First Course on Partial Differential Equations-II

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Lecture-31
Wave Equation-2

Hello everyone, welcome back, we will continue the discussion how the wave equation in higher dimensions. So, last time we obtained a solution of the wave equation in three dimensions, n equal to 3 looking for radial solutions, when initial data itself is radial. So, it is natural to look for a solution which is also radial. And that motivated us to look for spherical mean function of a given function.

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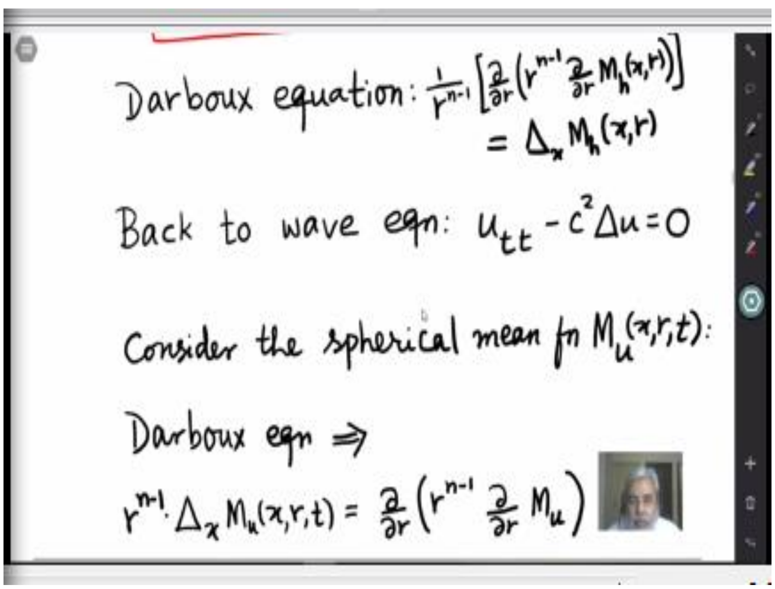

So, we will continue the discussion of the spherical mean function.

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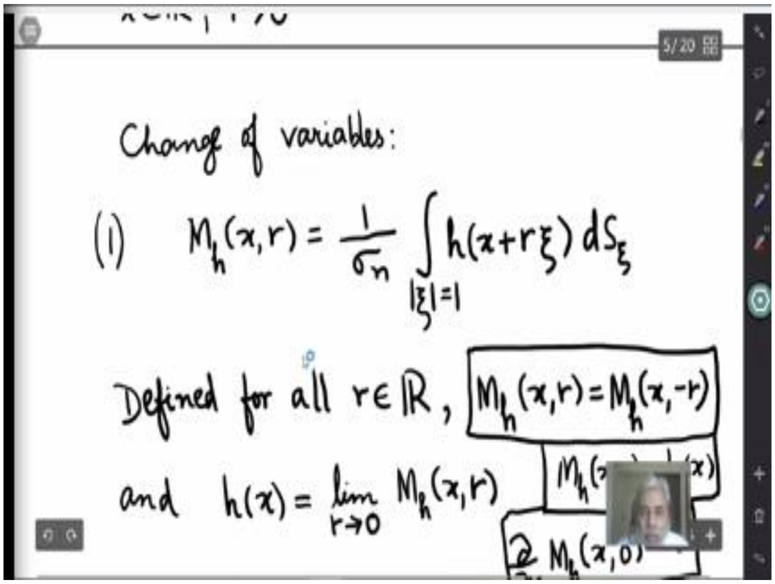

So, again just I will recall the definition of the spherical mean function and this we already seen in the discussion of Laplace and Poisson equations, so a very useful tool.

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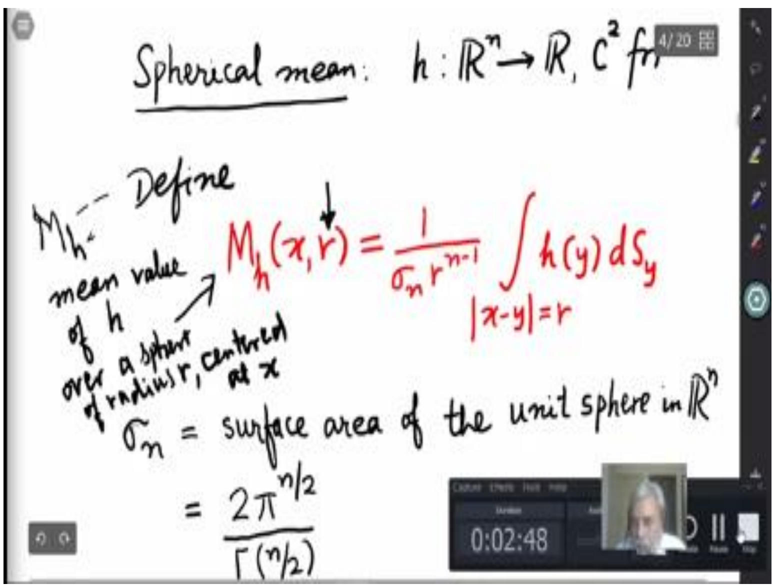

