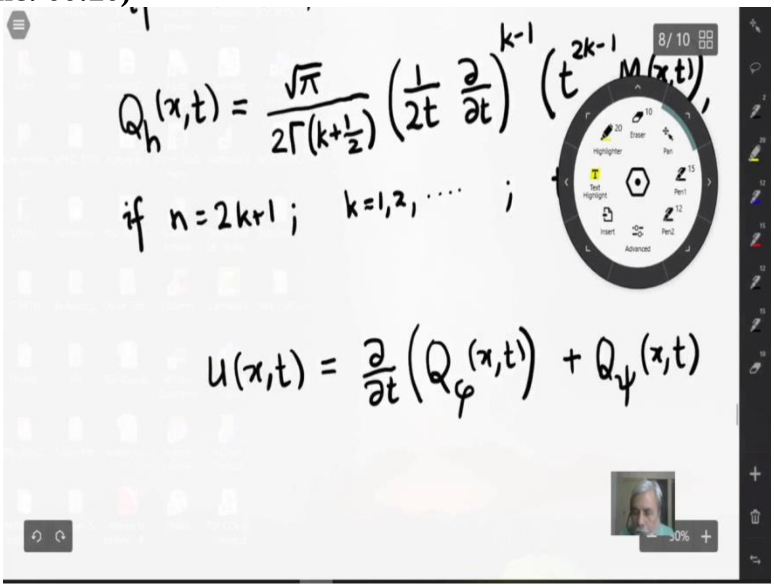
First Course on Partial Differential Equations - II
Prof. A.K. Nandakumaran
Department of Mathematics
Indian Institute of Science - Bengaluru
Prof. P. S. Datti
Former Faculty, TIFR – CAM - Bengaluru

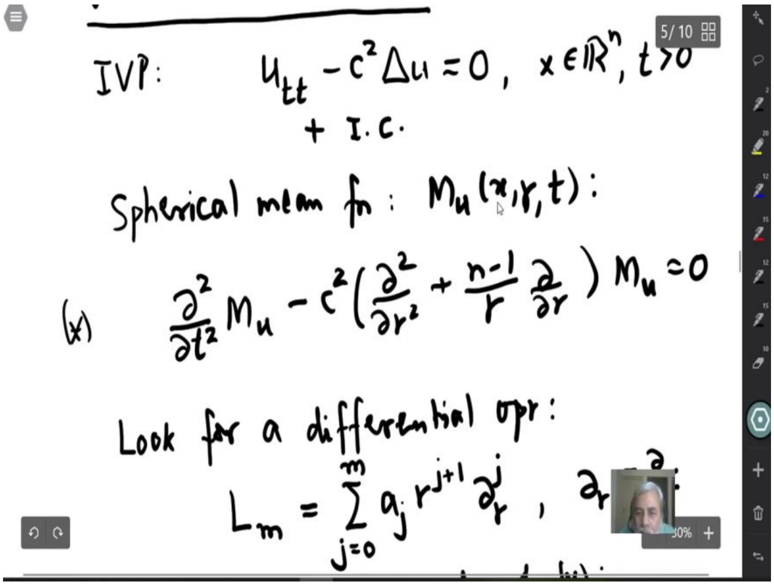
Lecture - 35
Wave Equation 6

(Refer Slide Time: 00:20)



Welcome back in the previous class we had discussed in detail the wave equation in 3 dimensions by the method of spherical means and then we derived the formula for the 2 dimensional wave equations using the method of descent. And to continue again the method of descent we also derive the formula for the solution of the telegraph equation in terms of Bessel function and we started the discussion on what about the case of n bigger than 3. So now continue the discussion on that.

(Refer Slide Time: 01:55)



