First Course on Partial Differential Equations - II

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Lecture - 34
Wave Equation 5

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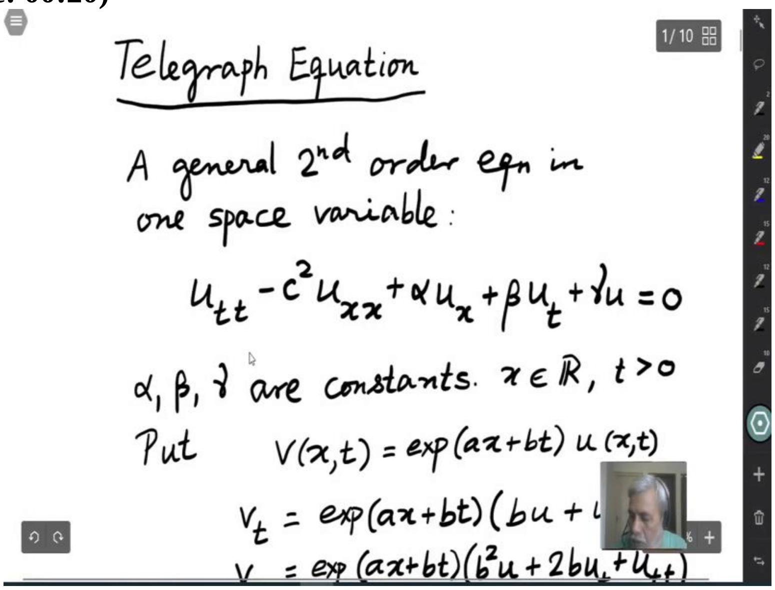

Hello everyone, welcome back we continue discussion on wave equation in more than one dimension. In this lecture we discuss another interesting application of method of descent to the so called telegraph equation. So, now we are going to descent from 2; dimensional wave equation; to a second order equation in one space dimensions. So, let me start with the problem. So, let me start with the general second order equation. So, the principal part that is the terms containing the highest derivative tilde wave operator.

And now, we add these lower order terms alpha u u x + beta u u t + gamma u u = 0, where all alpha, beta, gamma are constants. So, we make some simplifications. Some simplifications are possible here. So, we can get rid of certain terms by simple change of variables. So, for that purpose so, we put V of x t is equal to exponential of ax + bt into u u of x t. So, u u satisfied this given equation and now we will see what this V satisfies if we choose appropriate constants a and b so, a and b are constants. So, simple differentiation, so product rule and Lebanese rule.

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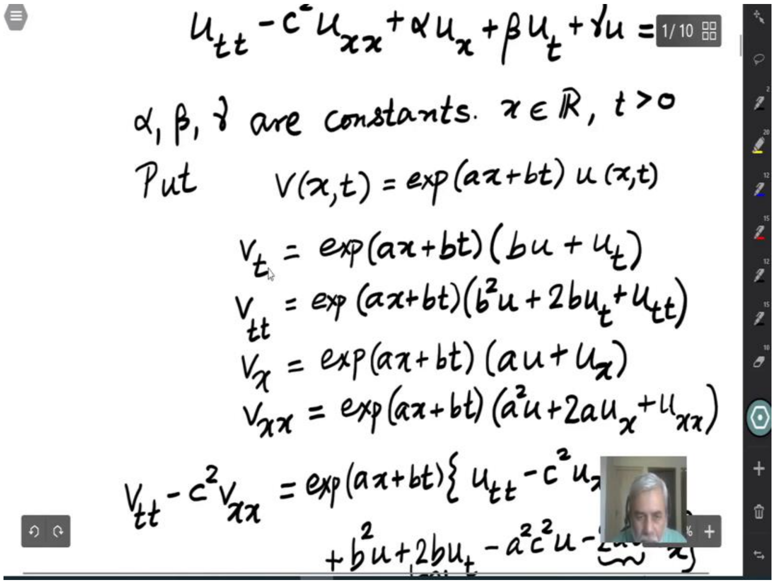

So, we find these derivatives of V with respect to t and x. So, they are very easily computable. So, here are the expressions for the first and second derivatives of V in terms of those of u. And now, let us form this V tt – c square V xx. So, looking at the expressions here, so, this exponential factor is common. So, you take that exponential term outside and then inside you have u tt – c square u xx coming from the second derivatives. So, V tt there is u tt here, there is u xx here, so that is fine.

But they also contain lower order terms u and u t, so those are added here. So, we have b square u + 2bu t - a square c square u – 2ac square u x. So, these constants a and b are at our disposal. So, we can choose them. So, here we make the choice. So, you choose 2b = beta. So, beta is in the given equation. So, alpha beta gamma are in the given equation.