So, that is just given any function from R n, R you just define this spherical mean function by taking the mean of the given function h over the balls around a given point x. Sometimes it is also useful to write thinking this as an operator, so instead of M h, so you think M as an operator. So, that is equal, I will write it.

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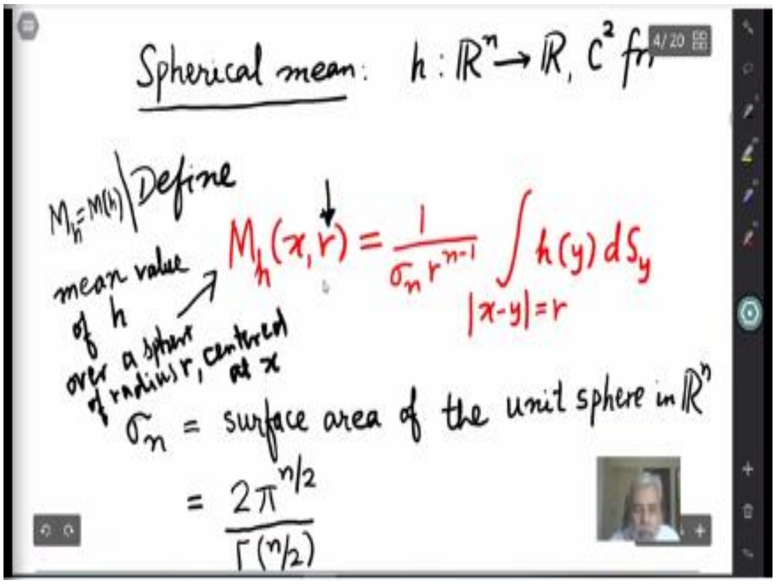

So, we will write this M sub h as M of h thinking that M is also an operator, it will be useful sometimes. So, given any function h, so this mean value operator M produces another function with one extra independent variable R.

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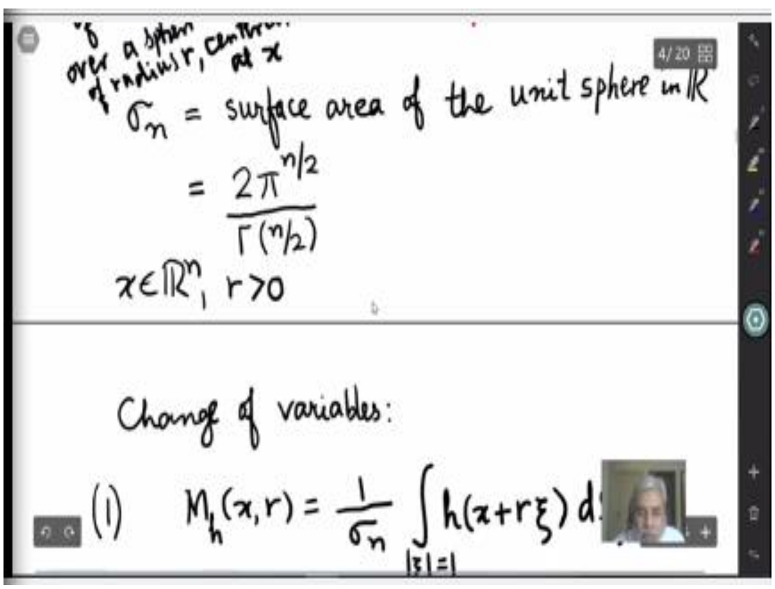

