First Course on Partial Differential Equations - II
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Lecture - 02
Weak Solutions 2

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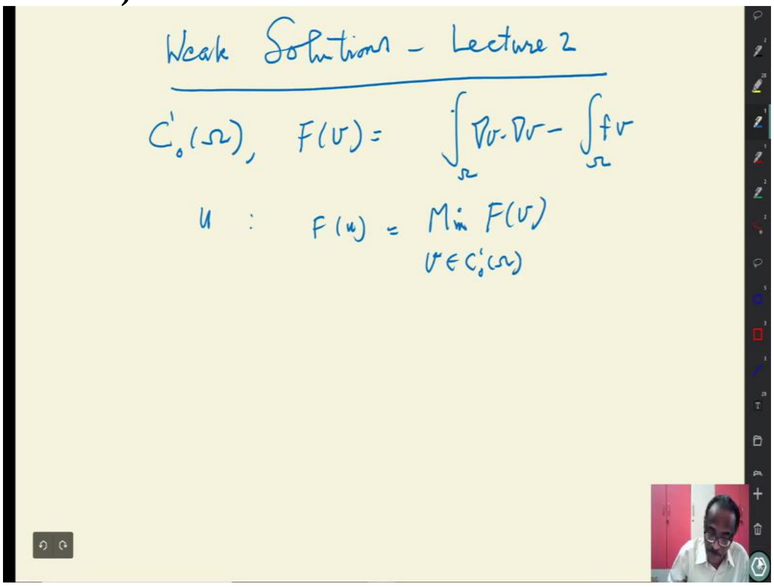

Good morning, in the last lecture on Hilbert space method and weak solutions, we were trying to introduce the notion of solutions in Hilbert space and we tried to give an algorithm what are the steps and what are the things you need to do it. And in the process we have introduced motivated you why we want to do that one throughout the course we have motivated. And in particular we motivated by a minimization problem.

And in the process of the minimization problem was in the space C 1 0 of omega bar and you want to minimise F of v was integral of the grade v, grade v minus integral of F is a given function to u. And then what we have seen that the if u is a solution such that F of u is equal to minimum of F of v, v is this also motivated by a concern that space that need not be the correct space, but we are motivated this one. Then we have introduced in the process and linearization the freshened derivative everything little was vague we did not give the definition here.

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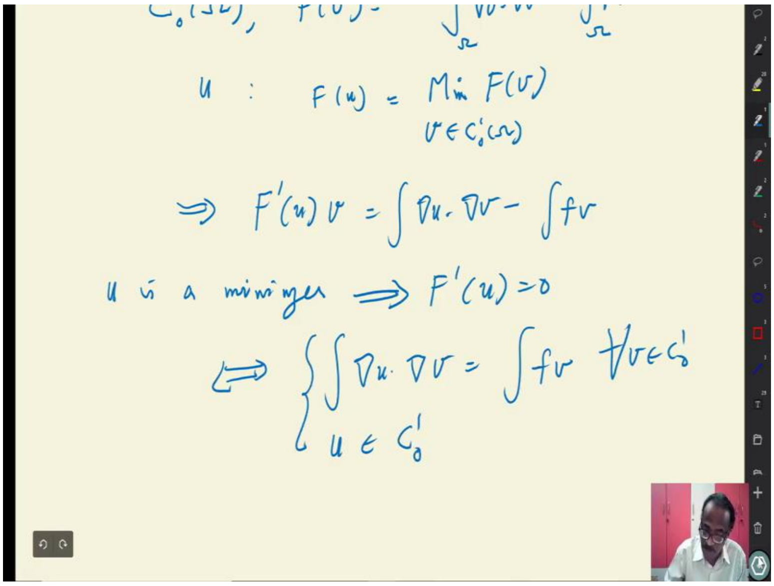

And the prime of F u v equal to integral of grade u grade v minus the integral of fv everything in omega. Therefore, u is a minimizer implies you have a prime of u = 0. That is equivalent to saying that you have this problem grade u grade v equal to integral of the fv we should look for all v in C 1 0. And you look for a solution u in C 1 0. And this is called a weak formulation is called 1Dirichlet formulation.

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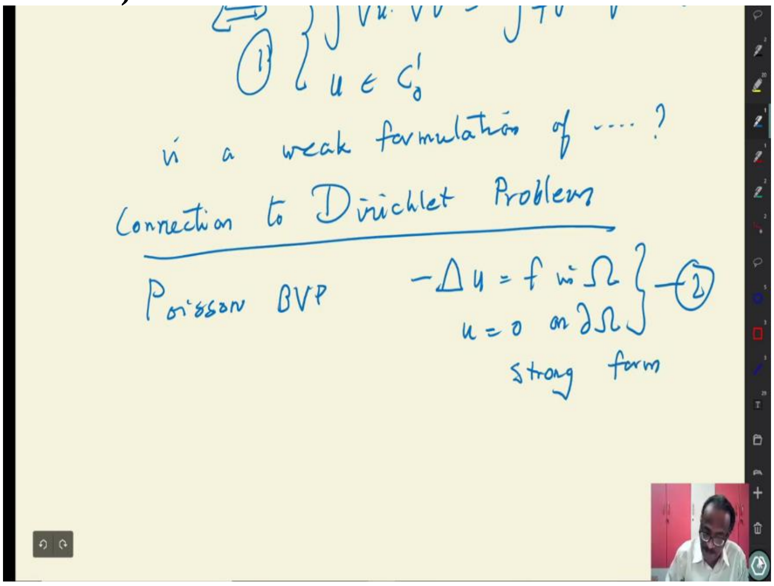

You have to justify your steps, which we have given in the last class. That is what we are trying to say. But weak formulation of what that is what we are going to say that now. And then, so that is why we are giving connection to Poisson equation, connection to Dirichlet problem. So, recall the Poisson equation Poisson boundary value problem minus Laplacian of u equal to so, I call this is the weak formulation 1 and this Laplacian of f = f in omega u = 0 on d omega. So, this is basically a strong form, this is your PDE is also known as strong form. Because you have a weak form so, you have a strong form.

