First Course on Partial Differential Equations - II

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Lecture – 01
Weak Solutions 1: Hilbert Space Method

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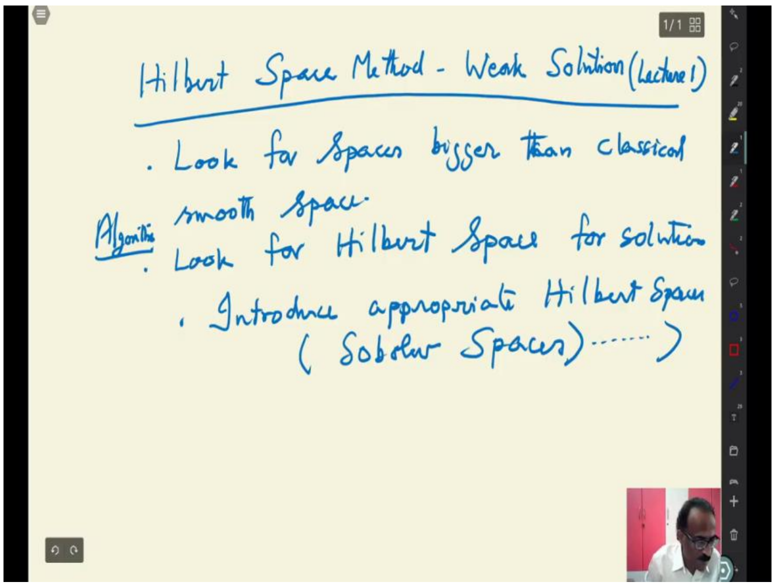

Good morning and welcome, in the present set of 4 lectures formulated a couple of hours you will see a set of lectures which are quite vague. The reason is that this is a setting is stage for a next course on partial differential equations. What we have seen is PDE 1 course we have covered some topics we extended many of the results it could not finish there and we have studied further on that one.

Now in the process of studying this PDE 1 and PDE 2 you would have observed that you would do not have the rather we do not have the comfort level of finding solutions in the usual differentiable spaces. On one hand you have a differential equations which are smooth you expect but what we have observed is that quite often you do not get solutions in the smooth class. The reason for that can be many but probably the reason is that the physical quantities which are formulating which have seen also minimization problems etcetera may not have the smoothness.

The PDE problem the PDE is the actual physical problems will not have smoothness at all differentiability property at all. But then you are formed modeling such physical problems in your partial differential equations. And if you are trying to solve that partial differential equations in a smooth class of functions, naturally you do not get solutions. So, this was an understanding right from the end of 19th century and the early part of the 20th century.

And even in our set of lectures you have seen when whether you are studying Conservation Laws and Hamilton Jacobi equations, we have some sense introduced some sort of weak solution without getting it into the integrities of the mathematical techniques. Even in this set of lectures, we cannot do that one. And that said basically, next course on partial differential equations, what is called an advanced PDE course. So, but we; want to give a glimpse of this theory and many theories developed lower in the last 120 to 250 years.

So, about understanding your function of PDE in the context of functional analysis and other mathematics developed. So, what we are going to so our lectures will be deliberately vague and many things you may not be very, clear about why we are doing that, because we do not have time or that is not expected of this course but at the same time, we want to blimp so some theory. So, one of the theories developed in the 1940's.

But no one shows maybe if that is in 1944 introduced the weak notion of a derivative, if a function is not differentiable in the classical sense, you can actually differentiate in a weaker sense using the theory of distributions. So, maybe in the last 2 lectures I will try to introduce the concept of weak derivative without again getting into the more functional analysis. But in the first lectures, let me give you in a slightly vague notions and this will be a little more clear when I introduce the other things.

So, that is; why the title is given what is happen is a Hilbert Space Method. So, what are we doing in trying to so, the motivation is very clear now. So, you want to look for spaces bigger than classical smooth, need not be Hilbert space all the time, but I am explaining thing and in fact that what I am going to explain everything in the setup of elliptic equation, but this is not restricted to only elliptic equation. So, this weak theory is can develop for other hyperbolic parabolic equations in simple fact.