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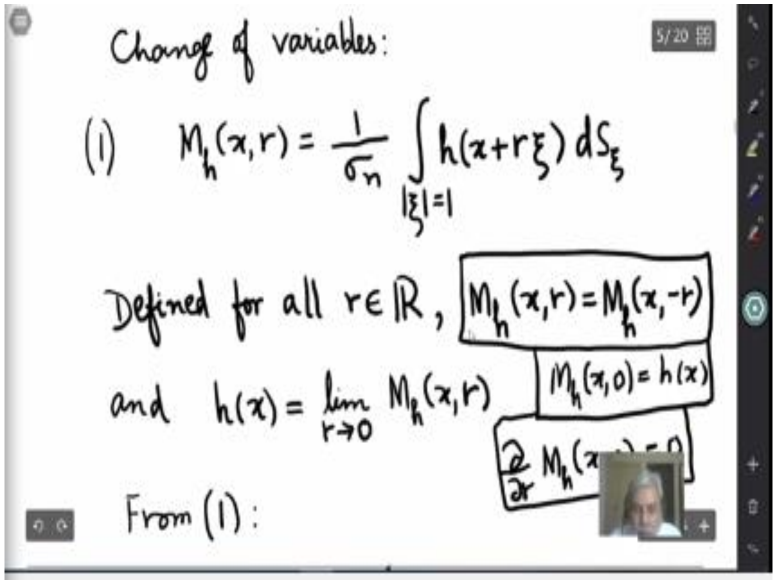

And then by change of variable rewrote the spherical mean function and so some important properties we discussed.

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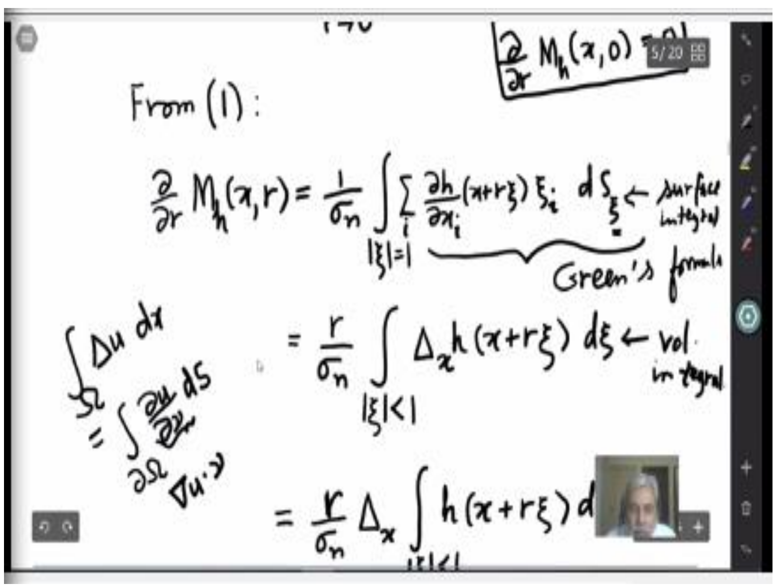

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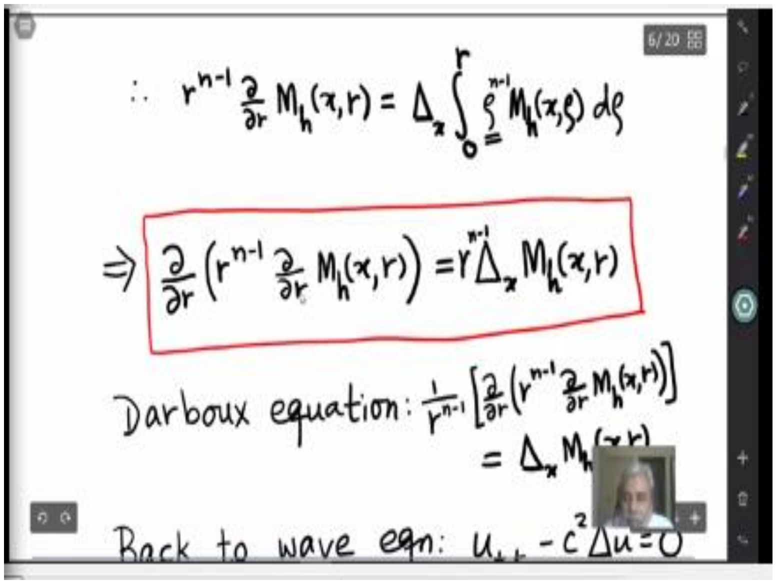

So, the important property is this Darboux equation that connects the Laplacian of the spherical mean function with a second order derivative of the same mean value function with respect to their extra variable r. So, that is an important thing and now we will see how that can be exploited in the study of this wave equation.