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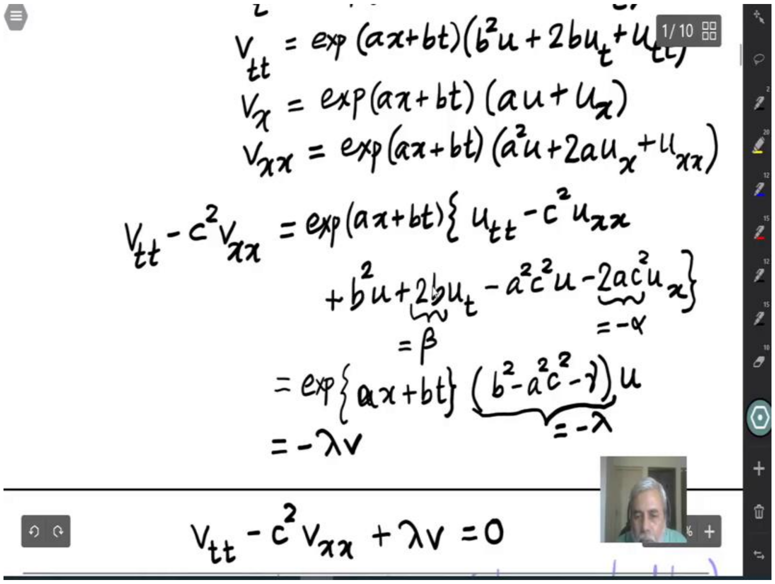

So, this coefficient of the first derivative of u with respect to t, so that is beta there. So, you choose this 2b = beta. And similarly, this 2ac square = - alpha. So, we this choice. So, now, since you satisfy this equation, so let us do that. So, u tt - c square u xx + alpha u x + beta u t = - comma gamma u and that is what I have written here. And these 2 things are coming from this change of variable. So, there are u terms also here. So, b square – a square c square – gamma u. And let is call this constant the coefficient of u as minus lambda.

And again so if I take that lambda out so this product is precisely v. So, with this change of variables, so this function V satisfies this second order equation.

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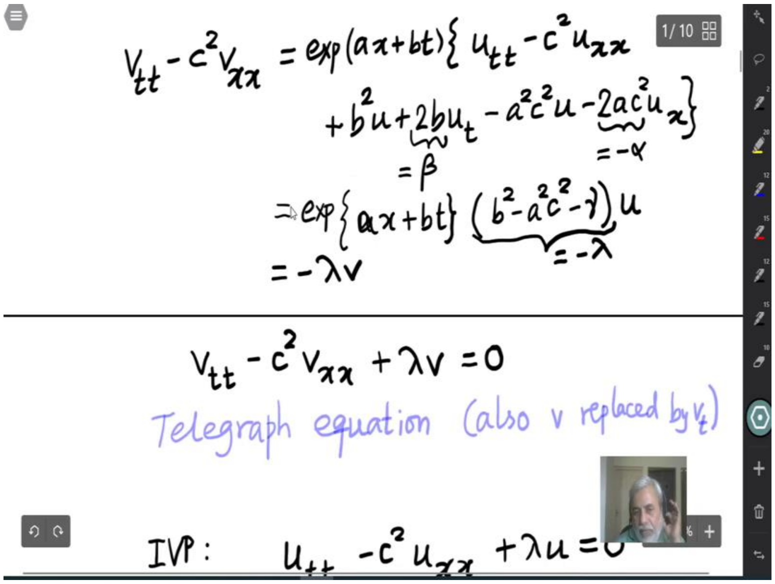

So there is, so it is not wave equation but it is there is a term lambda V comes in there. And this is called the telegraph equation. In the literature you also find that so, is this you replace this V by V t that is also called telegraph equation in some textbooks. In either case now, just now we have learned that even if it is V t, we can always transform that equation to an equation of this form. So, since a general second order equation of this form can always be reduced to the telegraph equation. We now consider the initial value problem for this telegraph equation. So, just remember here, so, this is the x is in r.

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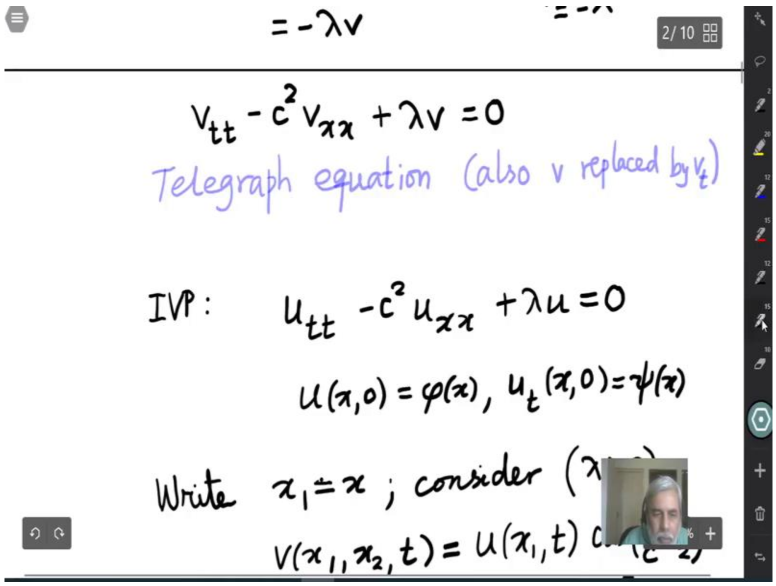

So, this is one space domain. Just remember that. So, the only difference between wave equation and this is that we have a lower order term namely lambda u = 0. And since your second order equation, so, we prescribe 2 initial conditions. So, this is the initial value problem. Now, try to obtain a formula for the solution using the method of descent.