The modern way of understanding is in that format. So, it should be bigger than classical smooth spaces. So, for example look for Hilbert spaces for solutions. That is so what is the algorithm we are looking at it. So, you are basically looking for an algorithm or a set of rules. So, look for Hilbert space for solutions and then you have to introduce appropriate Hilbert spaces. Later when you study advanced PDE, you will see what are called Sobolev spaces. I may indicate something later after a couple of hours, so whatever call it so there may be other spaces. This is one set of spaces where you are so you are looking for a solution in a Hilbert space.

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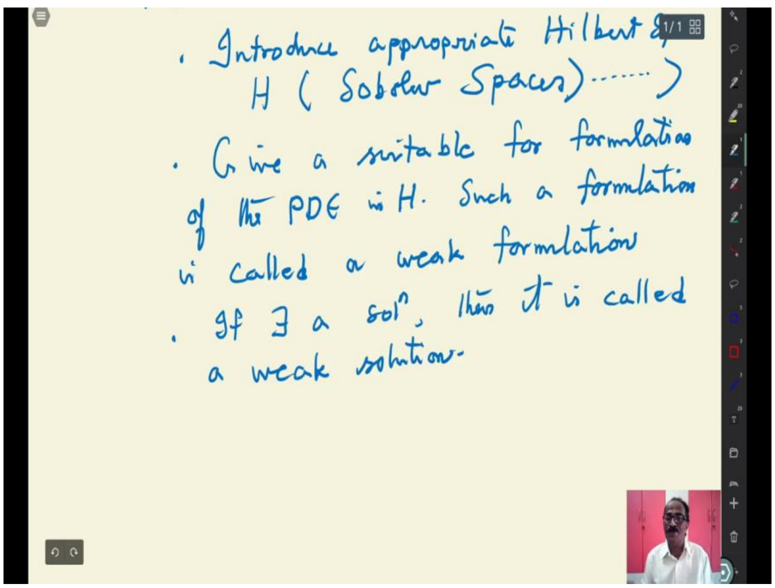

That next step so, I am giving you a very general algorithm. The next set of thing is to give a suitable formulation of the PDE suitable formulation. So, let me see Hilbert space H, as I said need not be Hilbert space other spaces. So, let me restrict to here give a suitable formulation of the problem on H, formulation of the PDE in H that is what is it is all vague such a formulation is called a weak formulation.

It should not have a, see it is not that when you have anything you choose any formulation you give and then you immediately say that this is my week formulation. That is not enough you have to have some relation to your PDE eventually. So that you will see that such a formulation is called a weak formulation. All these are difficult we do not know how to do it, but one of the things you have to see that we could as I said you can give some formulation and say that this is equal of them.

So, you have to check some equivalents, so naturally what is it when you say that it is a weak solution and the corresponding formulation. If you have a solution is called, if there exists a solution to that formulation is called a solution then it is called a weak solution. So, by the name weak solutions suggest it should be weak then your strong solution classical solution.

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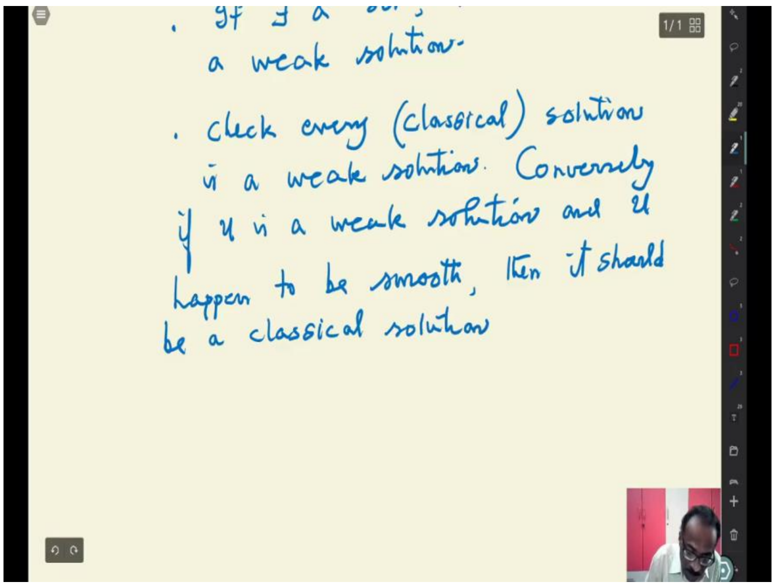