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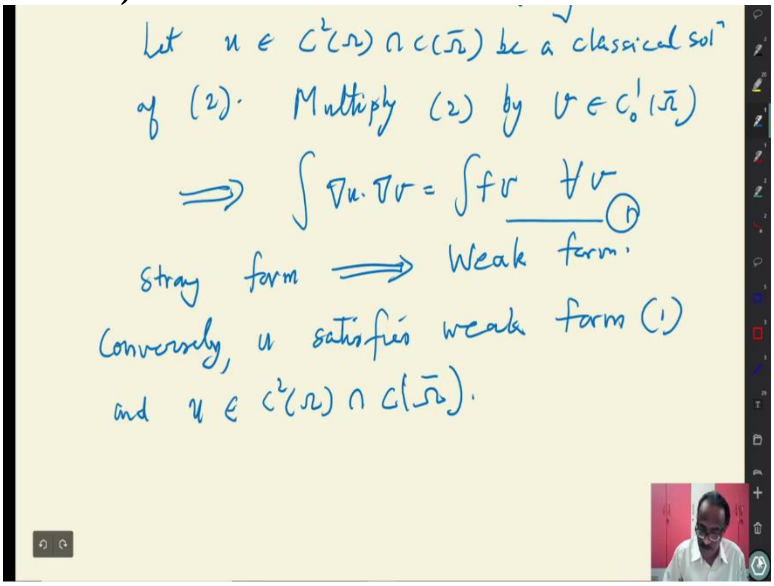

I want to see the connection between this one. So, let u be a solution again let u be you know that under suitable conditions seated there. So let u be a classical solution u belongs to C 2 of omega intersection C omega bar be a classical solution. This is a first we have to check both of this is a weak formulation the 1 is a weak formulation of 2. You have to first check if it is a classical, then it is a weak formulation classical solution of 2.

Then multiply 2 / 1, 2 / v in C 1 0 of omega bar. Integrate by parts use the divergence theorem that will immediately imply grade u grade v = f of v for all v so strong form implies weak form. You have to prove also conversely weak form. Conversely if u satisfies weak form so, this is the same of 1. Weak form 1 and u is in smooth suppose u is in C 2 of omega intersection C of omega bar you see.

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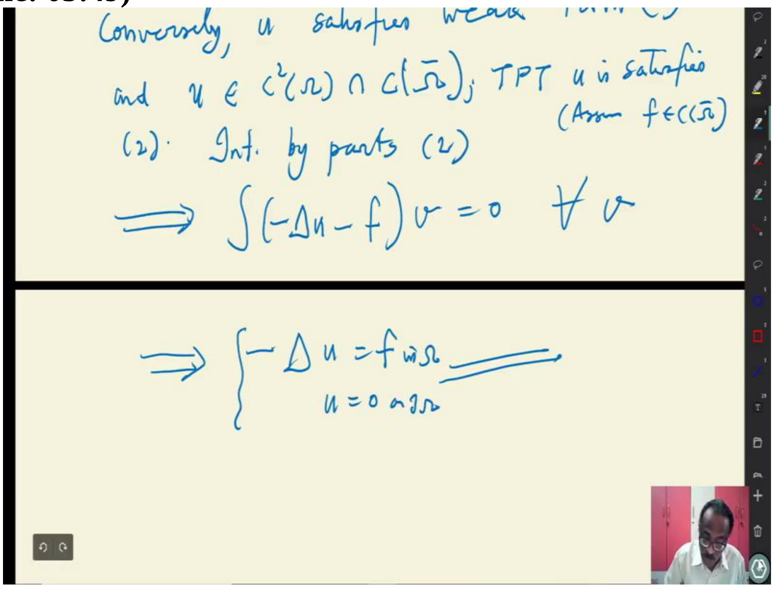

You have to show that you are to prove that u satisfies the strong form that is what we say satisfies 2. So, this is just integrate by parts because u is in C 2 you can integrate by parts 2 will imply integral of minus Laplacian of u if you integrate this 1 it will become minus Laplacian of u, u + f so I am taking. And then v will go here the v if you integrate by parts and there would not be any boundary term. So, this v = 0 for all v that will imply by continuity of u and other things and you assume a f is also assume.

Of course, assume efficiency omega bar assuming that implies minus Laplacian of this is true for all v so, and that is it. So, that is what you want to prove it. So, it is at the end of course, u = 0 because u is a solution is this in omega this is in d omega so, that means it is a corrective.

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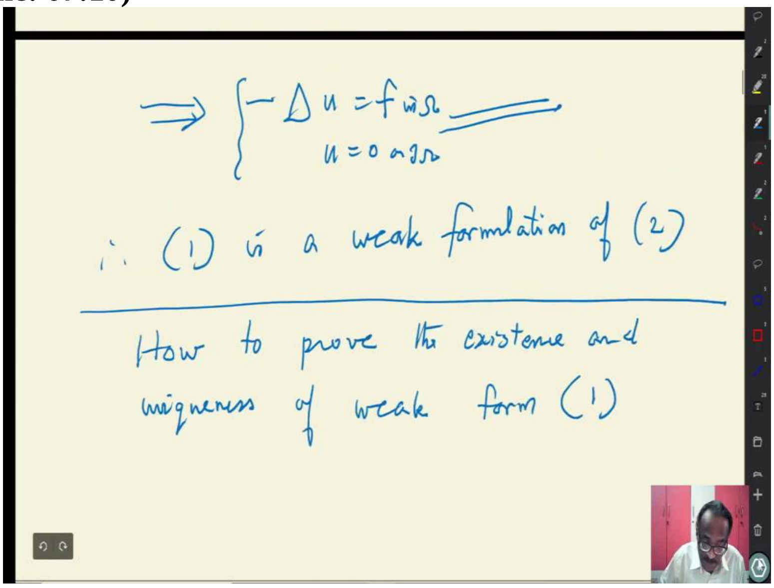

So, 1 is a therefore, 1 is a weak formulation of weak formulation of 2. This is what do you want to prove in any weak formulation you give if u is classical it should satisfy the weak form and weak form together with regularity it should satisfy a weak formulation of 2. In other words, and you see and but then you see this is the big strong form and this is the weak form this is a strong form this is the weak form.

