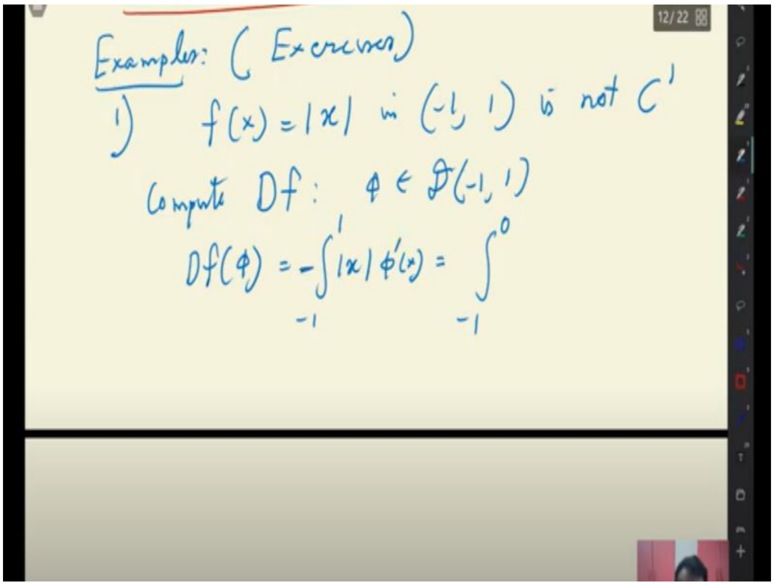

So let us do one example take f of x = mod x is not C 1 so looking for functions which are not C 1, if it is C1 that is no issue. Because you know the classical derivative in minus 1 to 1 is not C 1. So, you want to compute weak derivative Df. Because weak derivative is defined for a every locally integrable function meaningful or not, we will see so, what is your Df at phi? So, you take phi in Df of minus 1 to 1, that means phi vanishes in the neighbourhood of 1 and minus 1.

This is equal to minus 1 to 1 modulus of x and phi of x. And now you separate it because it is differentiable in minus 1 to 0 and 0 to 1. So, you do minus 1 to 0, then in that case, minus 1 to 0, it is minus. So, you have to compute Df is 0. Sorry, this will be by definition, this is minus, and this is phi prime. So that derivative goes here, that is what the definition.

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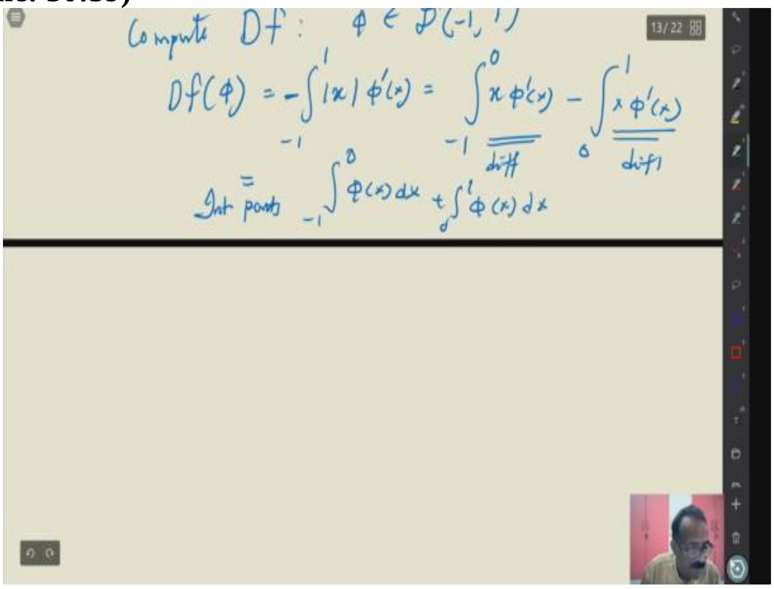

So, this is in minus, it is minus x, so x phi prime of x + integral 0 to 1, and that will become minus 0 to 1 x phi prime. Now do it in this, it is differentiable so, integrate by parts equal to integrate this is exercise parts, do it and integration by parts. You can see that this is nothing but integral minus 1 to 0 phi x dx + integral 0 to 1 phi x dx. So, minus may be here minus that equal to that one, please do that integration by part.

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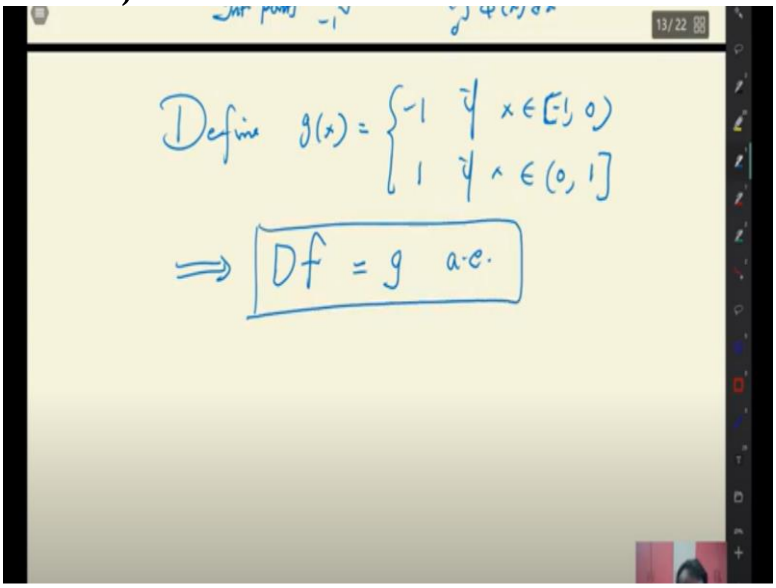

So, define this simple exercise g x = - 1 if x is in minus 1 to 0, and 1 if x is in 0 to 1, this endpoint does not matter. At 0 does not matter what we are doing that implies your Df is identified with this g almost everywhere. In fact, except that the origin and we always look for in an integration theory. So, in this case, it is a nice example.