So, the first thing you have to check every classical solution he said every classical I will not meet all the time when I say simply solution it is a classical solution. So, I will not use this but say every classical solution is a weak solution. This is what you should be checking for sure. And then classical solution somewhat connect even if you get a weak solution you should be connected to you are this is one way checking every classical solution is a weak solution. But even if you get a weak solution, you have some interpretation.

Conversely, if u is I am not even prescribed in the, if u is a weak solution. And somehow it happens that; u is happens to be smooth. So, the week solution may not give u is smooth, but somehow after fighting a weak solution from the weak solution and it happened to be a smooth. So, you have a weak solution, it is smooth enough to verify your classical thing then it should be a classical solution. Sufficiently smooth according to PDE is the first order PDE should be once differentiable like that.

Then it should be a, this is very important to see the equivalent, it should be a classical solution, this you should have whenever you form a weak theory, since the strong theory should be a strong theory you will see weak theory and then week theory has more facility classical solution. So, these are the kind of steps you have to prove it.

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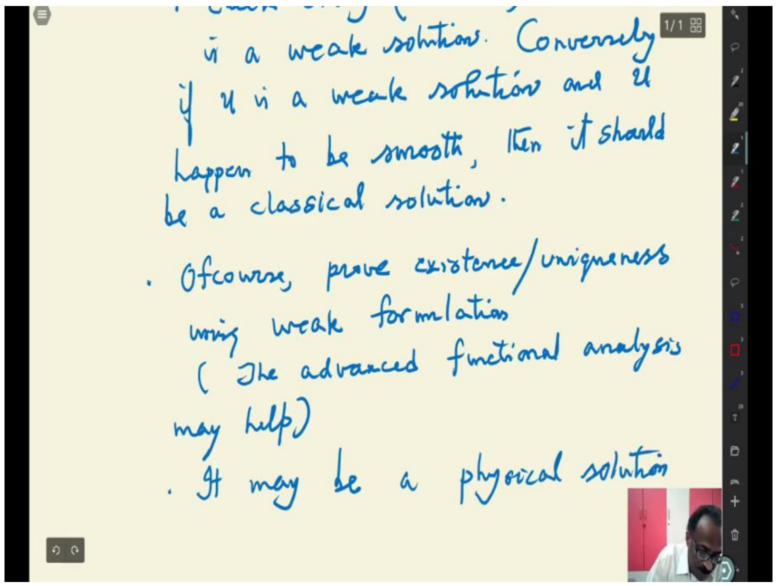

So, and of course then prove existence uniqueness using weak formulation. So, in this direction there are advantages one the functional analysis the advanced functional analysis may help. More than that it may be a actual physical solution I told you the few classic, many of the physical solutions may not be smooth enough but in a weak solution setup it may be a physical solution. I thing I will need more pages I will add one page is here for you. So, we have a set of problem so it may be a physical solution.