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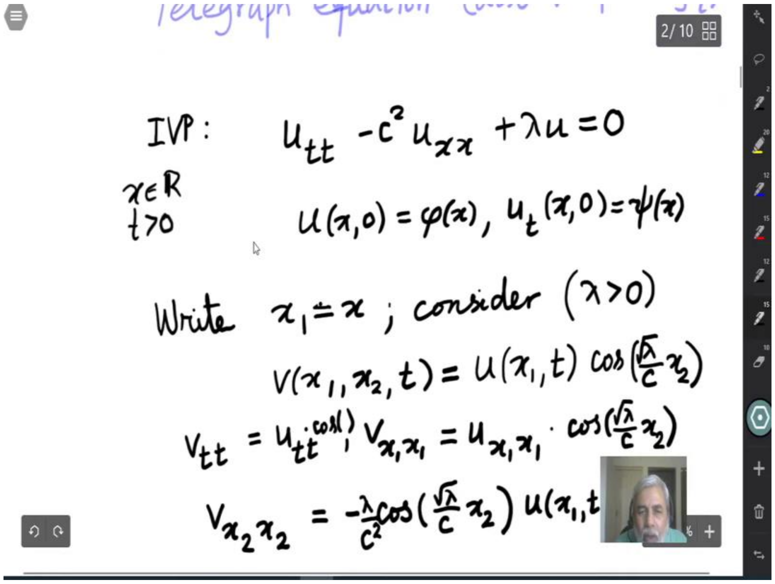

So, for that purpose so, just write the x 1 = x and consider this function so, first assume that this coefficient lambda is positive. So, if lambda equals to 0 this is just one dimensional wave equation and we already know that D'Alembert's formula gives us this solution. So, first assume this lambda is positive. And consider this function of 2 space variables now, V of x 1 x 2 t which is equal to u of x 1 t. So, again u is solution of that equation into cos root lambda by c x 2. So, this is you can say separated when you see x 1 and x 2 are separated here.

So, since I am assuming lambda positive so, the root lambda is also positive. And again simple computations so, x 2 comes only here. So, the second, derivative of V with respect to t same as second derivative of u with respect to t into this cosine factor. And similarly, the second derivative with respect to x 1 variable so x 2 is only there so, get u x 1 x 1 into cos of this variable. What about V x 2 x 2 the second derivative of V with respect to x 2 variable and that is all you have to differentiate this cosine function.

So, if we differentiate cosine function place we get back again cosine function with a negative sign. Of course, this constant also will produce a constant here that is lambda / c square.

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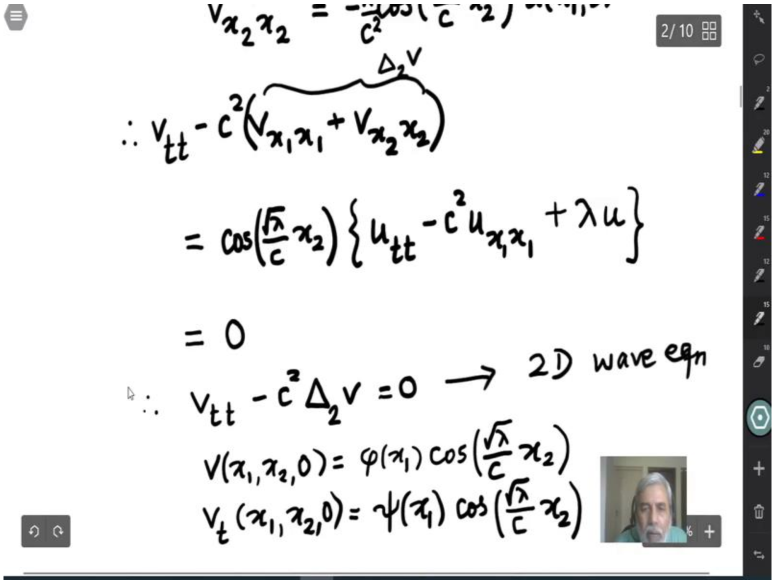

And now, you see that compute this wave operator on V. So, V tt – c square into V x 1 V x 2. So, this is nothing but in our earlier notation, so, temporary notation. So, this is the Laplacian in 2 dimensions. And you plug in these computations. So, again this cosine factor is everywhere so that comes out and in the bracket you have u tt – c square u x 1 x 1 + lambda u and u since you satisfy the given equation. So, that is 0 so u satisfy the telegraph equation then with this change of variables V satisfies the wave equation. So, this is 2D wave equation.

So, what about the initial conditions so, they are also very easy to compute. So, only t appears in this factor there is no t there. So, this u x 1 0 is just phi of x 1 and u sub t x 1 0 is psi of x and that just multiplies by this cosine factor. And important to notice again just look at this transformation we are using so, we get back