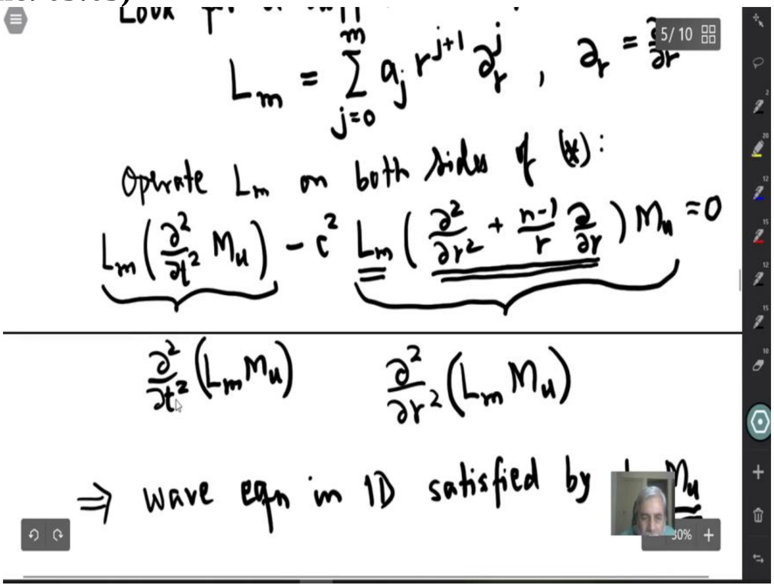
So more or less the ideas are already there even in the case of n = 3 so we continue the same discussion. So now consider again the initial value problem for the wave equation now when R n. So equation for x belongs to R n and t positive and prescribed initial conditions. So what we did for n = 3 So we consider this spherical mean function of u and then we observed that if u satisfy the wave equation this M u satisfies the second order equation but important reduction here is this only 2 variables.

So the t and r is the real variable that is introduced through the spherical mean function. So again I stress that this as far as the M u is concerned this x variable plays only the role of a parameter so it will not come in the analysis of this M u. So, M u the main thing is so satisfies a second order equation only in 2 variables. And what do in case of n = 3, so by a simple transformation we could remove this first order derivative with respect to r.

So that reduce this equation to 1 dimensional wave equation and then we can use the D'Alembert's formula etcetera. So we try to do the same thing again not try to remove somehow this first order part so namely n - 1 / r d / dr. And for that now we use as earlier it was only just a multiplication of M u / r. So, r M u that did the job of removing this first order thing and obviously know that not sufficient. So we look for a differential operator so this is r so used to variable coefficients.

So the coefficients r powers of r source a variable coefficient operator. And now let us see what we want this L m to do. So we operate L m on both sides of this Euler Poisson dot equation satisfied by M u. And again this is a linear operator and the equation is linear so everything is fine here.

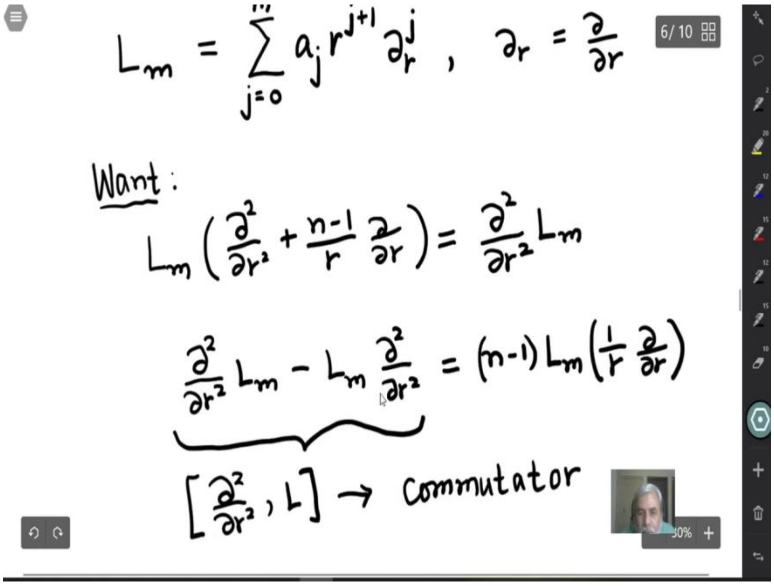
(Refer Slide Time: 05:03)



So since r and t are independent variables so this second derivative with respect to t I can just bring it out but this one so this is also a differential operator with respect to r. So in r variable and this is also a different second order differential operator with respect to r and since these are variable coefficients so generally they do not commute. So this I cannot push this L m that is just like we did here so this L m d square / t square they commute.

So I can just push this L m inside but that is not possible here. But somehow we want this L m to do that job. So when I push this L m inside in the outside I just want to have a second derivative with respect to r only. And so what makes it that possible whether this is possible or not we have to see in detail so again I am writing that thing this is what we want so just look at here.