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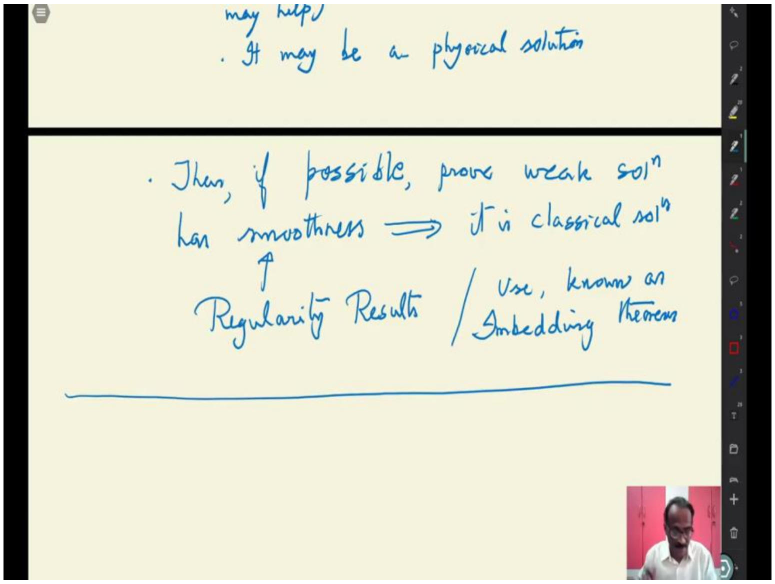

Then it is not how do you prove then? If possible that is a thing if possible prove week solution has smoothness. That is so by the earlier argument very smooth weak solution and such results smoothness that will imply it is a classical solution. So, this is also a method to prove classical solution. First proving solution may be easy but you have functional analysis available but then prove that and then smoothness and then show that then it will become a classical solution.

And proving such kind of smoothness is known as regularity estimates, regularity theorems. So, after getting in some Hilbert space show that it is a elder continuous it is twice differentiable and all that regularity results. And this is quite often use a notation use what are known as imbedding theorems. You imbed certain spaces in general in other spaces. We will not do this one in this class. But what I am saying is that what is again, so this is the basic algorithm which you follow in a, this is a very general setup I am going to do it.

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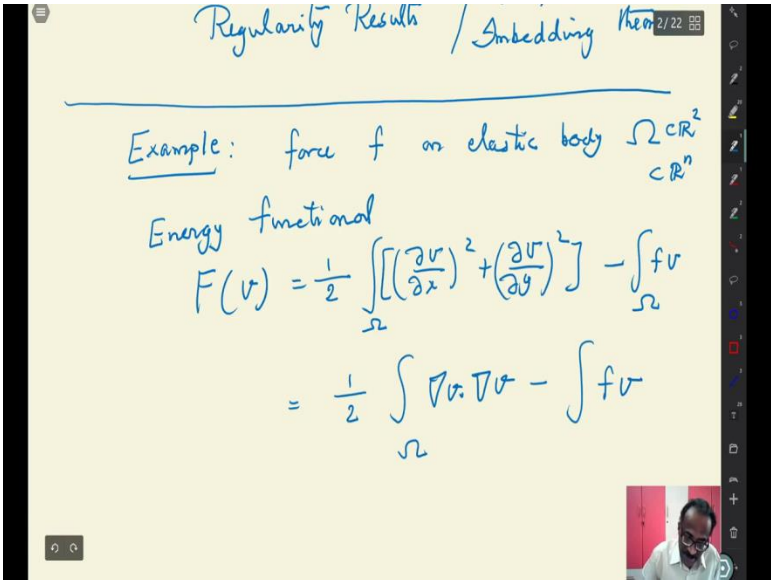

So, now let me start with some example to motivate you further. So, I want to how this week solutions are quite often physical. In the earlier lectures of calculus of variations, we had given minimization problems, and minimization problems you may not have second derivative. And then you same probably you will see it is a equations or Hamilton Jacobi equation which are all PDE if u is smooth and then you are seen that you may not be smooth and you prove that Lipschitz continuous solutions and uniqueness in that one.

So, you are seen so, already such a results so, we are going to given another result thing. So, you are applying a force F on an elastic domain an elastic body omega. So, I am taking omega is for simplicity you can take up 3 and 4 and for simplicity I am taking so, it is not necessary it can be R n. And then what is the energy functional, so you can introduce what is called an energy functional F of v = half of this the kind of strain energy dv / dx square + dv / dy square - integral of f of v inside of it I am not even putting conditions now.