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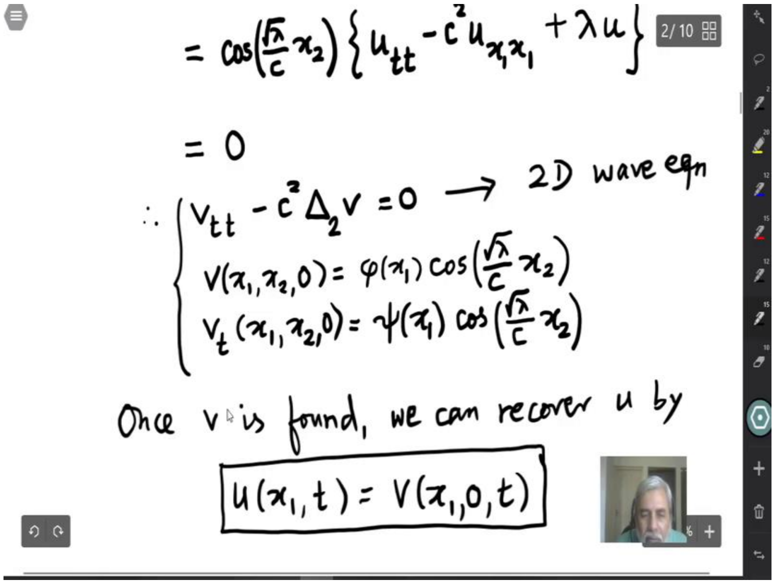

Once we know, once V is found we can recover u by this way very simple thing. So, V of x 1 you put there 0. So that cosine factor when you put x 2 = 0 produces 1 and we get back our original function. And now, since V satisfies 2 dimensional wave equation we can use Kirchhoff formula we have already derived and then in the solution which just substitute x 2 = 0, then we get back our required solution u.

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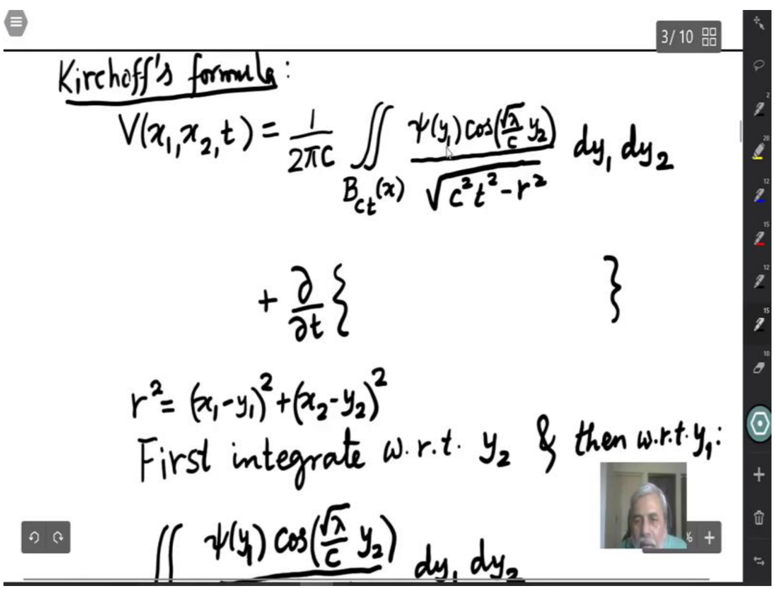

So, by Kirchhoff formula so, this is Kirchhoff formula gives us 2 dimension. So, V of x 1 x 2 t is 1 / 2 pi c double integral over this ball of radius ct centered at x into the initial condition comes into picture. So, initial conditions are here psi for V t. So, psi of y 1 cos of root lambda by c / 2 c square t square – r square root dy 1 dy 2 just recall. So, this we derived in the previous class and then the second term coming from the other data file. So d / dt, so same term except that psi replaced by phi and here again r 2 is x 1 - y 1 square + x 2 – y 2 square.

So that is the notation for this. And since this double integral and more or less the variables y 1 and y 2 are separated, they are still together here but at least in the numerator they are separated. So, let us write this double integral as iterated integral first integrating with respect to y 2 and then with respect to y 1. That is what I have written here.

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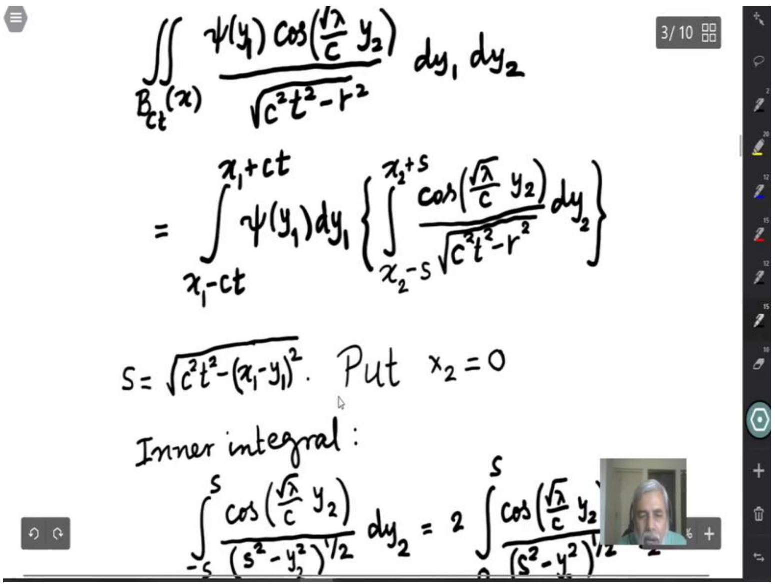

So, this double integral we are writing as iterated integral first with respect to y 2 variable and then with respect to y 1 variable. So, it is easy to calculate the limits in the integral of both variables y 1 and y 2. And again some notation here. So, this S is c square t square – x 1 – y 1 square. So, just writing the double integral as it gets. Now comes the important part. So, we are not interested in this solution V in the entire x 1 x 2 plane but you are just interested in the line x 2 = 0. So, that will give us just remember that u x 1 t and that is what we want.