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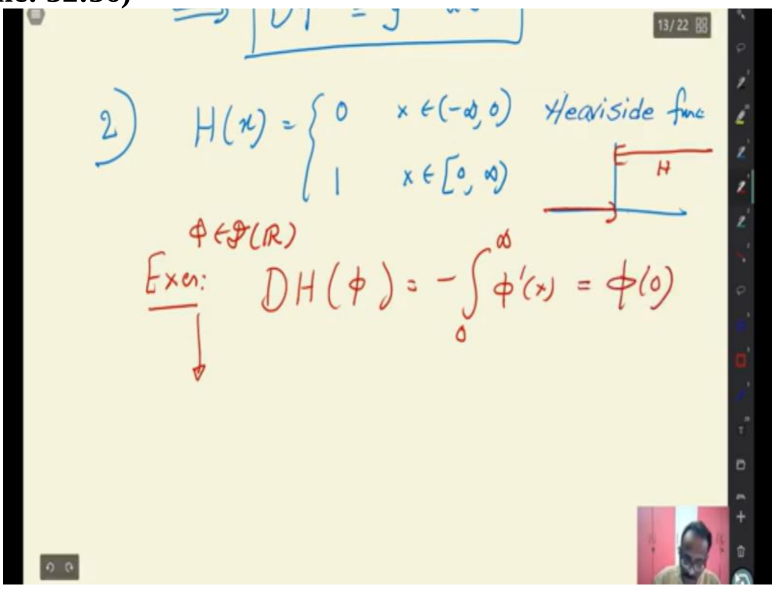

The next example is more difficult so, you define in this case; this is a very important example from the electrical engineering understanding the pulses and the signal moving. And this is also called the heavy side function. If x is in minus infinity to 0 and 1 if x is in 0 to infinity. So, this is called a heavy side function, and the function is this one. And I would like you to compute, so this is your exercise, immediately, you can do it.

So you will compute your D H acting at phi and this is nothing but minus because it is 0 up to that one 0 and phi is in D of R in this case, phi minus infinity to 0 and that is one phi prime x that there is nothing to do this one.

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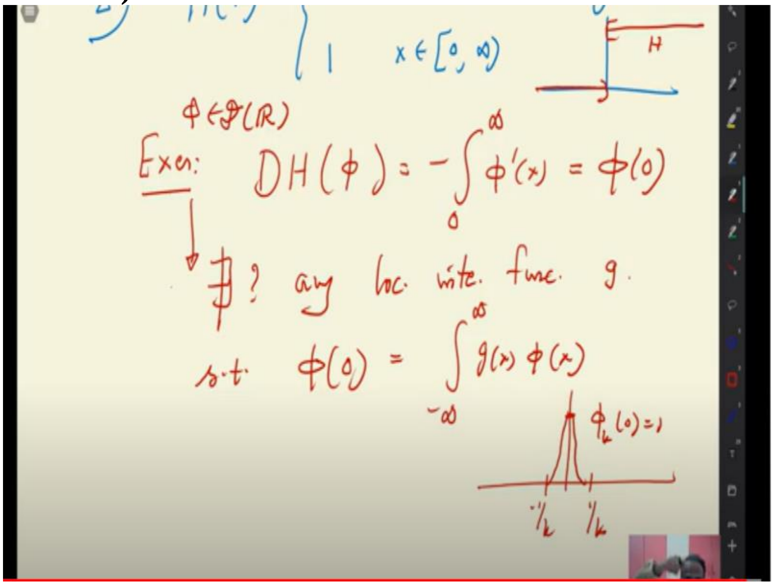

So, the exercise is below coming and here this is nothing but phi 0. So, the question exercise is that does not exist any locally integrable function. See this exercise I would like you to know any locally integrable function g such that phi 0, you need to do the D H of phi = integral minus infinity to infinity g x phi x. What do you do is that to prove this exercise, construct a sequence of functions, as I said that phi k 0 is 1 and all other phi k but it will go to a more and more.

So, you construct a sequence of functions like this, minus f minus 1 over k 2k 1 over K. So, you construct a sequence of functions like this, with phi k of 0 = 1 and then derive a contradiction. So, work out. So, you see there is a function for which we said is a complete singularity because it does not even have continuity and but you can compute it, so there is you will not even know that one. So, you can compute a weak derivative of this function

So, it is a global quantity it is not that point wise concept. And that is it. So, we will stop and continue and give the definition of distributions with this thing. And we are going to give a definition of the weak derivative for all these objects not just to function, what I established that there is a class of objects and in the weak derivative produces that. So, we will introduce that in the next class. Thank you.