First Course on Partial Differential Equations - II
Prof. A. K. Nandakumaran
Department of Mathematics
Indian Institute of Science - Bengaluru
Prof. P. S. Datti
Former Faculty, TIFR - CAM - Bengaluru

Lecture - 33
Wave Equation 4

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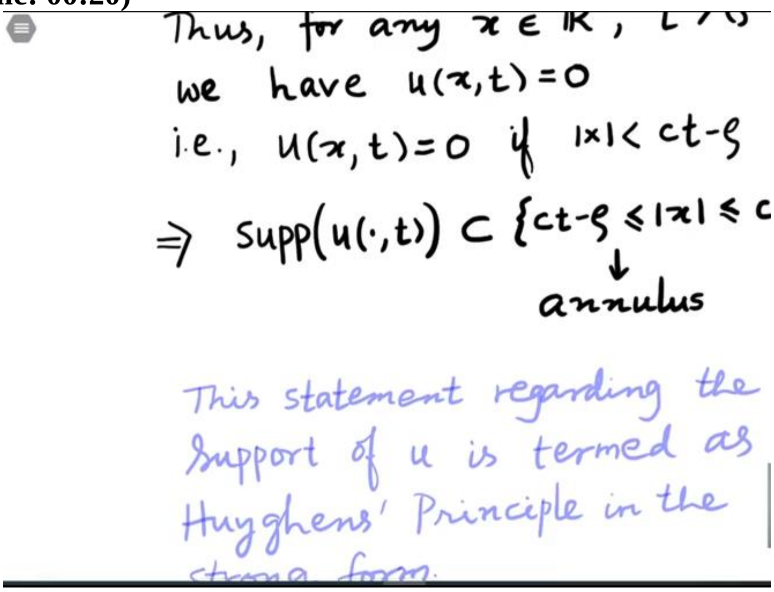

Hello everyone, welcome back so we will continue the discussion of the wave equation in more than 1 space variable in the previous class we derived a formula for the solution of the wave equation in 3 dimensions which is called Kirchoff's formula and then now we use the same to derive a formula for the solution of the wave equation in 2 dimensions and this is known as method of descent and is due to Hadamard.

So we start discussing of the 2 dimensional wave equation so in the previous class we also using the Kirchoff's formula we observe some qualitative properties like domain of dependence of the solution and range of influence of the initial data and Huyghens principle etcetera.

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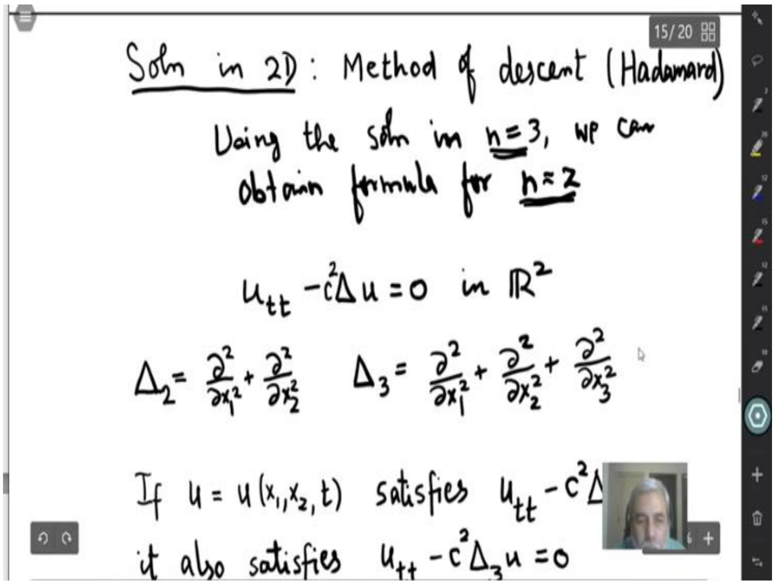

So now we start discussion of the wave equation in 2 dimensions so we use some temporary notations here. So, we write Laplacian sub 2 for the 2 dimensional Laplacian and Laplacian 3 for the 3 dimensional Laplacian so we will be going from 2 dimensional to 3 dimensional etcetera, so first we will make some simple observations. So if u which is equal to u of x 1, x 2, t satisfies the 2 dimensional wave equations. So, we can treat this u as an independent function of the x 3 variable so it also satisfies the 3 dimensional wave equations.

So we can just take that u is independent of x 3 so this del square u / del x 3 square will be 0. And conversely if we have a solution to the 3 dimensional wave equation v of x 1, x 2, x 3, t then if I restrict to the plane x 3 = 0 or any constant you can put that and then this restricted function now we said 2 dimensional thing. So, this u satisfies the 2 dimensional wave equations so these are the simple observation. So we are now going to construct a function which is satisfies the 3 dimensional wave equations and by this restriction we obtain the formula for the 2 dimensional wave equations.