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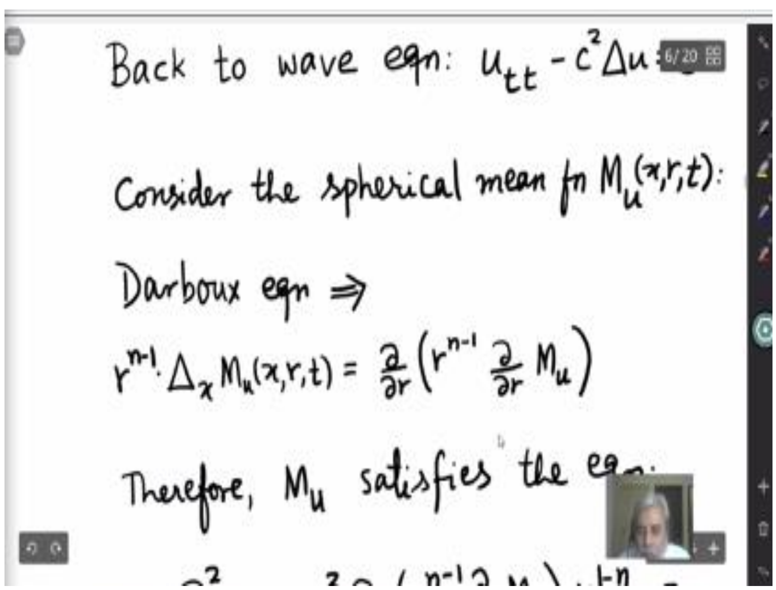

So, before coming to this wave equation let us compute do some computation.

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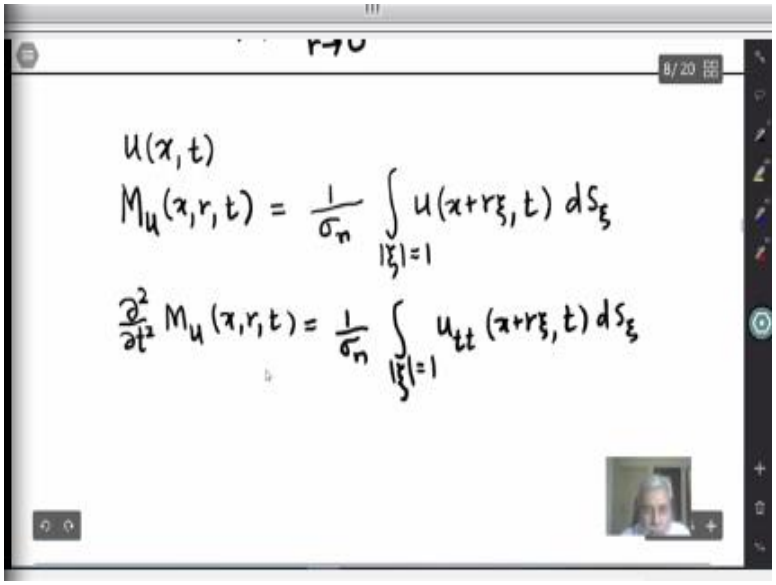

In all these things computations are bit lengthy but they are straight forward. So, now we are given a function of 2 variables x and x, x is in r n and t is positive. And now we form it is spherical mean function, so now this will be a function of x, r, t. So, we are taking the spherical mean only with respect to the x variable, so this is just let me write it again. So, mod Xi = 1 u of x + r Xi, so this is surface integral on the hemisphere.

So, as such if you notice here this integration with respect to Xi variable, so if u is smooth, so I can simply take this del square by del t square directly let me take that, M u x, r, t. So, this is simply 1 by sigma n mod Xi = 1, so u tt, 2 derivatives. So, you simply take the differentiation under the integral side. And similarly if you notice at the x variable, this x variable and Xi variable that is variable of the integration they are also not coupled.