This is also you can write it as in general even in higher domain this is nothing but grade v dot grade v – integral f of v.

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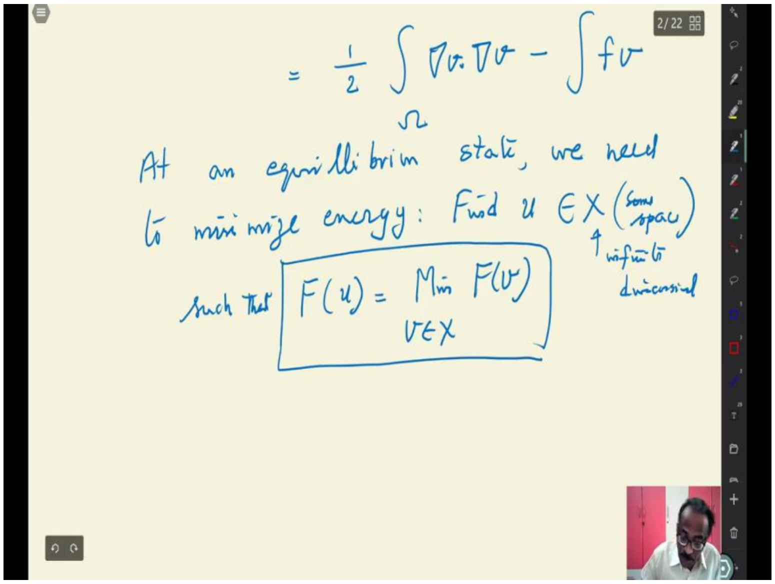

So, this is your, these are typical energy functions in an elastic body. So, I think equilibrium state the energy at an equilibrium state we need to minimize energy. In other words find u, we do not know where to find u is some space these are all the problem that to do it some space. I will typically take some X later to show you some analysis. So, find u in X such that F of u this we have already seen that what is in the other minimization problems of trajectory minimization.

So, you at see it but now, in this case this will be an infinite dimensional space in general in finite dimensional, so, we are doing something more than finite dimensional which you have seen it in the Euler Lagrange equations and the calculus of variations. So, you want to minimize F of v for v in X. So, this is what you are looking for at all, look at everything you can specify. Now what do you do?

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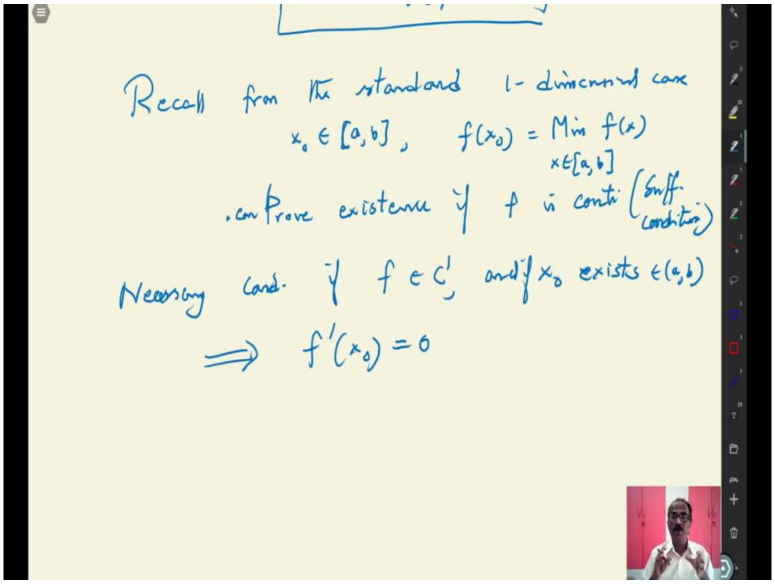

Let us call the one dimensional equation which I have explained recall from n = 1 from the standard one dimensional, these are already explained later but I am repeating case, suppose you want to find x 0 in an interval a b and you want to find f of x 0 = minimum of f of x is x is in to a b. So, what do you do? You can prove existence not the uniqueness of course there will be minimal, can prove existence if f is continuous, that is fine, but how do you find the solution existence if f is continuous. So, this is a sufficient condition but then these only they are you want to find the minimum thing.