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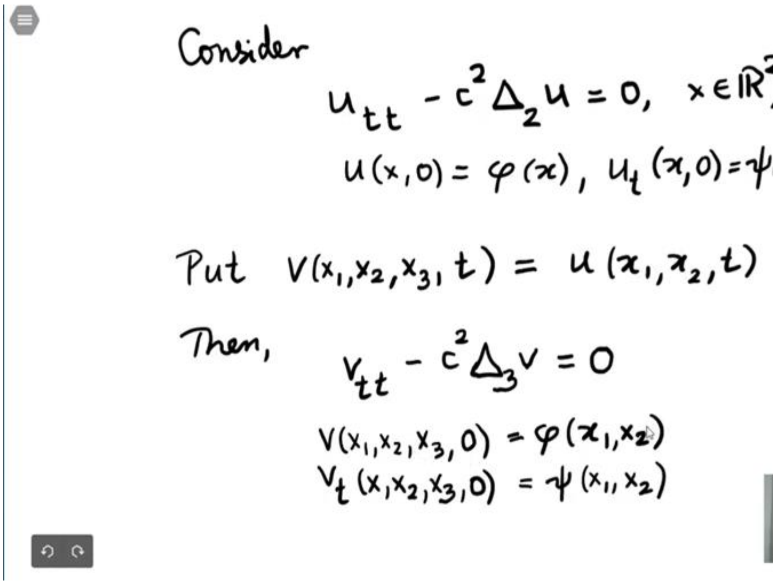

So now you consider the initial value problem 2 dimensional thing so as observed here so this if we define this v of x 1, x 2, x 3, t is equal to this u of x 1, x 2, t so u that satisfy the 3 dimensional equation and essential data are again independent of the x 3 variable and they are the initial data of the function u which we are looking for a formula.

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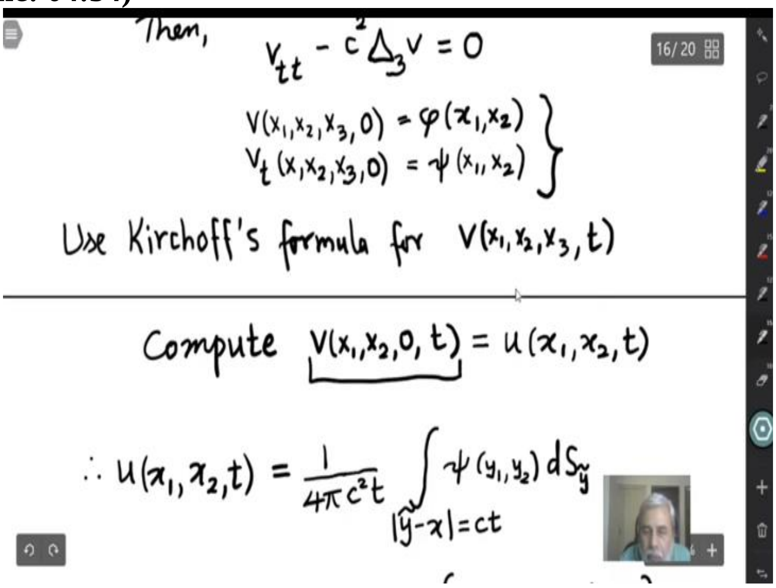

So now that V satisfies the 3 dimensional wave equation and now we find that formula for the solution of this 3 dimensional wave equation using Kirchoff's formula and then we recover our u by putting x 3 = 0 maybe a little hurried here. So this first you write this formula for the solution of the select so use Kirchoff's formula v of x 1, x 2, x 3, t using this initial conditions which are independent of x 3 variable and again we are more interested only in this V of x 1, x 2, 0, t.

So you go back to Kirchoff's formula which is so this is Kirchoff's formula now so this is a 2 dimensional surface integral in 3D so that so we use slightly different notation here so not to be confused. So let us write down the expression for v of x 1, x 2, 0, t using Kirchoff's formula.