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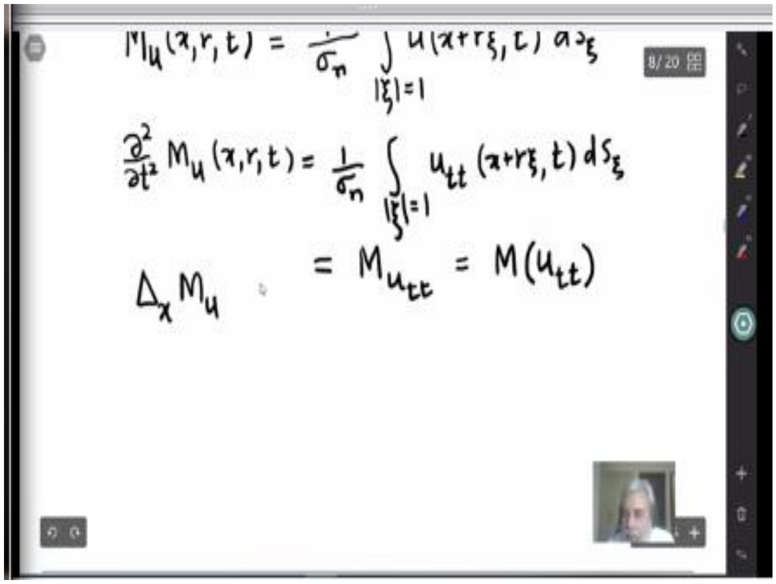

So, this we immediately obtained, so this let me stress now M u before that let, so this is nothing but the spherical mean function of u tt. So, if we think this as an operator, so this will write this as. So, the second derivative with respect to t variable of the spherical mean function is same as the spherical mean function of the second derivative of u with respect to t, so that is what it says. (()) (07:51)

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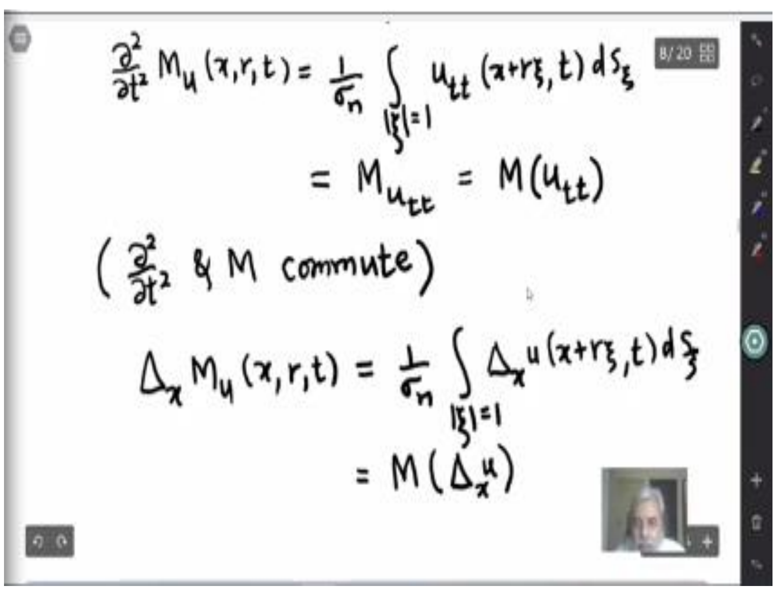

So, this in other words we write this as, so this operator of differentiation and this M they commute, whether you first operate M and then take the derivative that is what is left hand side or you take the derivative first and then take the spherical mean function they are same. And same thing happens with the x variable, so this just let me stress that M u x, r, t which is nothing but (()) (09:04) if x and Xi were coupled, so we cannot do this. But since in the integrand x and Xi are not coupled we can do this and that is same thing as M delta x u. So, the importance of the Darboux equation now comes into play. So, Darboux equation relates the Laplacian of the spherical mean function with respect to x with this operator.

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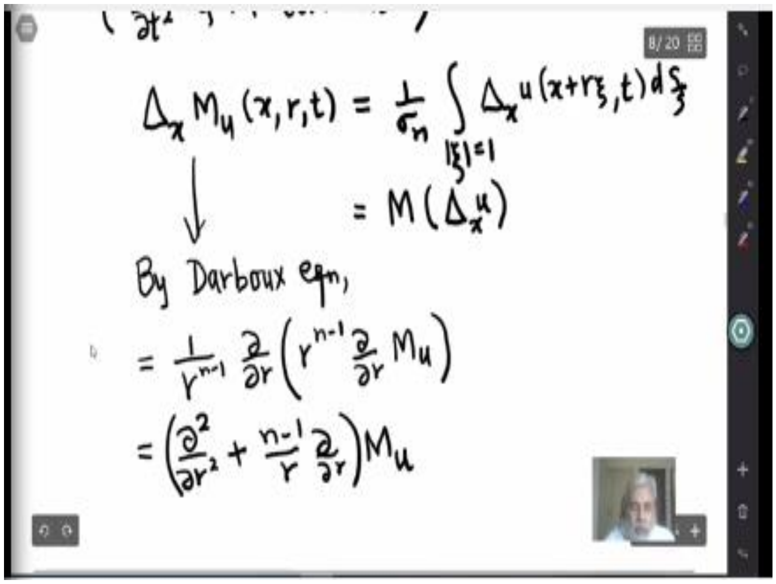