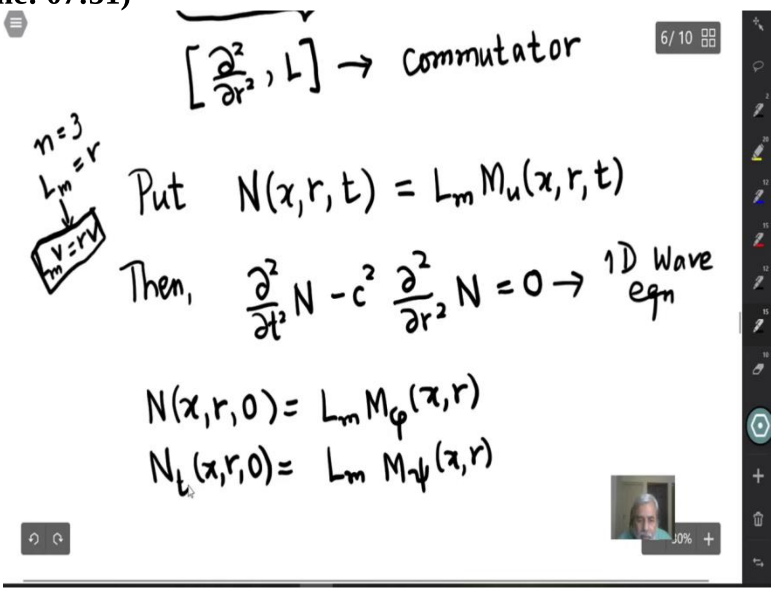
(Refer Slide Time: 06:35)



We want this so L m so this is a differential operator, this is a differential operator, we want this and you rewrite now so this how to see these things so you just operate on any smooth function so I want to the same quality as operators. And the rewrite this one so this L m d square / dr square you take the other side so d square / dr sqaure L m - L m d square / dr square and the n - 1 is a constant.

So this is just again linear operators so n - 1 L m 1 / r d / dr. And this left hand side is denoted by the square bracket and it is called commutator of these 2 operators d square / dr square and similar to commutator of 2 matrices suppose that is possible. Somehow we have found this L m support that is possible then what happens? Then it becomes very easy.

(Refer Slide Time: 07:51)

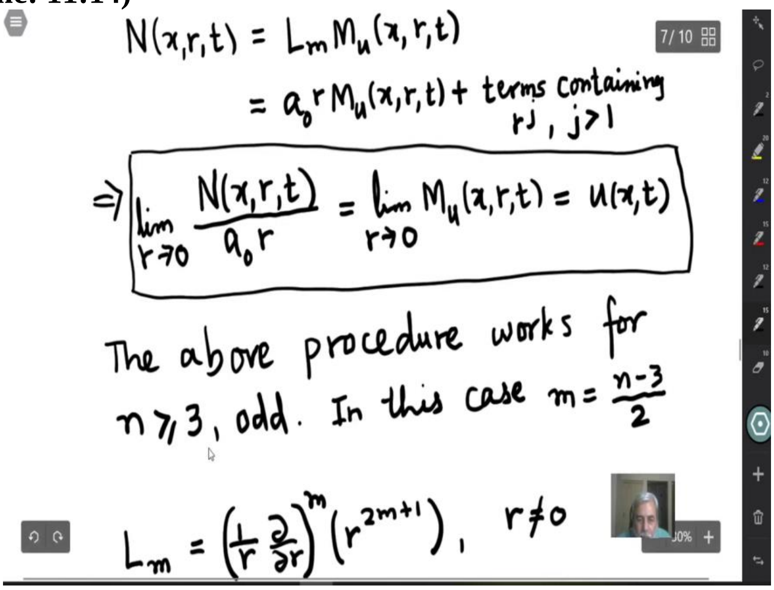


So now you define again N x r t. N as a function of x r t that L m operating on this M u x r t. So in case of n = 3 so L m is just nothing but multiplication by r. So this means L m if I operate on function v it is just r. So in this case it is a differential operator. L m M u x r t so assuming this commutator relation between these 2 operators is satisfied then again go back to this computation.

You see that this n satisfies the 1 dimensional wave equation and what about the initial conditions on M? They are just coming from this M u so N of x, r, 0 t = 0 you get L m, M psi x r and the first derivative with respect to t at t = 0 again in terms of M psi. The only difference is this you have to operate this L m whatever that L m. And then of course so since now this is just 1 dimensional wave equation so you just apply numbers we just apply D'Alembert's formula to obtain N.