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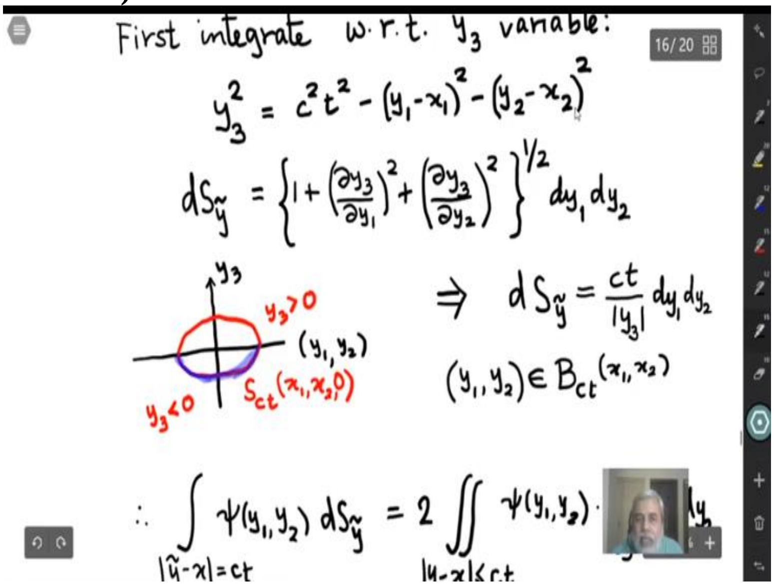

So this is nothing but so let me again stress that so this is nothing but V of x 1, x 2, 0, t so this is surface integral in 3D over this sphere and S the integrand this psi of y 1,y 2 just observe this one so this does not depend on y 3. So, we write this surface integral as a 2 dimensional integral and that is easily done so look at this surface so I am using here y tilde on this is every tilde here. So, this is in 3D and there is known x 3 here because we are taking the x 3 = 0.

So it is just y 1 - x 1 square + y 2 - x 2 square + y 3 square and that is equal to c square t square that is we are on that surface. So we convert this surface integral into a double integral and that is easily done. So we rewrite this expression as y 3 square = c square t square - y 1 - x 1 square - y 2 - x 2 square and so if you recall how the surface integral is defined so now this one so as y 1 and y 2 vary over this circle.

So y 3 over this sphere and so there is one upper hemisphere and there is lower hemisphere and both contribute the same amount to this integral suppose you to consider just one integral the other one is also very similar except for this time derivative. So, how do you compute this surface major. So as surface major simply is this formula so that is a double integral. So if you simplify this so using this expression for y 3 square you compute this del y 3 / del y 1 and del y 3 / del y 2 and do some simplification.

It turns out to be ct / mod y 3 again y 3 can you that square and this y 1, y 2 they are in this circle centered at x 1, x 2 and radius ct. So the surface integral, forget about this constant for the timing. So, the surface integral is nothing but twice as I said so there is one upper hemisphere, y 3 positive and y 3 negative they contribute the same amount so that is why it is 2 here and so this now it is a double integral. So let me just such that so it is a double integral and y 3 I am replacing by this expression, so what mod y 3 square root of c square t square minus those root of this.

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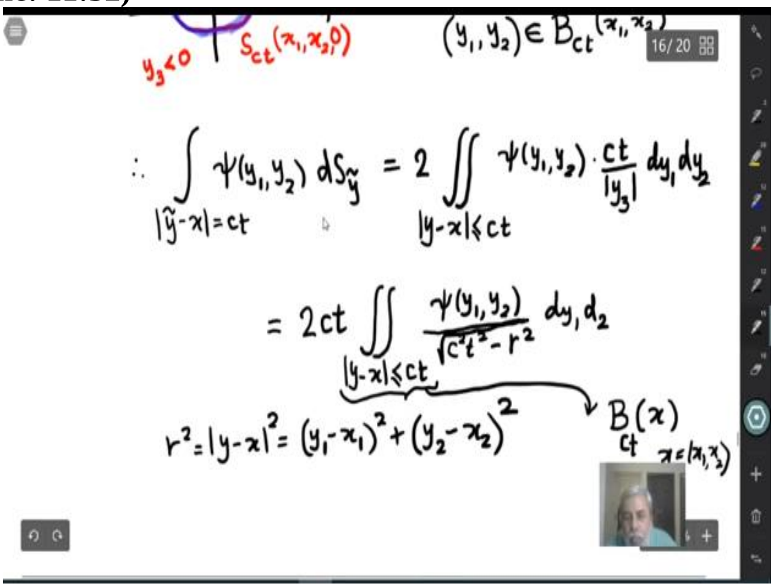

So let us write that in a simplified form so, let us write that as r square so r square is nothing but y - x square, which is y 1 - x 1 square + y 2 - x 2 square. So, note that this is the section c t just ball centered at the x and radius c t. So, let us use here just x 1 so similarly, for the other integral. So, phi replaced by xi there so we have another one here, so if you put back all this expression so now we have converted this surface integral and written as double integral.