So, now you apply the Darboux equation, so this one by Darboux equation. So, this equals to 1 by r to the n - 1 this operator del by del of r to the n - 1 del by del r of M u. So, if you perform the differentiation and expand it but this simplifies to. So, del square by del r square + n - 1 by r **del r** del by del r, so this operator second order operator acting on M.

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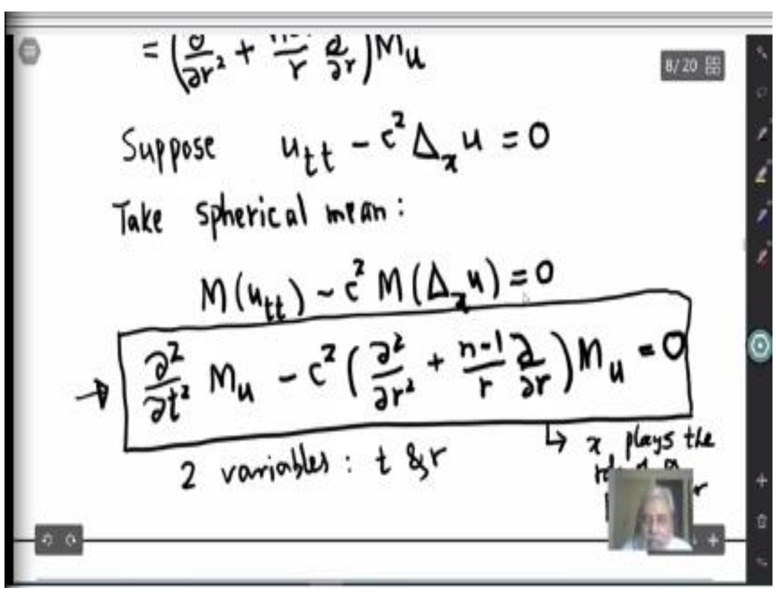

So, now suppose u is a solution now, the wave equation suppose u tt - c square, so let me just stress that dependence, so this is only with respect to x variables. And now you take spherical mean on both sides of this entire thing. So, spherical, so since this is just an integral, so it is a linear operator, so this is just M of u tt - c square, that is a constant M of delta x u. And now look at the computations we have done and this is just d square by d t square M u - c square d square by d r square + n - 1 by r d by dr M u – 0.

So, if u is a solution of the wave equation then it is spherical mean satisfies this equation, so this is our first observation. And in this equation notice that, so x plays only a role of a parameter. So, the spherical mean function now satisfies again a second order equation but in only 2 variables. So, that is what we want because we somehow want to reduce the whole analysis again to one dimensional wave equation, so again we can apply D'Alembert's formula and other things.

So, that is our goal, so we have achieved now, so this look at here, so the spherical mean function of u satisfies a second order equation but only 2 variables, so that is important, t and r. So, what about the initial condition? So, initial conditions are prescribed on the solution u at t = 0 and again taking the mean value. So, they transform to initial values for the spherical mean function and that is what I have written here.

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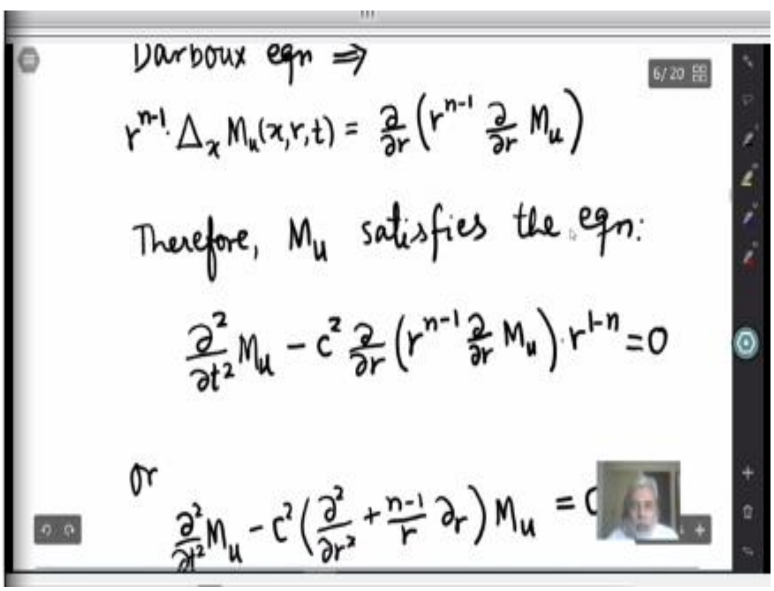