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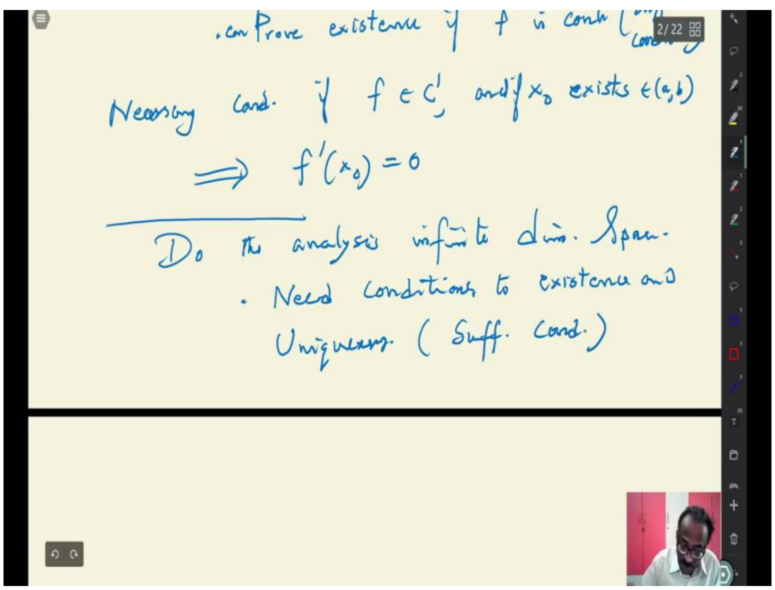

Then you have also a necessary condition. If f prime f is C 1, and x 0 is exist and x 0 exist, you are not proving x 0 exist, if f is even, and if it x 0 exist is in interval internally, you have all this condition, then that implies your f prime of x 0 = 0. So, you see so the f prime of x 0 = 0 that is all critical points are where so, if x 0 is a minimum, it is a critical point and as you know that critical points need not be minimal.

So, it is only a necessary condition in general, whatever it is and this is what we want to go it. So, you want to do this analysis in infinite dimensional space. In other words, first you prove under conditions that f is the minimum that is the first term. So, you need to have conditions on f to prove the minimum. So that jogs are too. So, need conditions which I am not going to do on this that is a convex analysis and other analysis you can do that need conditions to prove existence uniqueness of u.

Existence and that is not my aim is here with your uniqueness and this is its uniqueness. So, this is something like a sufficient condition you write it, there are conditions we know that.

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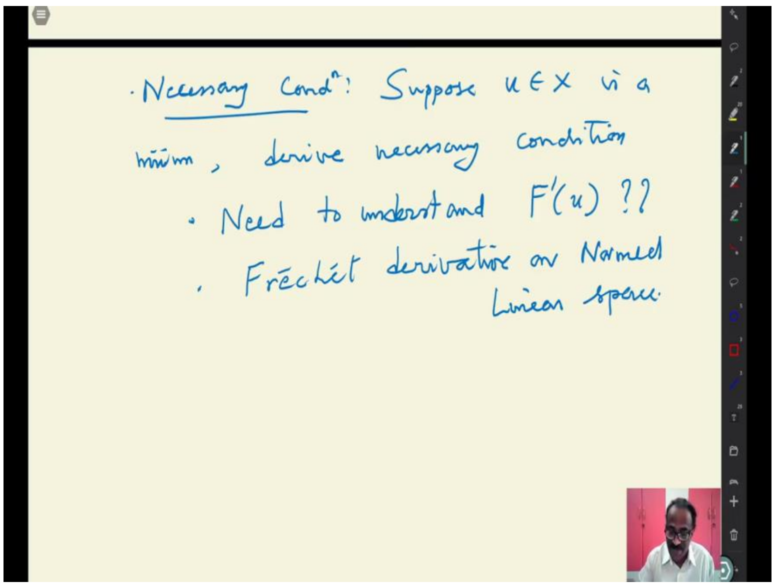

So, but what my interest here, so suppose necessary condition in this set up necessary condition, suppose u is a minimum we do not know yet suppose u in X is a minimum derive necessary condition. So, going with the one dimension case, you need to understand the derivative so, that means need to understand F prime u. So, these are there is a notion called the Frechit derivative on normed linear spaces.