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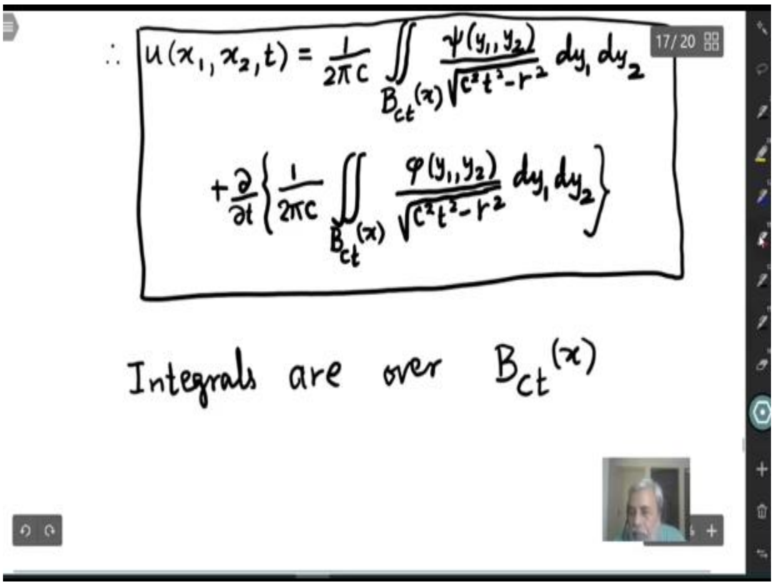

And finally we obtain this so in the 3 dimensional case the Kirchoff's formula involve a sufficient integral source, but in 2 dimensional case so you already start seen the difference. So this integral are over the ball so it is not circle but the ball just write this. So, this x this radius is ct, and the integral is over this 2 dimensional object and that makes the domain of dependence and other qualitative properties very much different from the 3 dimensional assumptions so for example, see the domain of dependence just mention that.

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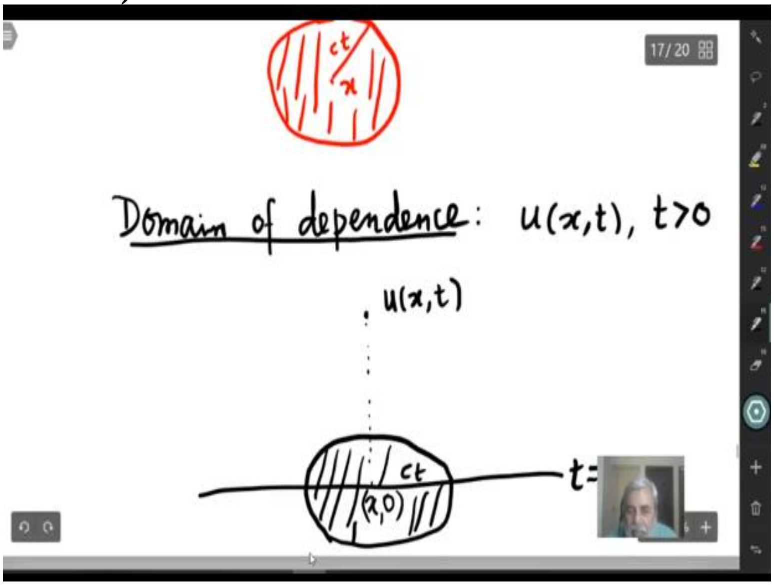

So we are interested to know how the value of the solution at a point u of x, t or t positive depends on the data are the initial line t = 0, the initial space t = 0. So, this is t = 0 so, we are interested in knowing how in this dependence of u on the initial condition. So, if you again look at the formula so, we required that entire ball so, this is x here x, 0 t = 0 so, this radius ct this entire disc earlier in 3 dimensional case it was only the sphere the circumference. So similarly you can write down the range of influence etcetera.

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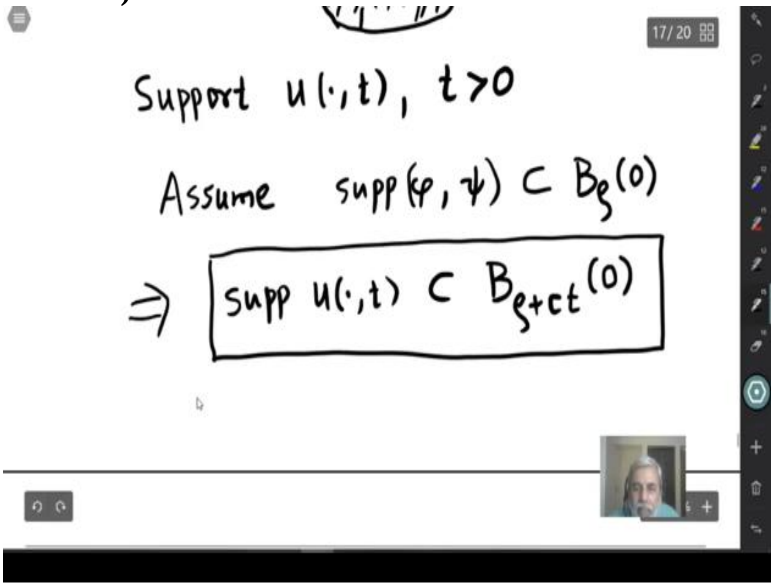