But our main concern is one to obtain u and that is again easy. So you just see what N is so N is N of x r t that is our definition of N. So if that is L m M u acting on M u and we are taking L m and this differential operator. So you first write the order so there is no derivative here.

(Refer Slide Time: 11:14)

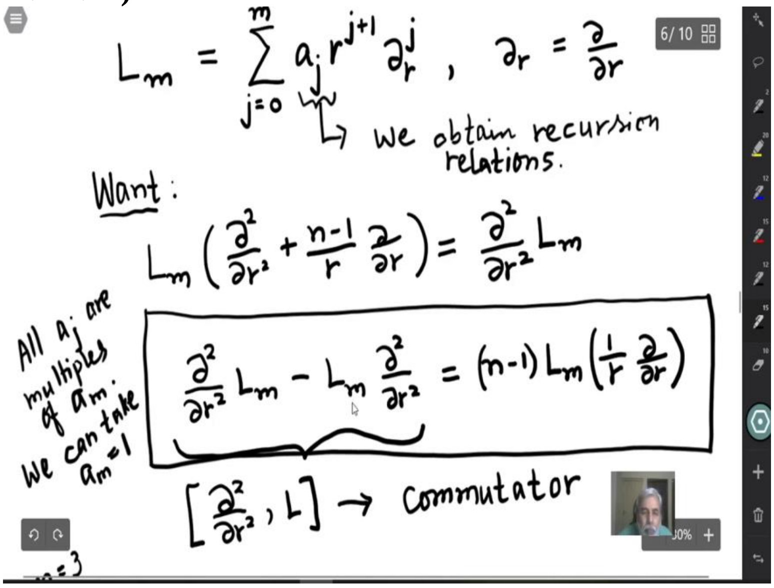


So you see these r j + 1 del r j so if I put j = 0 only r remains there. So a 0 r M u x r t plus terms containing r power j, j bigger than 1 j is bigger 1. So if I divide by a 0 r on both sides then I get this N x r t / a 0 r that is M u plus terms containing powers of r j, j bigger than equal to 1. Bigger than dividing by r so at least there will be r power 1 and higher powers. So this only M u remains when I take the limit r tends to 0 all these terms vanish and only this M u remains and you know that this limit is r tends to 0 of M u is u x t so this is very, very nice.

So once you get hold of this L m so it is possible to obtain the solution of the wave equation in terms of this N by some easy process. And it turns out that this above procedure works for n bigger than equal to 3 but only when n is odd and in this case we also see that m = n - 3 by some. So I am avoiding lengthy and somewhat tedious calculations what one should do is look at this commutator relation so this is our requirement.

So you just substitute this L m in this form and work out the details they are straightforward but very tedious calculation. So the first thing we notice that so these coefficients essentially we have to determine these coefficients.

(Refer Slide Time: 14:01)

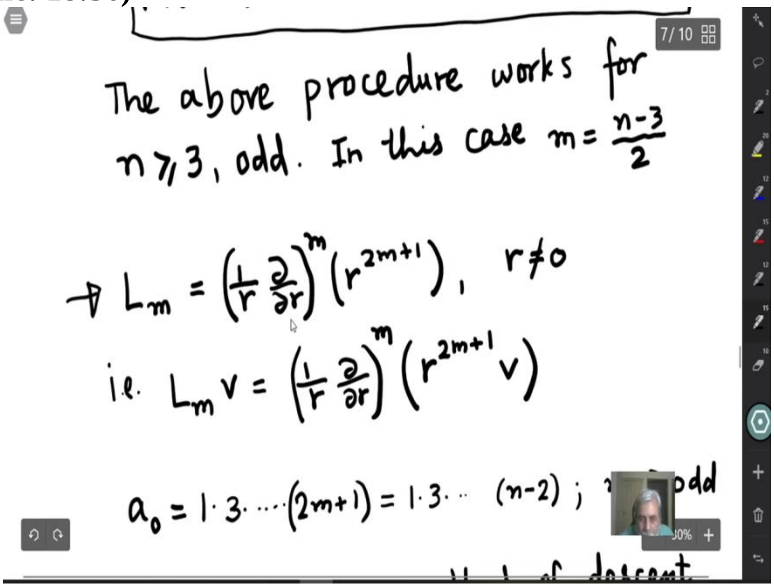


So, we obtain using this commutative relation we obtain recursion relation satisfied by these coefficients. So, when you substitute this and then you equate the like powers of the operator this differential operator del r j both sides so we obtain recursion relations. So all will be multiples of so all A j are multiples of a m. So that means you can take a m = 1 as I said I am not doing the computations it is lengthy computations.