And the weak form can exist, but strong form to derive your strong form you need extra regularity on u and you feel a physical solution with a minimizer, is divided only divide is defined in terms of the integral. So, you will see the minimum function is defined and that does not tell you anything about your first derivative second derivative so, that assuming a secondary derivative.

And looking for a classical solution is a more serious issue, because you are solution to this minimization problem may not have the second differentiability at all and only if there is a second differentiability it will satisfy your strong form. That is why that is hence, the weak form is more physical in this case, you see. Now, you want to go to the next step, the how to prove existence.

Now, let me how to prove this what are the algorithm which I have described how to prove the existence and uniqueness? I will not these are all I said is some sort of a motivation, you need more machinery eventually and that is what the next course on partial differential equations, the advanced PDE. How to prove the existence and uniqueness of weak form? So, let me start telling you some of the troubles and then you will see, first you want to understand the meaning of this term so, if you how to understand the meanings?

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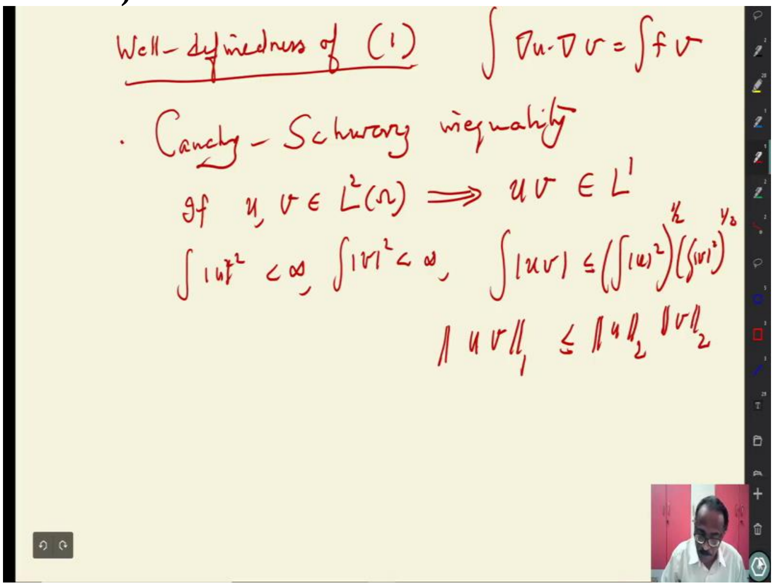

So, let me use a colour meaning of well definedness of 1 the weak form. So, let me call it here, weak form grade u grade v repeatedly writing so that you will remember that. So here we recall the Cauchy Schwarz inequality. So, I am appealing to some of the things from functional analysis. If u, v in L 2 then that implies the product uv is in L 1. That means uv in L 2 means integral of mod u square is finite integral of mod v square is finite.

And the Cauchy Schwarz inequality tells you that integral of mod uv is less than or equal to integral of mod u square whole power half into integral of mod v square whole power half. So, that means norm of uv this is left side is norm of uv in L 1 is less than not equal to this is the norm of u square integral functions norm of u and norm of v in L 2.

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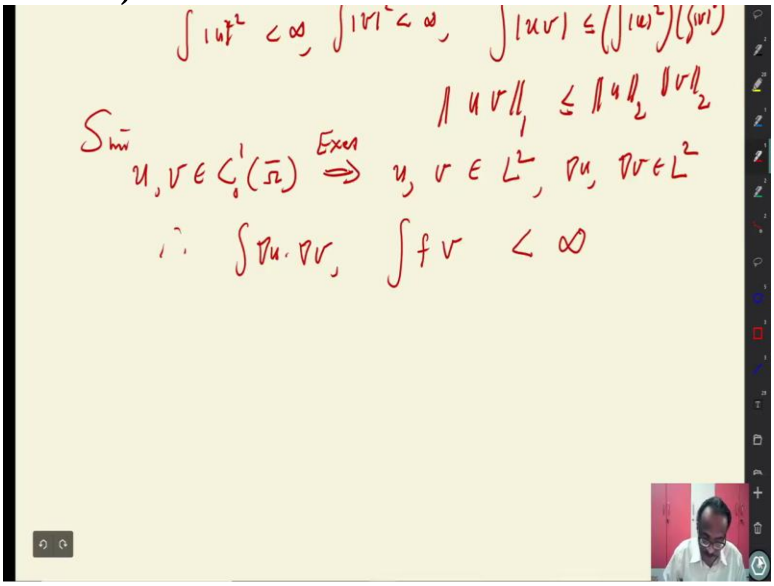

So, what you need so, the well definedness is fine, so is since u, v in C 1 0 since we have this. Since u, v that is why we have taken C 1 0 of omega bar you can verify if you are not familiar check this is an exercise if you are not familiar u, v is in L 2. So, all these therefore, these terms are well defined, because this is in L 2, u, v is in L 2. And grade u grade v is also in L 2. Because C 1 of omega bar the derivative is also there. So therefore, grade u the product and integral of fv are all finite, the same. So, therefore, the terms are finite quantities under this assumption.