And now let us see so the support of the solution at a time positive how it depends on the support of the initial line. So, again assume that support of phi and psi and same ball of radius rho centered at origin and similar calculations will show you that the support solution continuity rho + ct. So, this is all we can say in the 2 dimensional case so just cannot say anything further.

Because of this; integral which depends on the entire simple physical situation depicting this 2 dimensional wave propagation. So consider a large lake and you throw a pebble in the water under calmness conditions that wind is not blowing and other things. So, you see the waves never vanish though their amplitude will become smaller and smaller, but they will never vanish and this we call this Huyghens principle in the weak form and as we see little later that this persists in all u and t case this only n = t we are done here.

But if so, this Huyghens principle in the week for persists for all you u and t demand the wave equation and Huyghens principle in the strong form persists for all our demand for this. So, there is another interesting application of this method of descent to the telegraphic equation perhaps I will come to that point little later.

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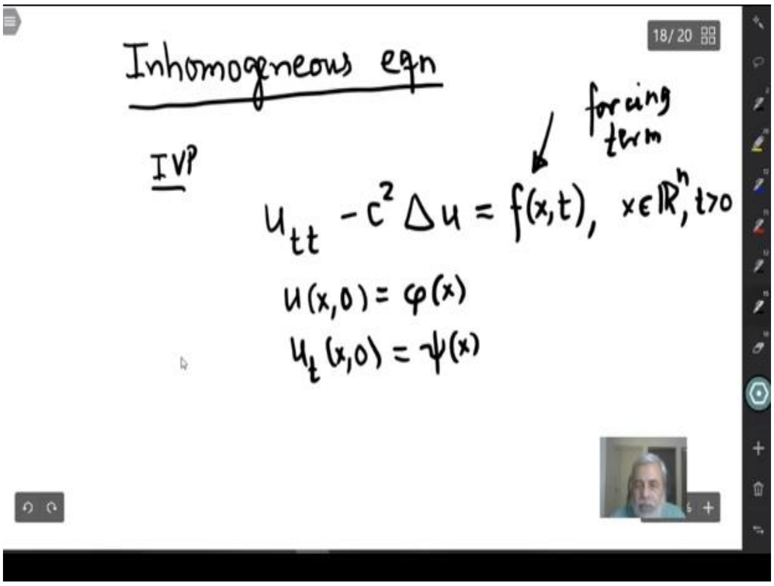

So, now let us discuss the inhomogeneous so if you recall in the first part of this course we have done this in detail for the 1 dimensional wave equation and same principle holds here. So again IVP, so u tt - c square u equal to now take a function of and so this is usually called forcing term or inhomogeneous term so this x is R n. So let me write them though we have derived formula for only n = 3 and n = 2 so, but this principle holds for all the dimension so this is also refreshing of what we have done for the 1 dimensional wave equation.

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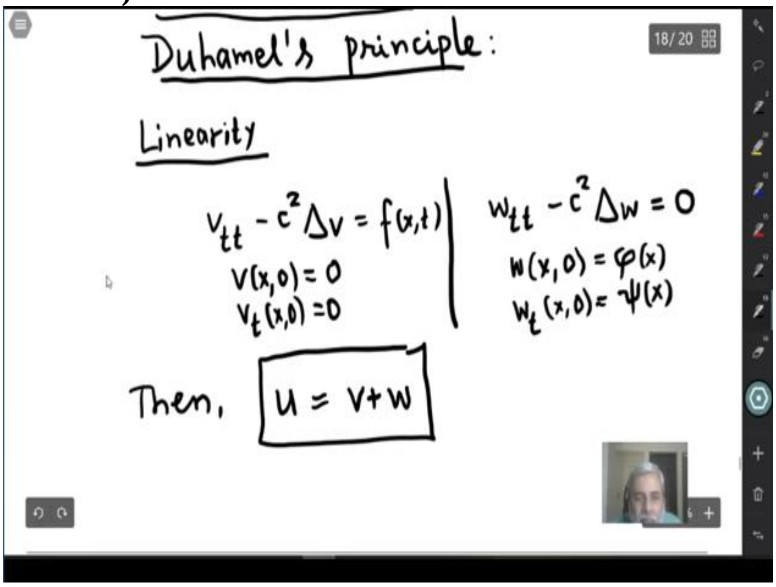

So, this is what we are now describing is called Duhamel's principle and is a very useful tool in obtaining a solution of the inhomogeneous equation from the knowledge of homogeneous equation not only for the wave equation, but many evolution equations u in one including head equation. So first observe so we exploit linearity so we consider 2 problems so, one is inhomogeneous equation with 0 initial depth and another one we consider homogeneous equation.