So, it can just look into our book there are some exercises they are straightforward but lots of calculations so you have to use Leibniz rule for the differentiation etcetera. So from the beginning we take a m = 1 in this operator and then all a j they satisfy recursion relations so they can be obtained one by one in terms of a m. And it is not easy you cannot easily write down the formulas for a j you see lots of binomial coefficients with positive sign negative signs.

So they are complicated expressions but what we require is again look at the end result we only need the coefficient a 0. We do not require the intermediate coefficients the exact values of the coefficient you do not need and the first relation we obtain is this. N must be r because we obtain this n - 1, n - 3 that means we obtain 2 m + 3 = n that forces n should be odd and easy. So, this works only for odd n which are together. So you see here again when n = 3 m is 0 so it is just L m is just the multiplication operator and when n is bigger than 3 we get more and more terms that is fine.

(Refer Slide Time: 18:36)

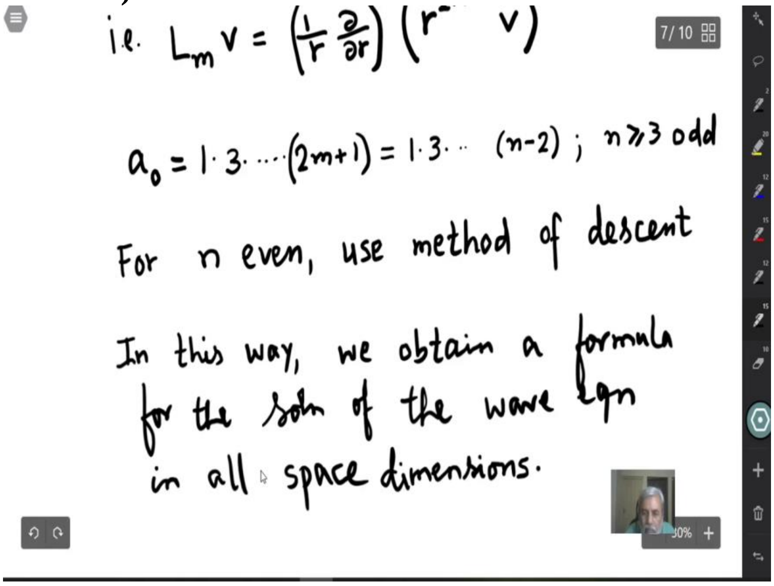


So we have to just obtain in a 0 so one more computation we have to do that. So it is again most of these arguments follow from induction process. So we have to do by induction and that L m can be easily expressed not easily but you have to some work in a need for so L m turns out to be this operator 1 / r d / dr to the power m into this r 2 m + 1. So, since we are dividing by r is r is not equal to 0 you have to take but again we expand it is preside the up this form.

So, whatever r factors come in the denominator they all get cancelled. So it is of that r so again this is proved using an index. So this means so when you operate on a smooth function v so L m of v is given by 1 / r d / dr of this function r to the power 2 m +1. So v is a smooth function of r and in this form we readily see that so this we want to see a 0 coefficient that a 0 is given by 1 into 3 into 5.