So, you need to understand, so, I am doing without introducing I am trying to drive, I said this is going to be I am not going to introduce all these concepts, but I am doing it in a slightly vague way, but it is more understandable without introducing all these notions. So, we will start so, we take as a special case take for example, you need to define a F of v for example, to define this integral you need v to b a grade v to be a L 2 function also you need v to be differentiable. So, you keep that minimum requirement and then to avoid any boundary conditions on the thing.

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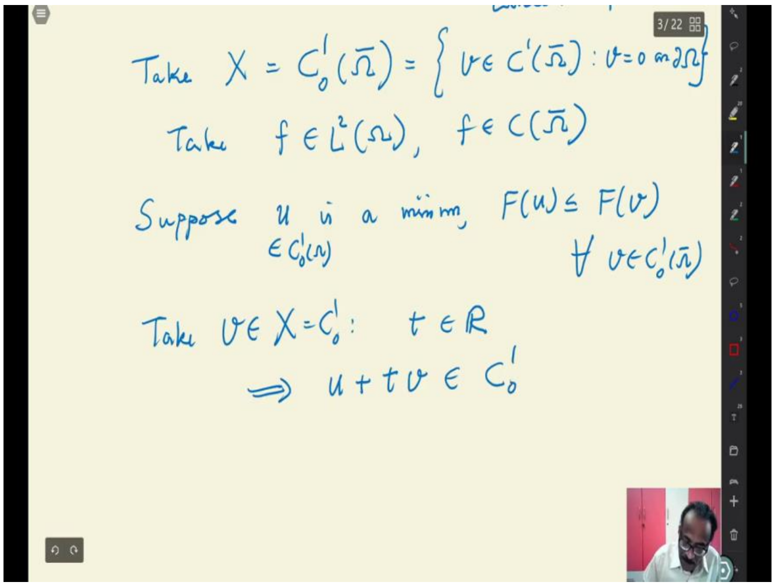

So, I will take I start with a space X = C 1 0 of omega bar to avoid any difficulty, so that the integral should be meaningful. So, this is a set F of v that is the minimum thing from the definition my definition to be valid for v, and then F should be some L 2 function of continuous and you can take it and that with the C 1 0 means and v = 0 on d omega. So, you may ask why taken because this is the minimum so that my integral and maybe take f is an L 2 function if necessary that is enough.

Because v is a continuous function and so, hence it will be integral so, f is in L 2 so that or you can take f equal to be a continuous function you want it no problem so, or not this function. So, I want to understand my variations. So, if I take my v that way, so suppose u is a solution so, I leave it as an exercise for u before they come suppose u is the solution minimum u is a minimum then F of u is a minimum in C 1 0 of omega bar and that is x here then a F of u will be less than equal to F of v for all v in the C 1 0 of omega bar that is true.

So, I will compute in some so, you take v arbitrary v in X the C 1 of 0 and tv remember t is in r that implies my u is the minimum my u + tv is also in C 1.