So, taking the spherical mean function of the solution u.

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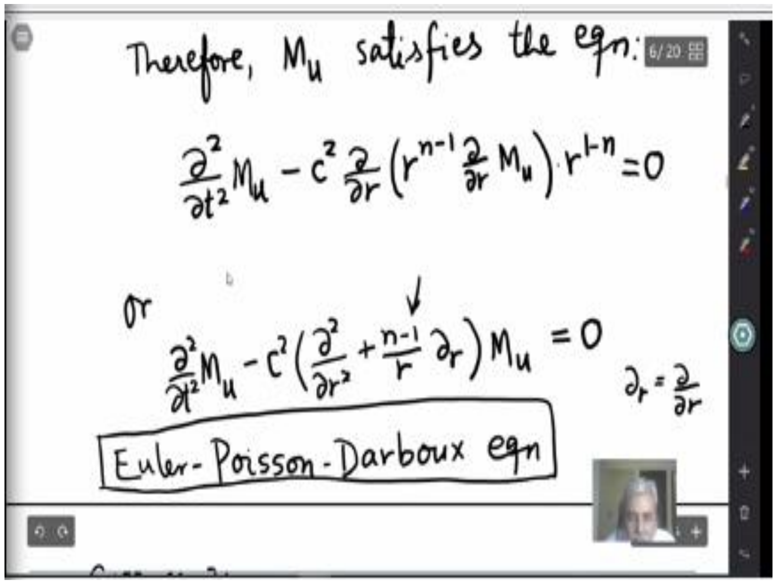

We just saw that it satisfy this equation and this has a name, it is called Euler-Poisson-Darboux equation. Again if this term, this first order this is just del r is, so I am mixing that same notation del by del r. So, if this first order derivative were absent then we simply get a wave equation in one dimension. But in the presence of this, so there are some difficulties and that we already seen.

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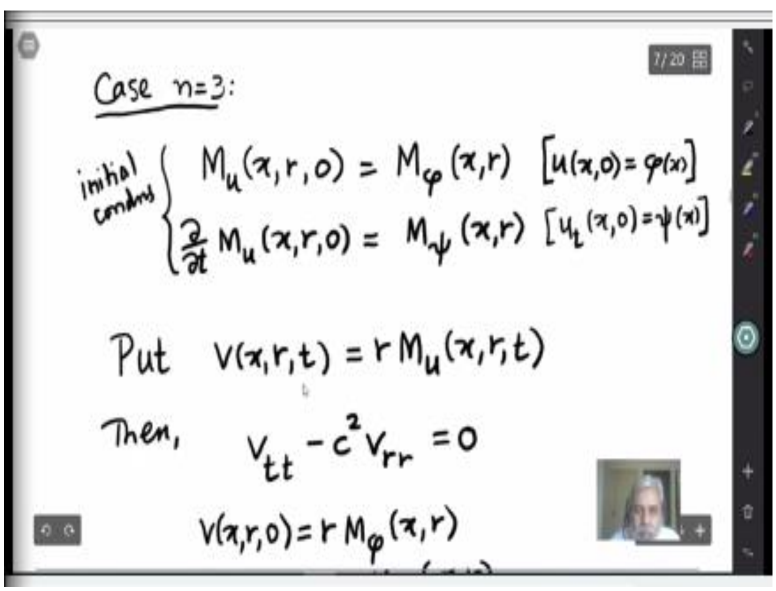

So, again we go back to the case n = 3, so the initial conditions at t = 0 on u now are transformed to initial conditions for the spherical mean function, so this initial conditions at u = 0. So, here sorry let us stream level here, (()) (16:44) so we are given these initial conditions of u and now we are just transforming them to r. So, again first consider the case n = 3, we will come back to the general case of M later. Because the transformation is little more complicated, so for n = 3 it is very simple one. So, again you put V of x r of x, so the function of x r, t, this r into M u x, r, t.