So, if you put x 2 = 0 the inner integral just look at the inner integral here. So that becomes minus s to s cos root lambda / c / 2 s square - y 2 square and put the half. So, this one now we have written slightly different here. So, taken this x 1 – y 1 square separately in that S and only y 2 there. And you will see that this integral is an even function of y 2 and this is symmetric interval about the origin. So, we can write this as 2 times integral 0 to s and this form suggests that to use a change of variable.

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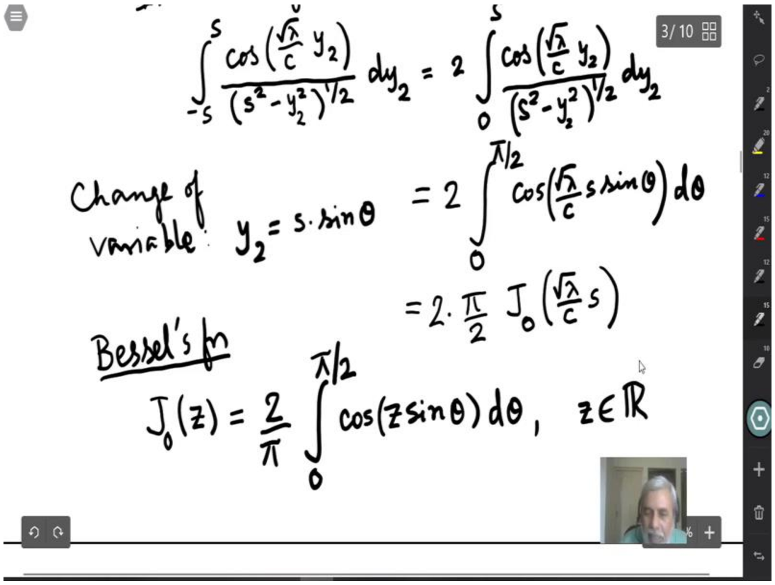

Very simple change of variable namely y 2 = s sine theta and that removes this new denominator and we get a very simple looking formula namely integral 2 times that remains the 0 to pi / 2 cosine root lambda / c S sine theta d theta. So, we are using this change of variable here. So, y 2 = S into sin theta. So, when y 2 is s you get pi / 2. So, that changes and this one this integral is again Bessel and that is related to the Bessel function of order 0 of the first kind.

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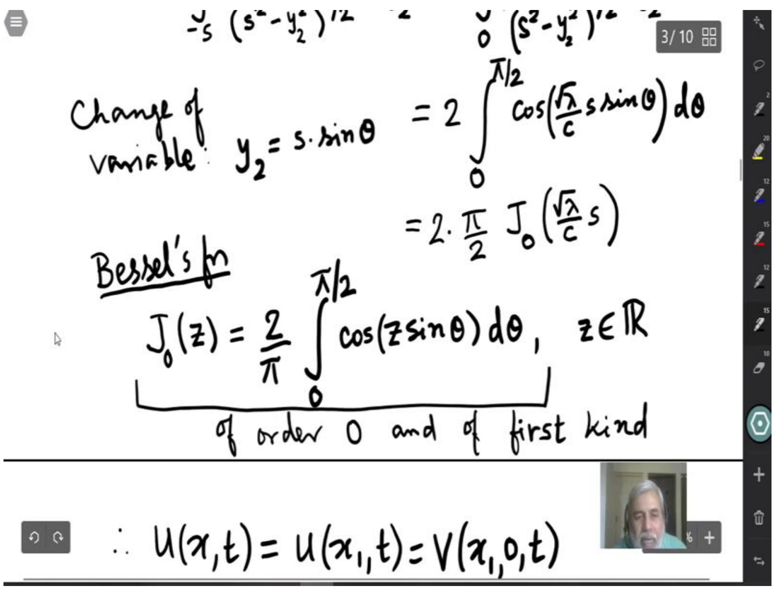

So, Bessel function has different representations and one such representation is this given by this integral. So, this Bessel functions of order 0 and of first kind. So, some material regarding this Bessel function so, we have been using given heat equation and it has again cropped up in this solution of the telegraph equation. So, that those things will be written down and provided for your reading. So finally, so just so the same inner integral that we have computed now, so thus is in terms of the Bessel function, so now we just put together so, we get the required solution.

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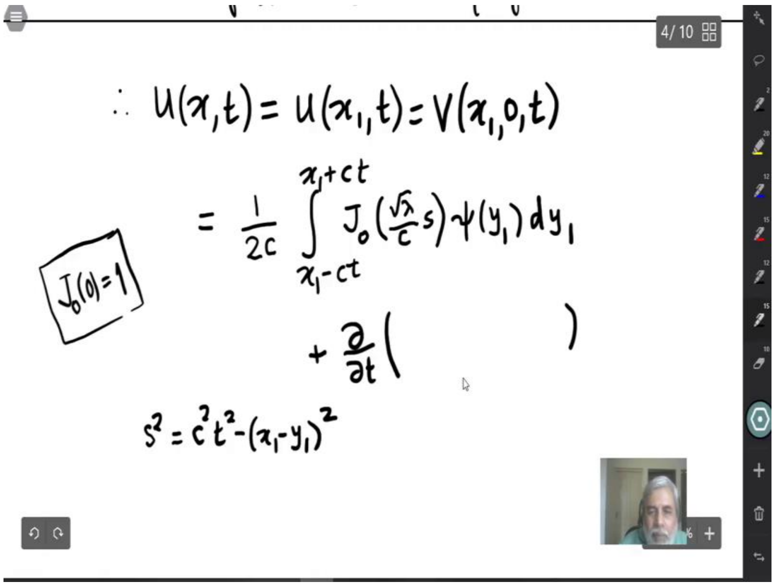