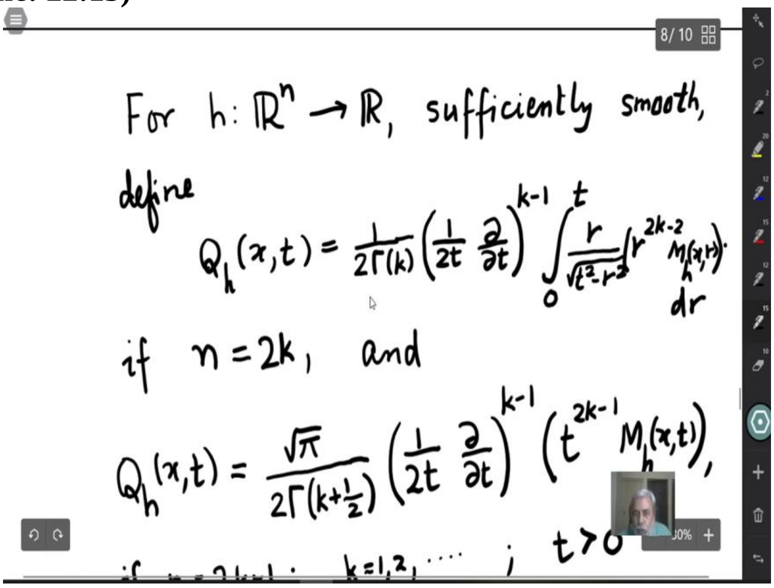
So all odd integers starting from 1 up to 2m + 1 and now we know that m is n - 3 / 2 so, you can express this in terms of n minus this 2m + 1 you can write as n - 2 and remember again your n is bigger equal to 3 and r. So once we obtain the solution of the wave equation for n bigger than equal to 3 and r and again we can use the method of the descent Hadamard method of descent and then obtain the solution for n even us.

(Refer Slide Time: 20:51)



So in this way we obtain a formula for the solution of the wave equation in all space dimensions. Of course when in case of n = 3 and n = 2 then the formulas are not simple compared to for example D’Alembert’s formula is very easy to handle but not in higher dimensions.

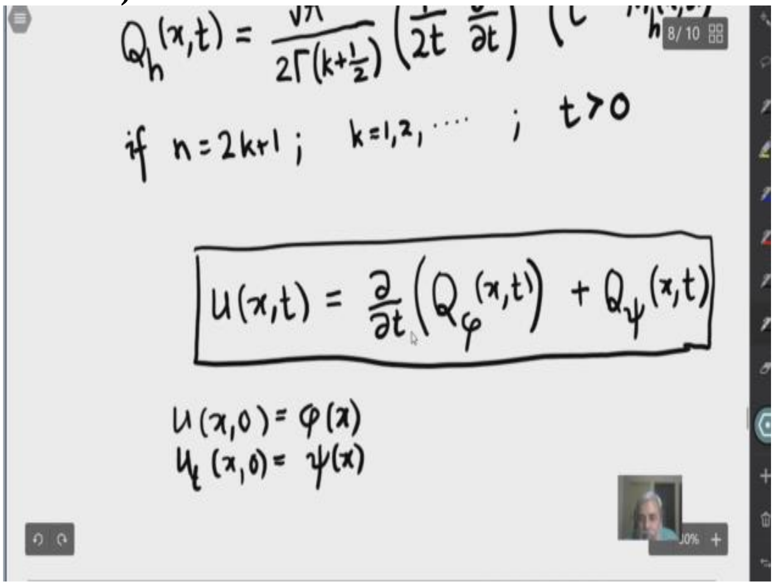
(Refer Slide Time: 22:15)



So now we summarise so that we write it in neat formula. So, we hide all the complications in a symbol and write the solution. So for a smooth function h from R n to R define these quantities so, again this is a function of x and t. So, when n is an even integer n = 2k showing defined Q h x t as 1 / 2 gamma k and this complicated operator so and this you remember it also comes here same thing now r is replaced by t so 1 / 2t d / dt to the power k - 1.

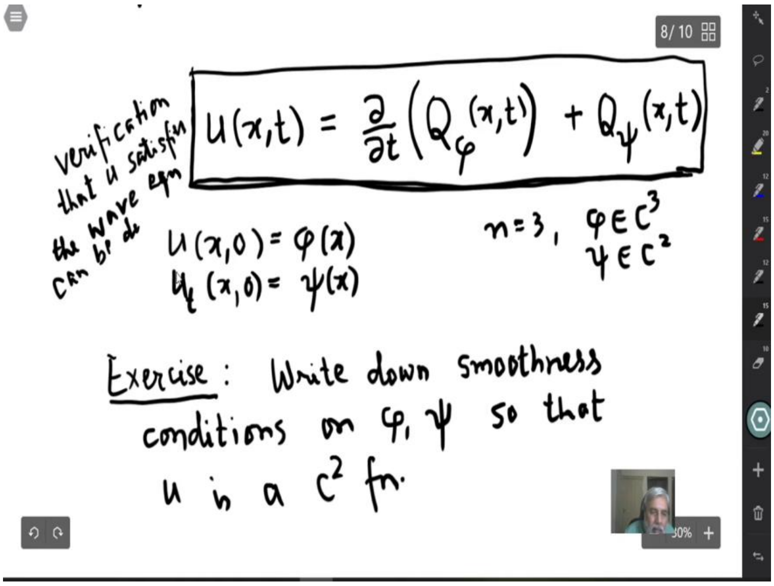
So, even in division you also see that is more complicated case and this integral is in one. And for odd n so when n = 2k + 1 so Q h has a different expression. So this is integral and this is Q h defined for the positive and for all k 1, 2 etcetera.