So, let me do that w, now homogeneous equation but with general initial conditions so, we split the given initial value problem. So, inhomogeneous equation and arbitrary initial conditions; into 2 problems one inhomogeneous equation with 0 initial data and another one homogeneous equation with given initial data. So phi and psi appear here and this is seamless so using linearity of the equation it is very easy to check that then u = v + w.

So, whenever there is linearity will always try to exploit and generate simpler problems to solve. So this one at least for n = 2 and 3 so the solution is given by Kirchoff's formula so, it is sufficient to consider this inhomogeneous equation with 0 initial data and for that, we apply Duhamel's principle.

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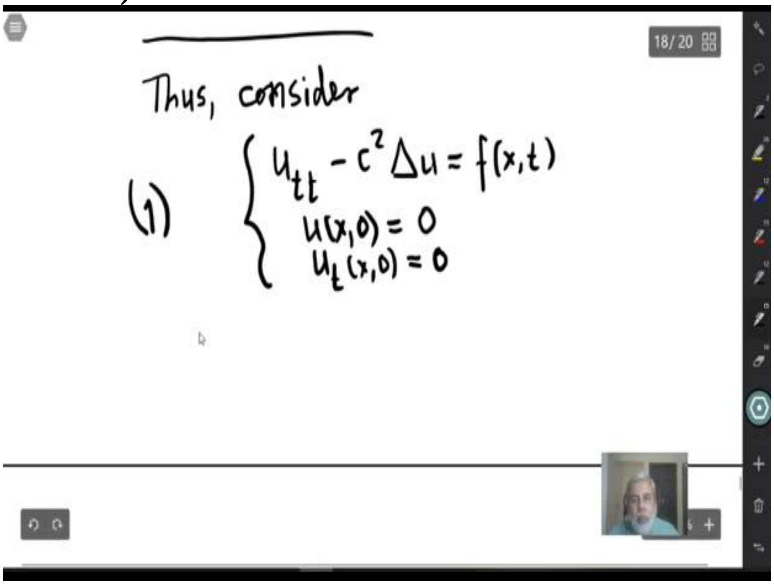

So thus consider again let us go back to thus consider this simpler IVP so this is u tt - c square Laplacian u and f of x, t is 0 initial. So, let us this one and as solution of this IVP 1 is obtained by reducing it to an IVP with homogeneous equation and some special initial data and that is what is referred to Duhamel's principle.

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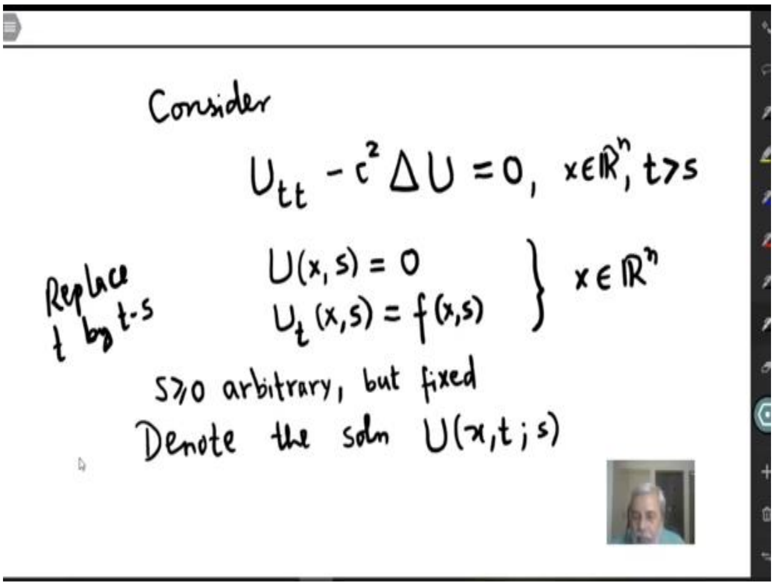

So, this is the related I repeat so consider this different rotation U tt - c square Laplacian u E 0 so, homogeneous wave equation and very this so again R n not t positive but t bigger than s. And we describe initial conditions not at t = 0 but at t = s so U t x, s this is given by f of x so this one x is R n. So, this is only a translation in t variable so you just replace the formula by so you will replace t / t - s and this is same as this f so this.