So, again go back to the original variable u of x t so that temporarily we put it as x 1 and this one is given by V x 1 0 t and that is just now computed. It is 1 / 2c x 1 – c t to x 1 + c t J 0 root lambda S sin psi and same thing comes here if psi replaced there by phi. I am noticed that if lambda is 0 this J 0 of 0. In our definition is to solve a normalized to be 1. So, if lambda is 0 the telegraph equation reduces to the wave equation. So, if lambda is 0 it reduces to the wave equation and that you can see given in this formula.

So, when lambda is 0 this is just one so this is presiding what is there in the D’Alembert’s formula. So, this formula now involves some additional term in the integral unknown function. That is important so this Bessel function of order 0 it is very well studied and all these properties are recorded. Now what about the situation, so here we started with lambda positive. So, what if is negative or less.

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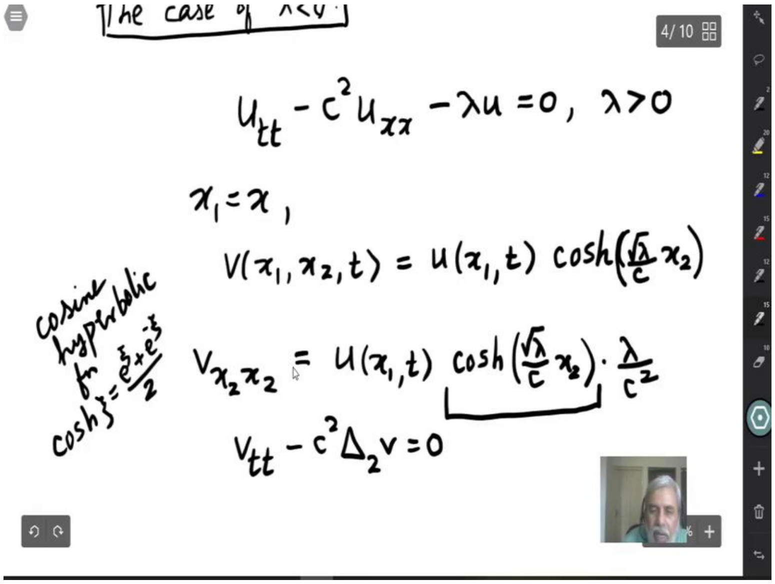

So, the case is very simple again. So, now write the equation u tt - c square u xx, so instead of writing plus lambda you are assuming lambda positive so you just write 0. And now again you write x 1 = x and consider the function V of x 1 x 2 t is equal to same thing. So, now we will just instead of cos you just use cos hyperbolic root lambda / c x 2 that is the only difference. So, all the computations are same. The only difference now is if you compute the second derivative with respect to x 2 variable of this function V.

So, this is just cos. So, this will not change the sign. Earlier we had since we had only cos, so second to produce a minus cos, but for hyperbolic function so this is cosine hyperbolic function. So, just recover what that is, so same computations. So, now you see that this again this function V satisfies the 2 dimensional wave equation. And now the initial conditions on u get transform to V with the additional factor of cosine hyperbolic.

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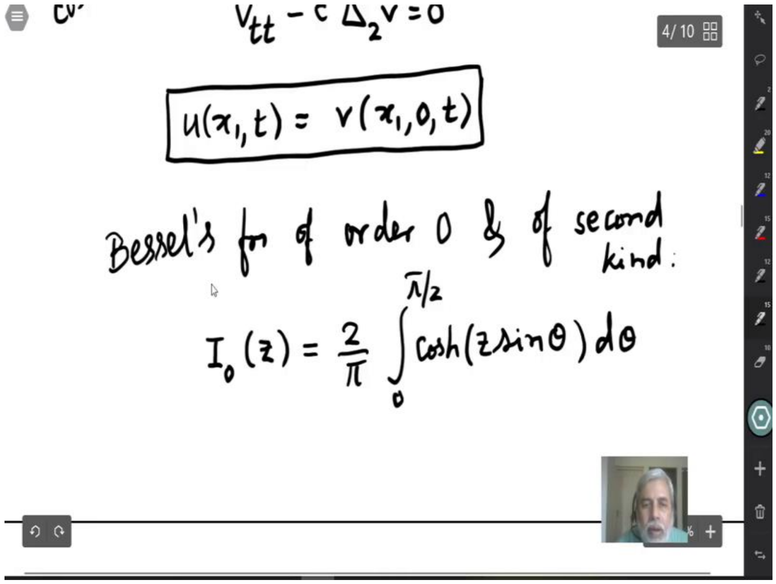

And again important thing is so we recover this; our function just by putting x 2 = 0. So that is you want easy recovery of original function. It is important. And if you do the same computation again so V satisfies now this V everything is same except that now you get instead of cos you get cosh. That the only difference and that is also called a Bessel function of the second kind of order 0 and of second kind.