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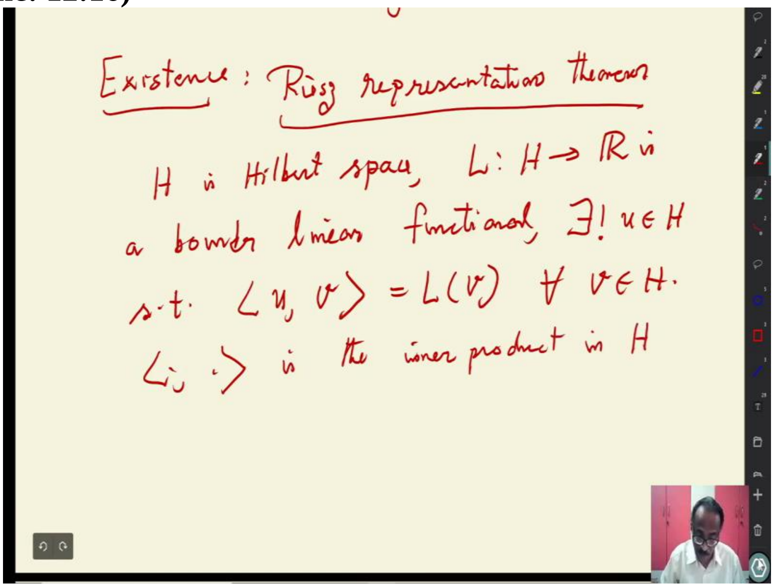

Now, let us look at the existence I am not going to prove these existences because we do not have the machinery yet. I am only trying to locate the difficulty. So, here we are going to appeal what are called Riesz representation theorem in functional analysis. And you need whether we have sufficient machinery to apply Riesz representation theorem. So, I am only discussing some simple cases.

So, a more difficult PDE you can handle and the distribution theory tends to such things. What is Riesz representation theorem tells you? Suppose H is a Hilbert space. And L from H to R is a bounder linear functional then there exists a unique you can identify the linear functions by elements from H there exists a unique u in H such that u, v = L v this is true for all v in H. And what is this one? This is the inner product in H.

So, we have so if you look at your weak formulation, if you recall your weak formulation you are interested in finding this one using. So, we can see an inner product like that. So, if you introduce an inner product like this, and if this happened to be a bounder linear functional, then you will get the existence of u.

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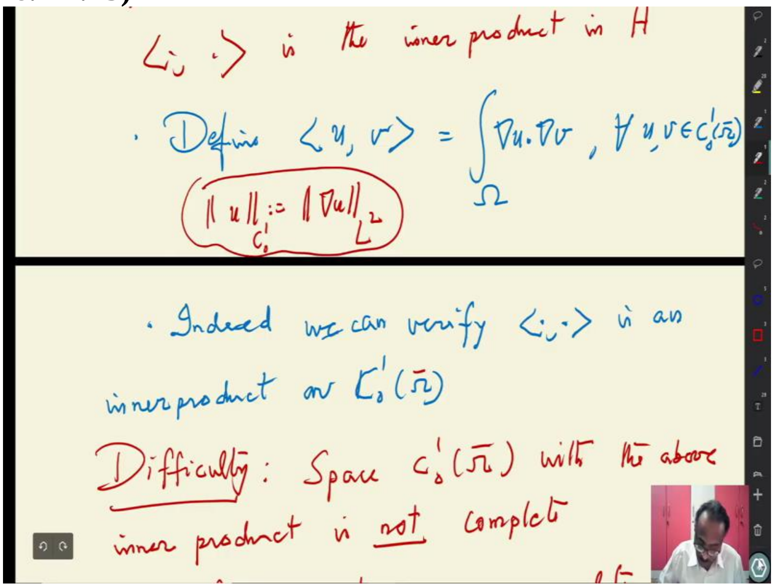

So, that is what we are going to do this, thus define u, v is equal to integral of grade u dot grade v over omega for all u, v in C 1 0 of omega bar. Because our space is this C 1 0 of omega bar. Indeed we can verify is an inner product on C 1 0 of omega, that is not the difficulty. So, what is a major difficulty? Let me alert to you that is the major difficulty.

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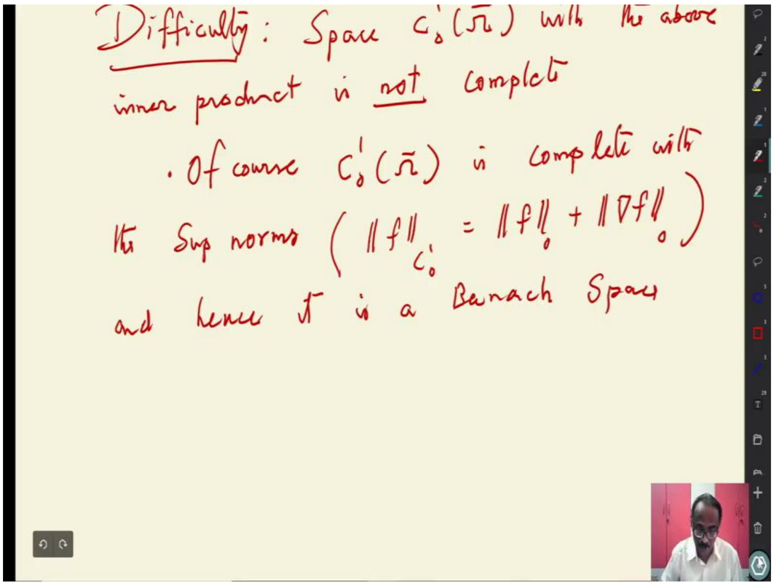

See this space the C 1 0 omega bar with the above inner product is not complete. Of course it is a banach space under different norm. Of course, that is what so far we are used. Of course, C 1 0 of omega bar is complete and is a banach space is complete under a different norm not with the inner product complete with the super norms. But here we have and in infinite dimensional space, the norms need not be collect. So you are had defining C 1 that is why the strong functional analysis and other things are coming with respect to super norms.

For example, for norm f in C 1 0 is something like u define norm of f in super norm plus, you define the grade f and you can define such things. And hence, it is banach space under this. But the hence it is a banach space this is not what we wanted we have defined a different norm, which suits our formulation so our formulation is this 1 we want. So in this case, the norm of u is nothing but L 2 norm of gradient of u, you will see the norm of C 1 0 here, this is a different norm. This is norm of in L 2, so it is a different norm; it is not the super norm.