(Refer Slide Time: 23:48)



And then the solution of the wave equation is simply expressed as this. So as usual again so the initial conditions u x 0 is phi x, u t x 0 is psi x. So it is important to notice that so no matter what the dimension is the form of the solution is very, very same. So it was in one function Q phi and one derivative and another Q psi. So this at least the final form of the solution looks simple. But the expression for Q phi Q psi they are not simple. They involve these complicated derivatives and in case of even in even dimensions you also have submitted.

(Refer Slide Time: 25:20)

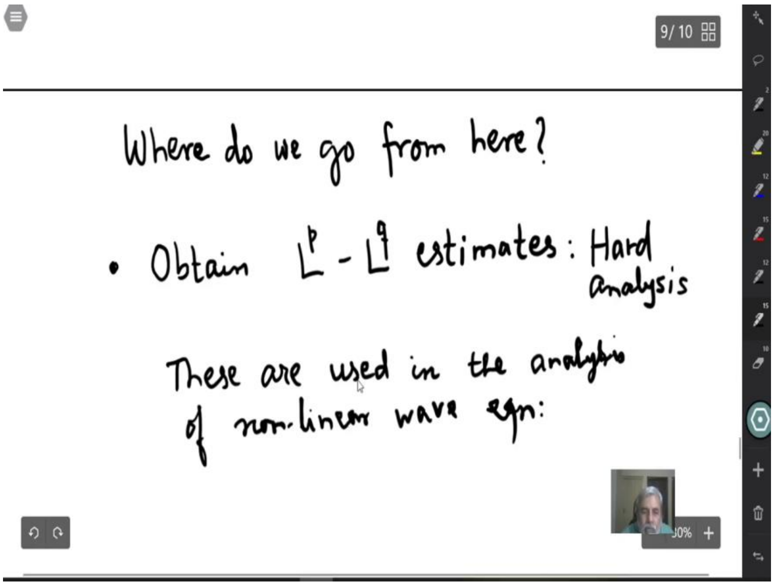


So here we get an exercise so write down smoothness conditions so I have not written there. Again smoothness conditions depend on whether n is odd or even on phi and psi so that looking for a classical solution. So that u is a C 2 function we are looking for a classical solution so obviously we want you to be at least C 2. And the smoothness conditions on phi and psi are determined by the formula Q h with this final Q phi and Q psi should be well defined.

And obviously phi requires one extra smoothness conditions that psi because there is a derivative involved with the formula that just like the 3 dimensional disorder. That n = 3 that is what we saw phi C 3 and psi C 2 and look at the expressions on this Q and figure out what smoothness on phi and psi are required. So that the solution is issued to function again the verification that direct verification so for given phi and psi, I can do to find u by this formula and verification is can be done that u satisfy the wave equation.

And that is struggle little bit derivatives and other things so this Q phi and Q psi they already involved. So many derivatives so one has to take care of all those things, but it can be done. So I do not want to go into details because they require lots of computations. You can do them very leisurely. So that is and from here once we thought the formula for homogeneous wave equation using D’Alembert’s principle we can also handle in homogeneous equation it is so that is not a problem. And what is next?