So, the second I 0 it is denoted by I. So, the first kind denoted by J, so the only difference in just replace this by cos. So, I just want you to there this in mind because we will be considering similar problems but the Laplace equation and may be one path heat equation. So, what we have shown here is just is telegraph equation. But it is more like an Eigenvalue problem.

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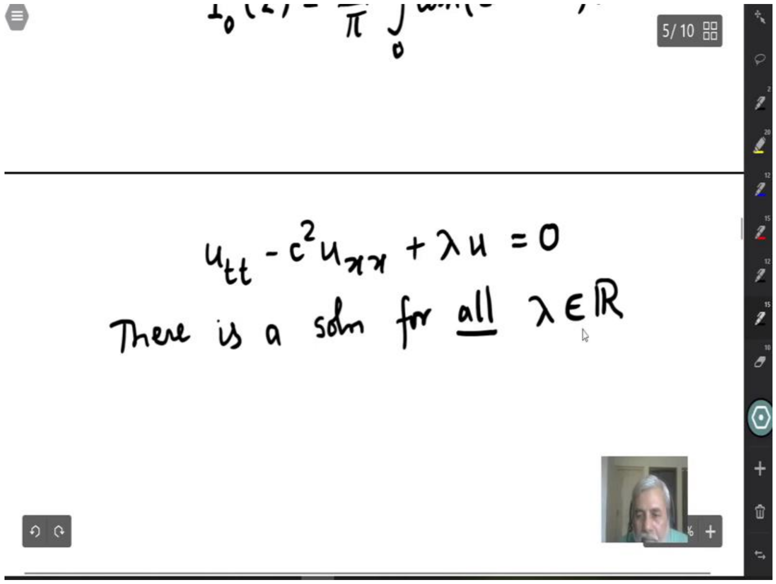

So, what we are shown is there is a solution that is unique for all lambdas. If you think this as wave operator and so we are showing that so each real number is an Eigenvalue on that operator. So, I want to just comment on that. So again, then we will come back to Laplace equation, we have occasion to recall this. So that is fine. So, just so far what we have done.

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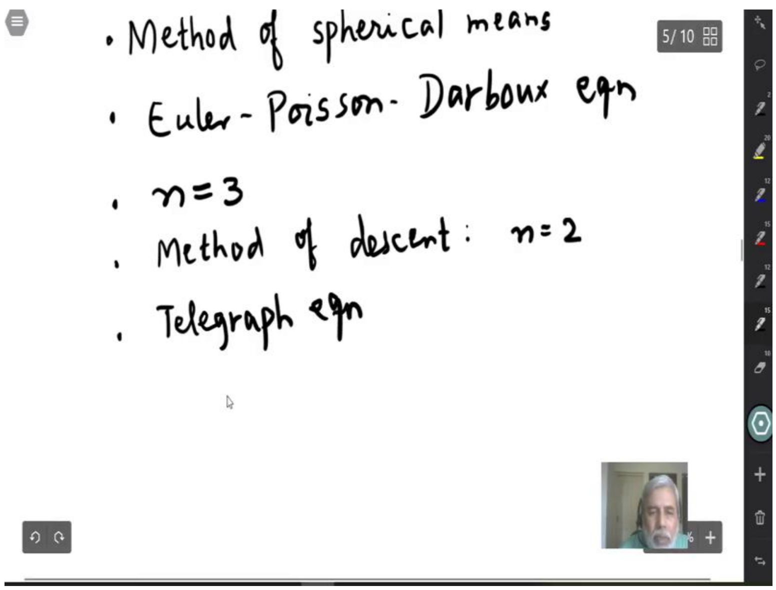

So, just introduce the method of spherical means, just summarize. And then we derived this Euler-Poisson dot equation and with that help you could obtain solution case n = 3 by simple transformation and reducing the whole problem again to one dimensional wave equation and then by method of descent. So, we obtained the formula for the n = 2 case and using again the method of descent. So, we also form for the telegraph equation.

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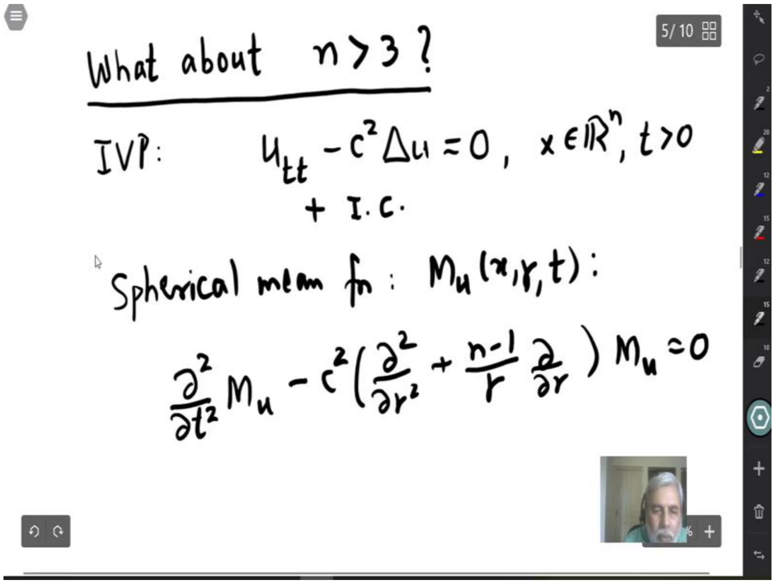