But this norm is different in C 1 0 so the 2 norms are different doing so with respect to this norm, it is a banach space, but the space what we do not want to deal with, because you are weak formulation is that one grade u dot grade v and this is what so, what do you do with that, so, you need to take what are called the so if you can do this thing. So what you need here is a completion.

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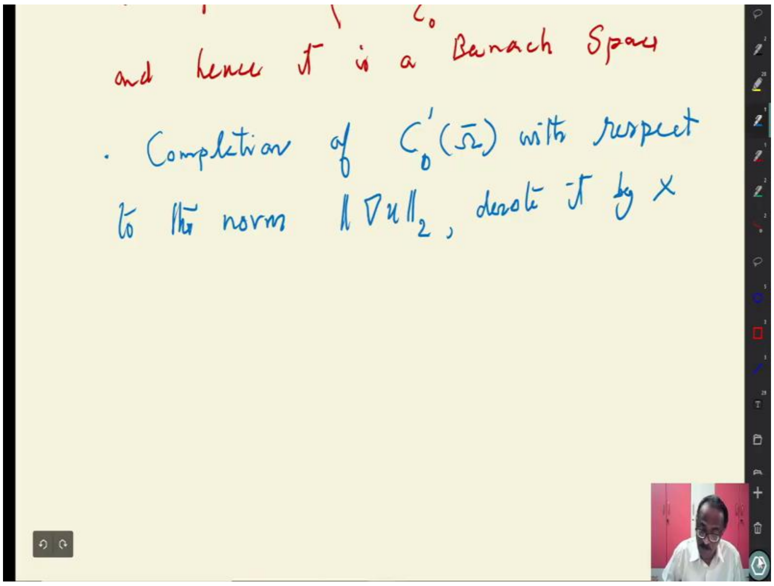

Completion of C 1 0 of omega bar with respect to the gradient norm with respect to the norm gradient of u in L 2 norm, this is your L 2 norm with respect to that the norm between let us denote it by X.

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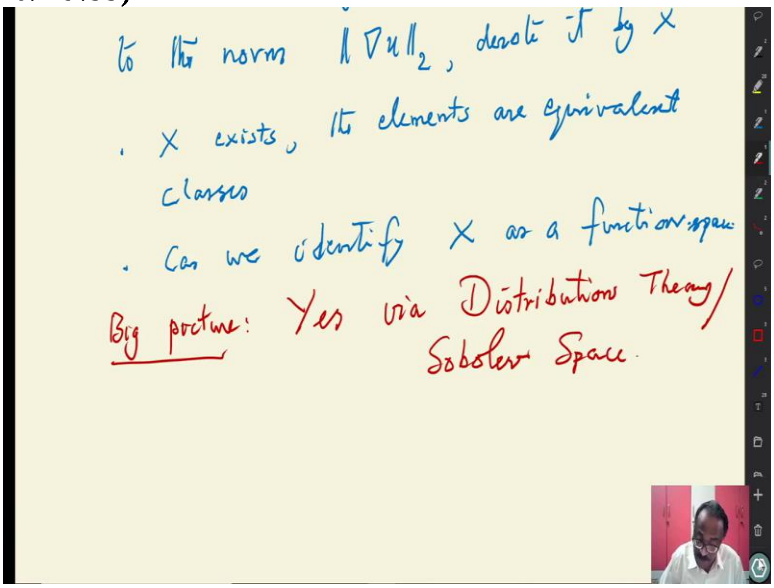

From functional analysis, we know that X exists and then the elements are the equivalent classes that are not function spaces. Now, the elements are the equivalent classes and that is not enough. So the question is that can we identify X as a function space and this is part is the big picture of function space. So, the bigger picture yes via distribution theory and Sobolev spaces distribution theory we may give you in the next class not in a detailed way. I told you this is the material for the next course or anything distribution theory or Sobolev spaces.

So, let me write it with you may not understand now, but let me try that. So, in those cases becomes Hilbert space X exists as I can identify. So, it is right now, it is a set of all equivalence classes that is so, how will you complete your normal linear space thing.

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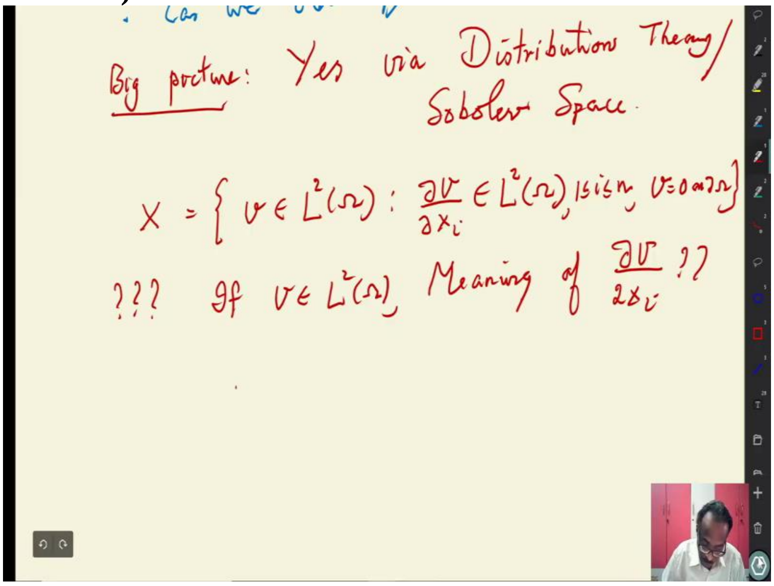

So, X turned out to be something like that. So, let set of v in L 2 of omega. So, this is not understood where dv / dx i is in L 2 of omega and for 1 less than or equal to i less than or equal to n and then v = 0 on d omega. But then the big question you may immediately ask if v is in L 2 of omega. What is the meaning of the dv / dx i?