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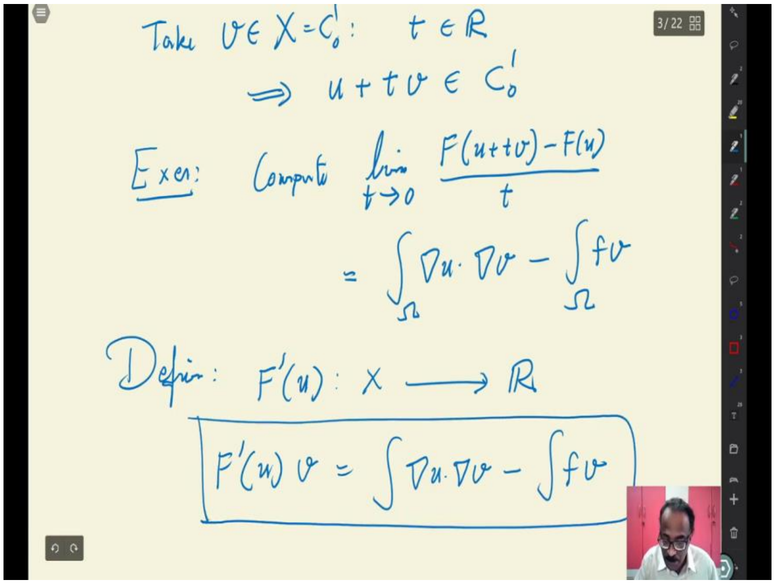

So, my jog what I want to do, this is your exercise fully compute this limit of t tends to 0 or F of u + tv - F of u / t you compute this by t is nothing but integral of grade u dot grade v – omega you see integral f of v of omega. So, my definition I will define so I can define anything, so I will define F prime of u, given u is fixed, u is the minimum solution F prime of u is defined to be a mapping from here to here prove this you can prove using the mean at thing F of u.

You can exactly get this compute to this limit t tends to 0. So, F prime of u this is my derivative I am defining a derivative in this fashion evaluated of v = grade u dot grade v – integral f of v. So, what I will now immediately know because u is the minimum so, when t is positive, this is you can actually this is your derivative. So, you want your derivative to be 0 at a minimum point. So, we want it.

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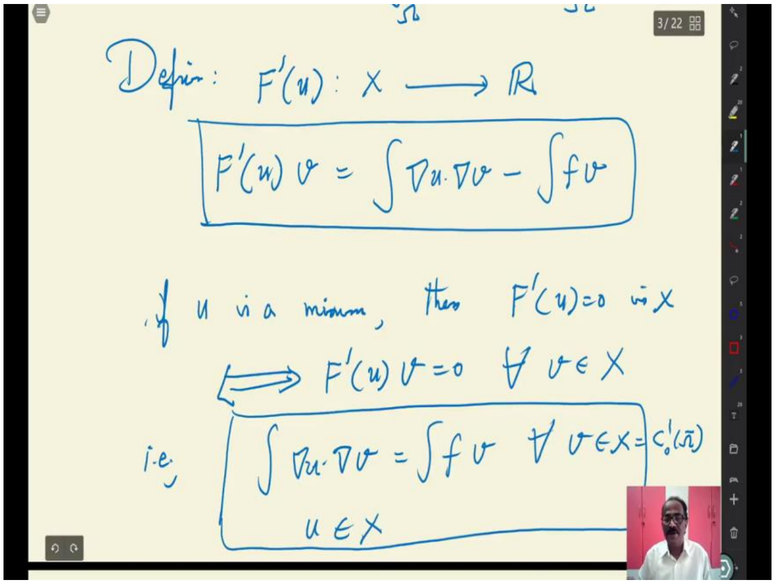

So, if u is a minimum then F prime of u = 0 in X equally in F prime of u v = 0 for all that is when a function F is 0, f x = 0 where a functional is 0 it will action on that will be 0 into X. That is you get an another formulation, integral grade u grade v = f of v, for all v in X so, you have a different formulation now and u is in X so, you have a new formulation if u is a solution, then you have this is equal to your what you are seeing is F prime of x = 0 and then you look for a solution here. So, we will discuss more, this is a problem to be solved X = C 1 0 omega bar.

So, next class we will tell you it is connection to your Poisson equation, when this represented by a Poisson equation what is additional thing you need to get Poisson equation from here. So, and this will turned out to be you one way of week formulation of your Laplace equation but the solubility of this is what eventually will lead to have a new mathematics to be dealt which we will do a little bit, but not too much. Thank you.