So now we are switching this s so this is just a function of x, s and we will try to determine the solution for t bigger than s and s is greater than or equal to 0 arbitrary but for the time being it is fixed so we as you we vary s we get a family absorptions working so denote this solution by U of x, t and just to stress the dependence on the variable s you also write this as U of x, t.

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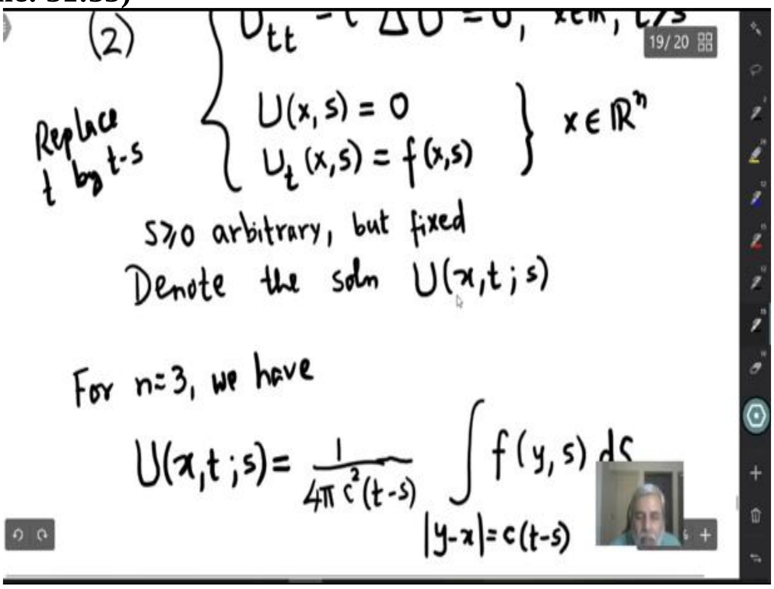

So for example for n = 3 we have U of x, t, s I read from the Kirchoff's formula so here phi 0 and psi is given by this function. So this go back to remember constant so here it is by example 1 / 4 pi c square t. So now we have to replace t / t - s and now this integral there is no change there so this is again surface integral so this y - x = c t - s. So, wherever t is there just to replace it by t - s so this is surface integral.

So now this one this f of y, s ds so that is surface major on this sphered so, this is the solution of this auxiliary initial value problem. So, from the given initial value problem we construct for this one homogeneous equation and the initial condition is coming from the inhomogeneous equation term original problem if you want to recall it so this is solution now. So, this is again for t bigger than s so what is the connection of this big U? And our solution small u so here comes the theorem.

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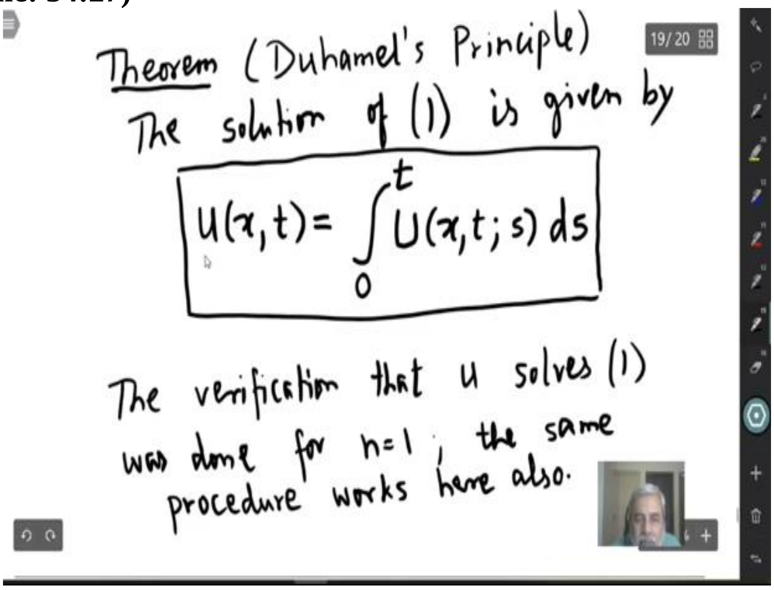

So this is the Duhamel's principle the solution of one is given by so u of x, t equal to so integral 0, t this x, t s ds. So, in the case of 1 dimensional wave equation in detail we verified that this is indeed a solution of problem 1. So, similarly in this case also one and to do that it is not immediately clear why this U given by this integral is solving this problem so inhomogeneous equation and with 0 initial conditions.