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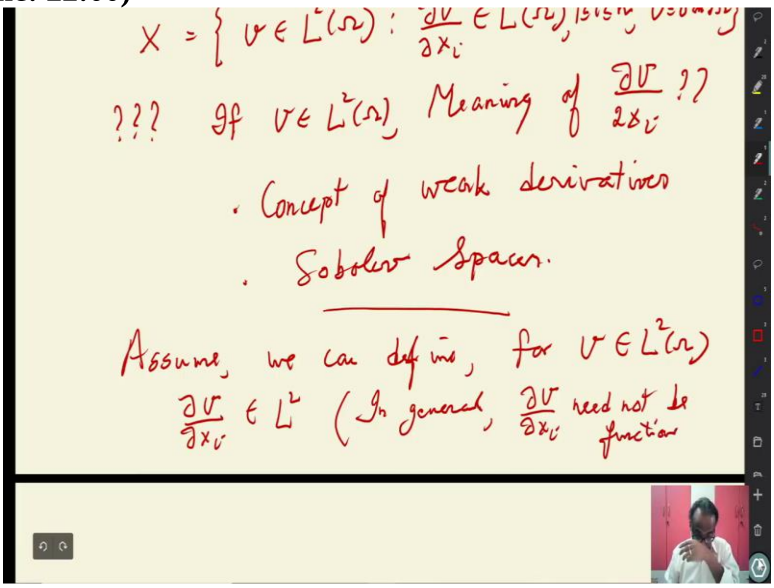

This is what we need a concept of a concept which may not in elaborate way, but we will explain to you a concept of weak derivative. And then using weak derivative we define spaces like X and are called Sobolev spaces. So, I may do a little better about a week derivatives and weak notions in the next 2 couple of lectures, but let me do a little more something like that, assuming that it is a so, we will stop in this portion we will stop here. So, assume x is a Hilbert space now, assume we can define for v in L 2 of omega dv / dx i.

What in distributed theory as an element in L 2 in general derivative of a function need not be a function in general dv / dx i need not be a function. But what I am telling you is once in general because for every function in L 2 you can define the derivative but you will not be able to get it as a function you will see that later, but you collect all those functions v for which you are dv / dx i is also a function and what we are telling is that the completion X can be identified with this space.

So, you can identify the thing with only those v for which dv / dx i in L 2 of omega and with that setup now, you have a weak formulation telling that 1. So, now we can show that with this you can have an existence of course uniqueness also. So, we will have a kind of existence.

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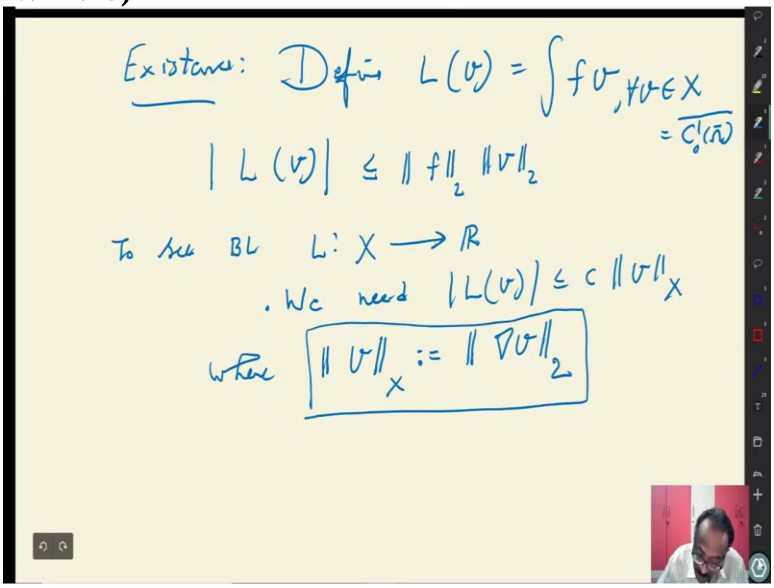

So, now look at it define norm L v is equal to integral of fv and then your modulus of L v for all v in X now, not C 1 0 of omega which is a completion of C 1 0 of omega bar with respect to that space with respect to the gradient norm, it is not with respect to the now. So, for all v in x this 1 and modulus of L v apply Cauchy Schwarz inequality this is norm L 2 in no v at L 2. But you want to see it is a bounder linear functional L is from H norm is gradient norm H to R that means, what we need to apply Riesz representation theorem.

We need modulus of L v less than or equal to some constant into norm v so, X here. So, what is norm v in X? Norm v in x is a gradient norm is equal to norm of gradient of v in L 2. This is by definition because you know that v is in X you define grade v in L 2. So, but you have only this inequality.

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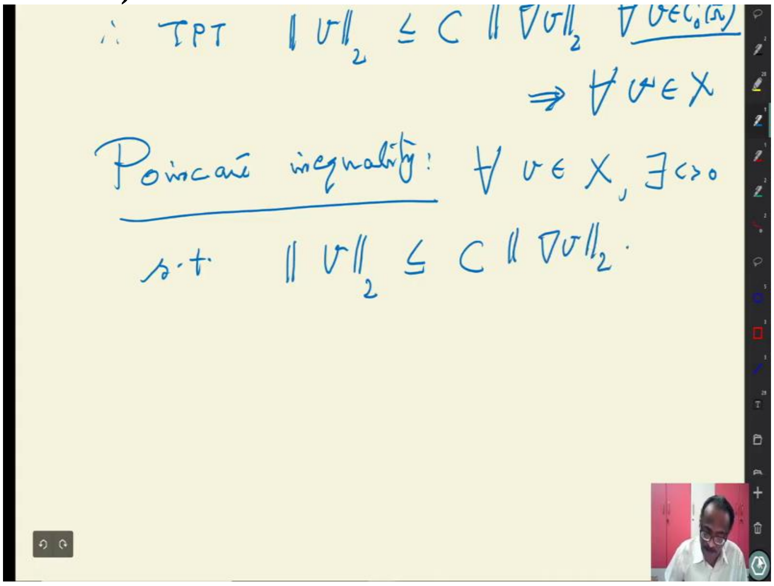

And then thus we need to prove therefore to prove that norm v at L 2 is bounded by some constant into grade v at L 2. This you want to prove it for all v in C 1 0 of omega bar and for all v it is enough to prove this then you will get it which will imply for all v in X its completion and this is provided by what is called a Poincare inequality. So, let me give you Poincare in the simple case Poincare inequality. What is Poincare inequality? So, for all v in C 1 0 for all v in X of course, X is not understood properly there exists a constancy positive. Such that norm of v at L 2 is less than or equal to constant into norm grade v at L 2.