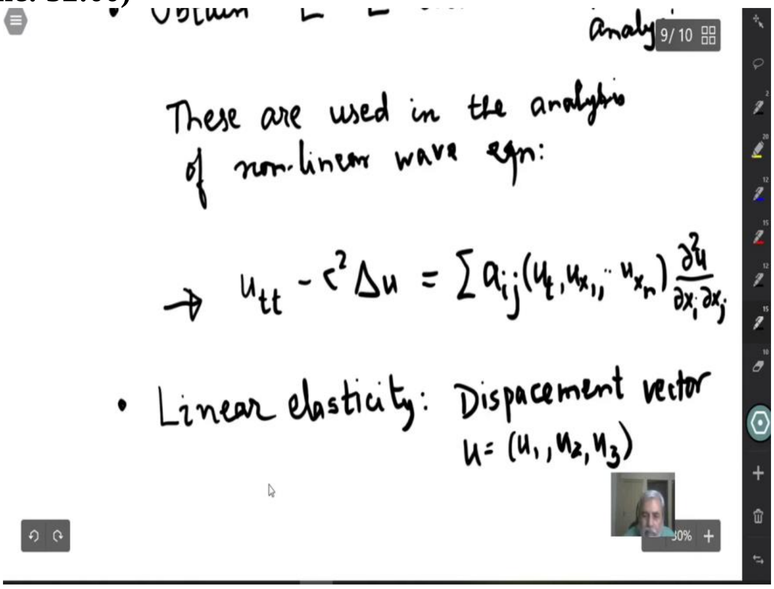
(Refer Slide Time: 29:25)



Where we go from here? We have obtained formulas for the solution. What is next? Now that question cannot be asked. I just want to remark on 2 things here. So obtain L p - L q estimates. So, assuming that the initial data are in some L p space p can be cube so for various range of p and q. So, obtain similar estimate for the solution using the Kirchhoff formula.

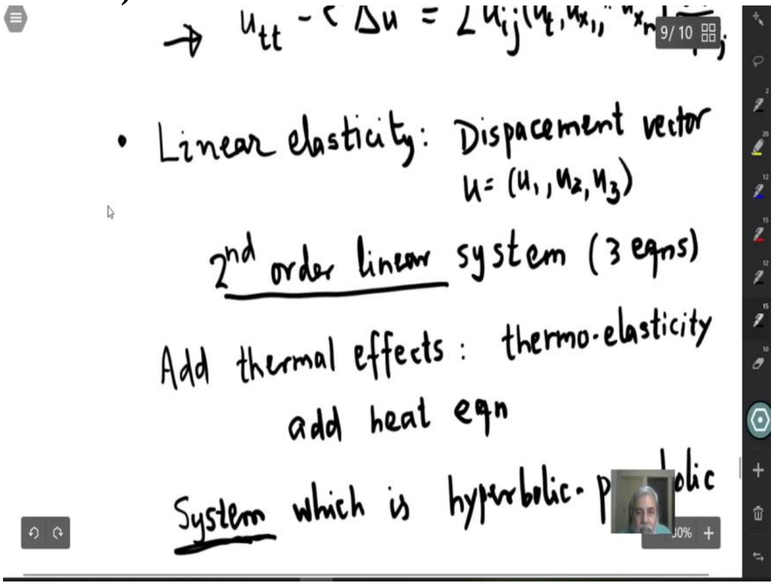
And this is already hard analysis so, not at all even though we have an explicit formula for the solution obtaining such estimates is not at all easy. And where are these used? These are used in the analysis of non-linear wave equation. So, which are perturbations of the wave equation.

(Refer Slide Time: 32:00)



So, we have this part and now you perturbate by this a ij. So, these are functions of the first derivatives, u and x have this. So, the analysis here means want to prove the existence of a solution and for how long it exists whether it exists for all time or not all those questions and in the analysis of that these L p estimates play an important role. And the second one I want to mention so for this is linear elasticity. So, the displacement vector so this is in the physical space r 3 and the displacement vector u also has 3 components.

(Refer Slide Time: 33:43)



And this is again given by a second order linear system. So, now it is no more an equation but it is a system. A system of 3 equations they are all coupled between this u 1 and u 2, u 3. But some by suitable transformation we can reduce those equations to wave equations. So, again the formula we derived comes handy even in handling this system of linear elasticity. And there are also studies you can see in the literature. So, now you add thermal effects so this is generally referred to as thermal elasticity in the literature.

We want to see the effect of heating the elastic material and then you add a heat equation. Apart from that system you also have add heat equation. It is no longer cosine system but it is no longer hyperbolic. So now you have a system which is mixed one hyperbolic parabolic. So, we have to combine the techniques of both the wave equation and the heat equation to analyse systems.

Not only our systems; so far we are not talked about systems at all our single equations. So that is what I just mentioned 2 things here, There are many, more things and in the next class we will analyse an example of a mixed problem for the wave equation. Thank you.