So, you just try to look at the proof in 1 dimensional case so the verification straight here verification that u solves IVP 1 was done for n = 1 the same procedure works. So, when you are differentiating with respect to t we have to exercise a little caution because t is also in the integral limit and also it is in the integrand. So, one has to differentiate with change but with respect to x there is no problem because there is no x in the integral sign. So that which can simply take the differentiation with respect to x straight way inside the integral sign.

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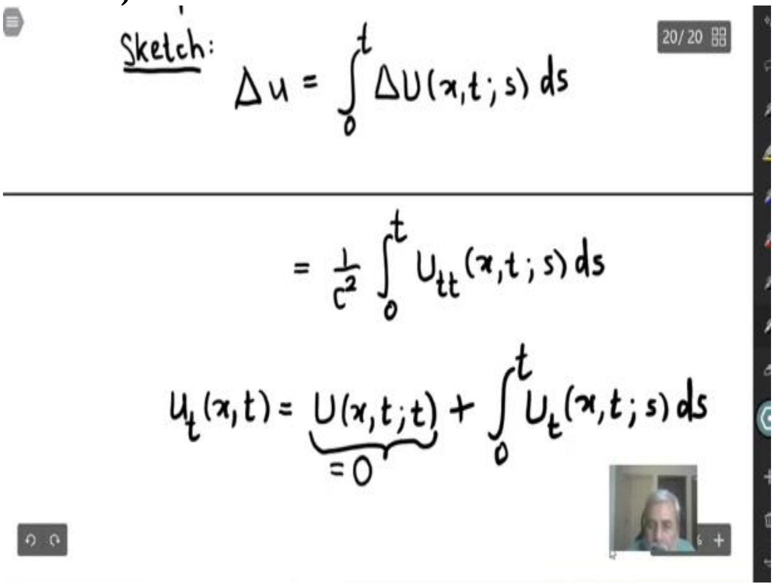

So let me just do that little bit so for example here so, sketch how you do that so as I said there is no problem with x variable. So, you just take Laplacian u simply 0 to t Laplacian U x t, s ds, but the capital U satisfies this homogeneous equation. So, you immediately see that Laplacian u is equal to so, this is 1 / c square 0 to U of t t x, t, s ds now just the 2 derivatives here so, you can just so similarly you can work out this U sub t what happens to sub t?

So u sub t x, t so first u differentiate this t in the integral sign and that produces so, U of x, t, t plus you have the other one so, U t of x, t, s ds and this one by the initial conditions now this is your because you are taking this 0 then t = s this is 0. So, instead of s we have here tt so, this is just 0 so this is 0.

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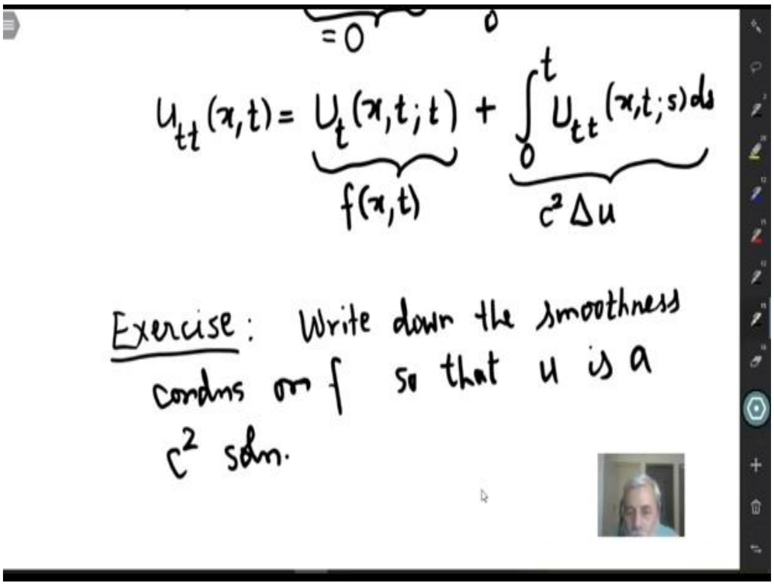

And the next one so u tt x, t I know this one is U t x t t + 0 to t U tt and this is where you will get this f of x, t and this one as we have seen already here because it is just c square and that completes the verification. So, once we know how to solve a homogeneous equation, so it is not difficult to solve any associated inhomogeneous equation so I have not written here the precise conditions on the right hand side the forcing term f, but you can write down so what conditions were required in order that this u is a solution.

So that is an exercise so write down the smoothness conditions on f so that u is a c square solution. So with that I will just conclude this lecture. And the next time we will see another interesting application of the method of descent to telegraphic equation, thank you.