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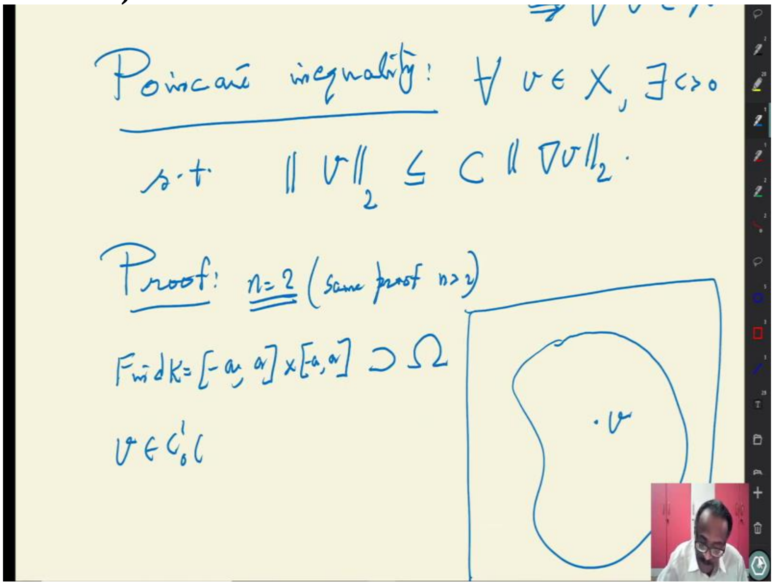

So, let me give you a thick proof before ending this lecture. Proof once we prove this your Poincare n is a bounder linear function and you have to think this proof is simple. Let us take n = 2. The same proof work for n greater than or equal to 2. So, your domain is a bounder domain. So, you can find a rectangle or a square you find a square contained like this. So find a square k.

So, find minus a to a cross otherwise we take n times minus a to a is finding a neighbourhood closed set k this contains omega. So I can write so you have a v defined here. So, for v in C 1 0 of omega bar so we proved this inequality for v in C 1 0 of omega bar then by density it is true for all X.

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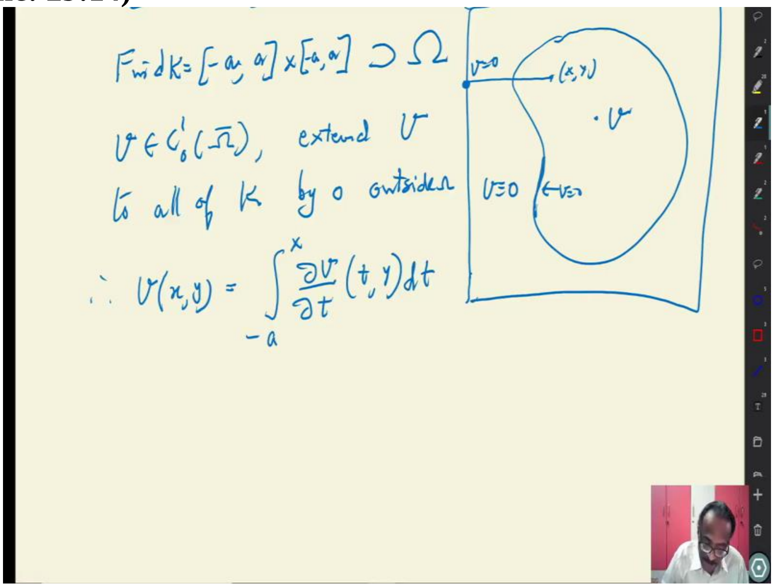

So for v in C 1 0 of omega bar, extend v by 0 outside extend v to all of k by 0 outside omega. So v is given here which is a continuous function. So, you have is a continuous it is 0 here, v is 0 here. In this point, so, define v 0 here you extend it. And then this extended function so, extended v and we denote by the same v. So, you can apply v to which is a continuous function and then it is also different all properties you can see. So, I take any point x, y here so, I can connect this from here and this is the point from here.

So, and v is differentiable so, you can integrate. Therefore, v of x, y and v at this point is 0 here. So that you can apply the fundamental theorem of calculus you get to minus a to x. So, I am integrating with respect to x and you are differentiate the dv / dt t of y and integrate with respect to t. So, I can integrate and there is this is nothing but v x y - v - a 0 and which is 0 on the boundary any side you can take it any integration you can take it.