And what about other n? That is our next question. So, at least we should answer little bit regarding that, I will just explain and we will continue next time discussion on that. So, this is the IVP for the wave equation. So, this is actually R n and t positive plus initial condition, let me not bother about that now and what we did was you this method your spherical means, so, that spherical mean function M u x r t and that satisfies Euler-Poison dot wave equation. So, this is again the second order equation but only in 2 variables equal to 0.

So, this is Euler-Poison dot wave equation, so when n = 3 by simple transformation by multiplying this spherical mean function M u / r we could reduce that to one dimensional wave equation. So, when n is bigger than 3 certainly that will not work. So, we have to look for a different kind of transformation. So, what we look for is an operator?

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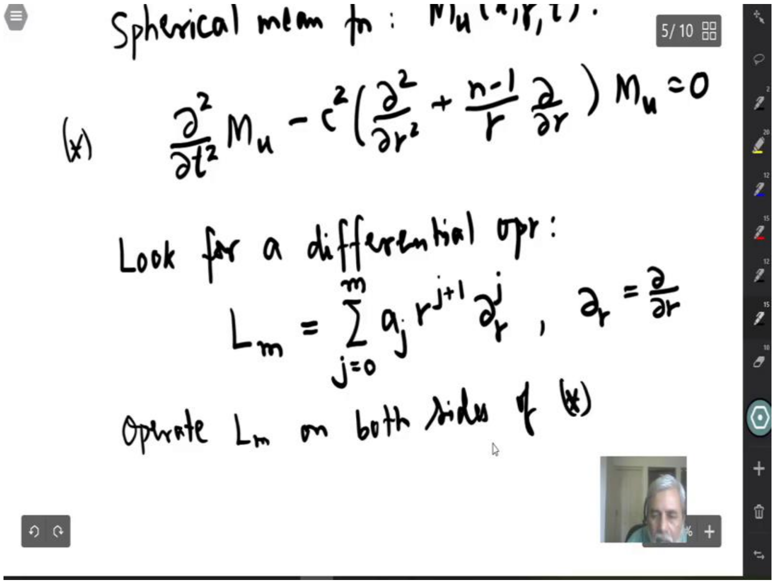

Look for a differential operator I denoted by L m. So, this is j = 0 to m a j r to the power j + 1 d r j. So, this operator d r is essentially an originary differential operator because only r variable there but since we are also dealing with t. So, let me denote it by the partial derivative. So, what we want to do is this operate L m so if you call this star there. So, operate L m on both sides of star, so what we get is just write that and then continue next time.

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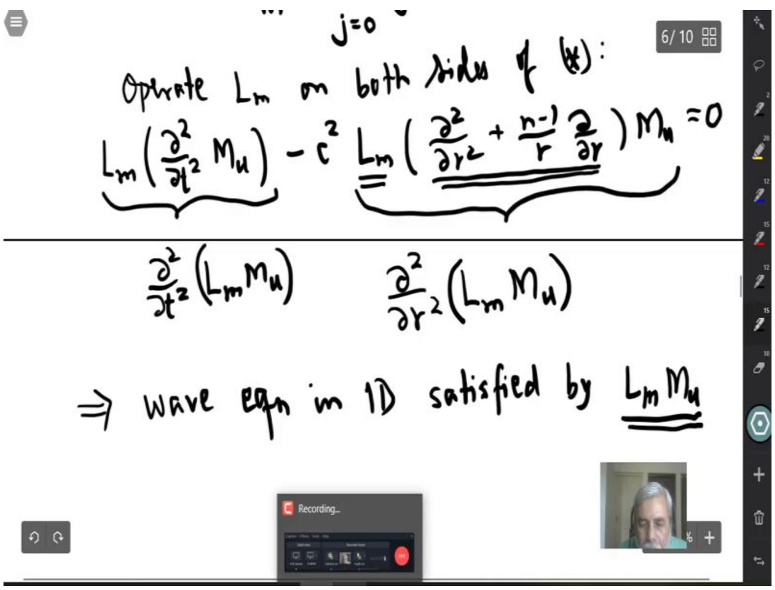

So, this L m so that is a linear operator. So, given star is also a linear equation. So, there is absolutely no problem. So, the L m this operator, there is absolutely no problem here this will just become del square / del t square because this t variable and that is r variable. So, there is no problem so L m M u. But this is also a differential operator containing r derivatives and this is also differential operator in r variables, so in general, there will not be commutativity, so I cannot just push this L m inside just like I did here. So, there will be some extra terms that will crop up.

And that somehow you want to reduce it to remember this is our work L m M u. So, if we succeed in doing that, so, when I operate the L m on this operator, I should get this del square / del r square L m M u and then we get wave equation in 1 D satisfied by L m. So, the first task is to find this L m. And then we have to also get back our original function u from all these operations, so that should that process of getting back u also should be simpler. And we will discuss these things in detail in the next class. Thank you.