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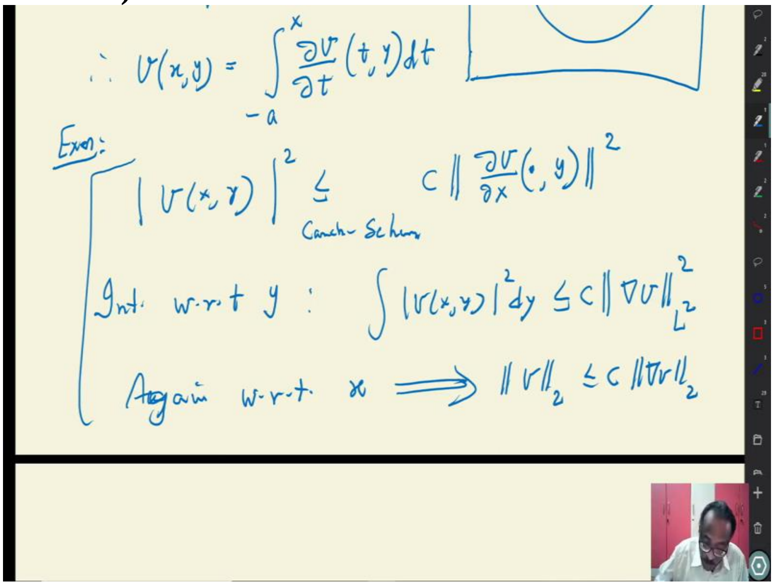

So you just if you want you can do a small exercise and you can easily see that modulus of v x, y is less than or equal to if you apply some constant you take applying Cauchy Schwarz and with one as other function, so you get some number. So, I will leave it as an exercise that you coefficients. So, I will get so if you get it as a function of which you are integrating here. So that first term there is no x anymore here and integrate and then I will get the square. So you what you get is that dv / dt.

That is nothing but the dv / dx at dot with respect to dot y square and this is with respect to x variable you are integrating. So this is so if you square it, you get this by Cauchy Schwarz. Now integrate with respect to y so, left side what you get it integral v x, y this exercise I am basically doing in dy it will be so u integrate with respect to y also already you made an integration with respect to X and that will give you less than or equal to some constant except i will come and 1 derivative only coming so that can be bounded by all derivative.

Now there is no x here do again, integrate with respect to X that is all. Please do this complete exercise with respect to x implies norm of v at L 2 is less than or equal to constant into grade v in L 2. So, we have all the setup except understanding X properly. So once we know how to define x, you have your Riesz representation theorem to basically prove this result so that is it.

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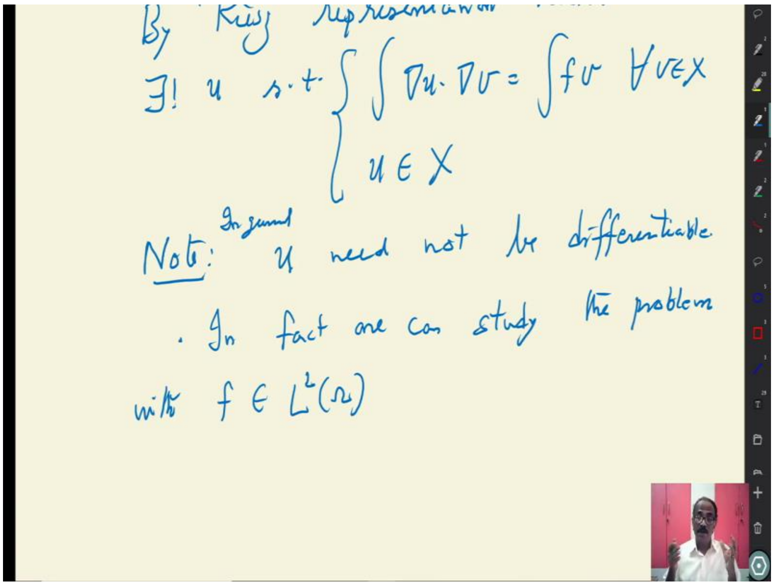

So then you can define so by Riesz representation theorem there exists a unique u such that grade u grade v = fv. This is true for all v in X now with u in X and this is very careful, u need not be a smooth function. So u is in X. So note u need not be differentiable u in general you need not be differentiable differential in general integral. So you need in fact one can study the problem with the more general with the general f with f in L 2. So, you see, the whole theory in L 2 of omega, the whole theory in the classical theory.

We assume the f is continuous to solve the problem. So, in this setup, you do not need it. So, in the generalised to set up, I can solve the problem with just f in L 2, in fact, I can have a bigger space a bigger problem, other Sobolev spaces, where it is need not even be function. So, I will be able to study such; a big formulations in very general theory like regularity.

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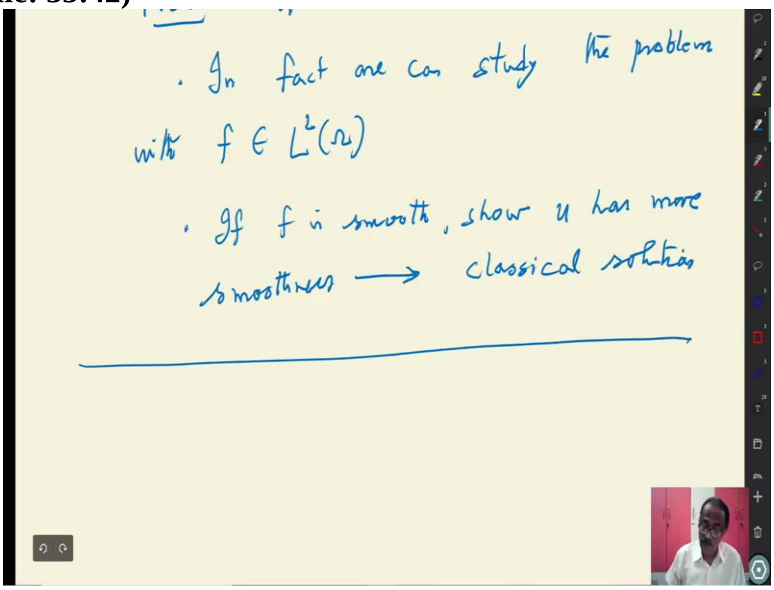

Then of course, if f is smooth show u have regularity, u have more smoothness this is what regularity results and then give classical solution. So, I will stop here this lecture. So, in the next class, what we will do we will define we will try to motivate you how to define weak derivative. So, as I said we will not be giving a complete lecture on distribution theory, but we will give you the glimpse of or the motivation behind the distribution theory in the next couple of lectures. So, which will motivate you to study PDE and the further analysis. Thank you.