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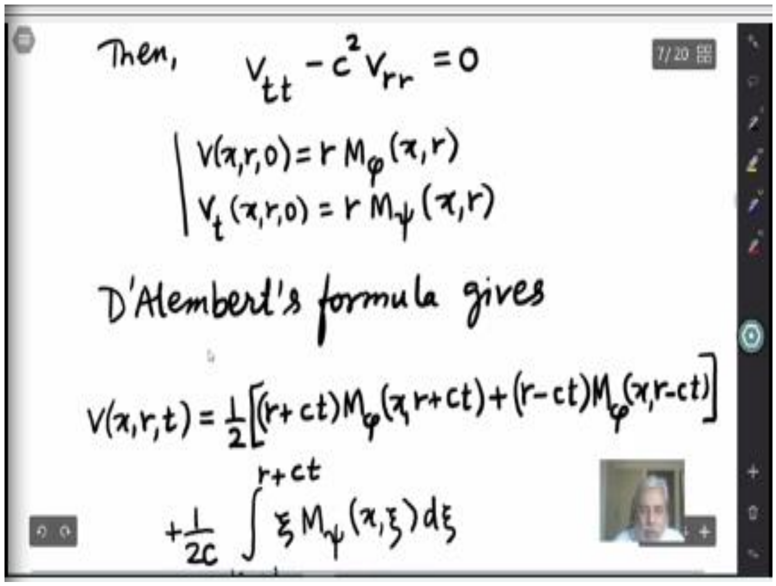

And as we observed earlier, so this V satisfies this wave equation in one dimensions. And what are the initial conditions for V? They are almost same as this spherical mean function M u, x (()) (17:45) out for this factor r so V of x, r, 0 is our M mean value of phi and the first derivative of V at t = 0 is r M psi. So, again you just observe that x is just a, it is role is only as a parameter. Now V satisfy this wave equation in one dimensional wave equation with these initial conditions. So, we can immediately use the D'Alembert's formula and write down the solution. So, these are the initial conditions, just here.

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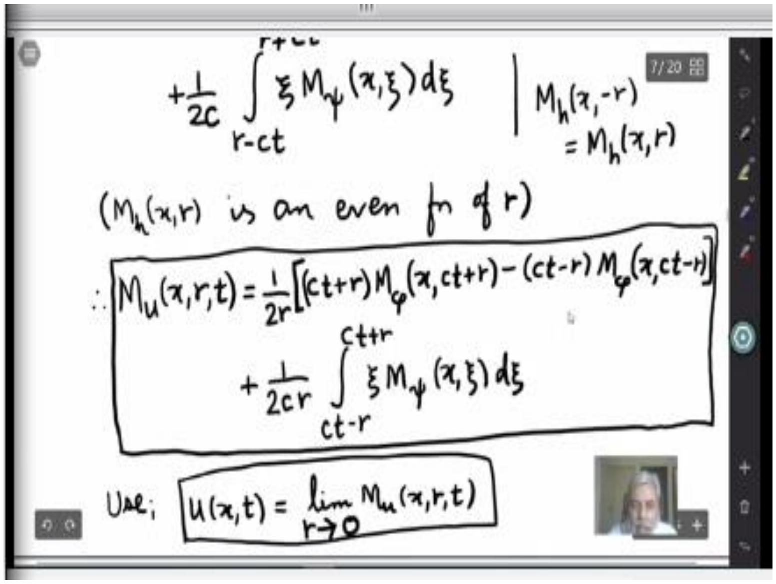

So, this is first part coming from the initial condition at t = 0 and this is the derivative. So, this is just by D'Alembert's formula. Now you want to rewrite this little bit, so here, so recall this the mean value function is any one function of r. So, that we already observed just by the definition, so this M phi x, r - ct I can write as ct – r. So, to keep the same notation here, so I want to write ct - r also here but that produces a negative sign, so that is what I have written here.

So, there is no, this is same as M phi x, r - ct but this one I write as ct – r, so it will be clear why you are writing that. And again same thing, so this is an even function and we are multiplying by an odd function Xi, so this is whole thing is again an odd function. So, if you substitute and do some work, so we can replace this r - ct by ct – r, so that is what with the observation. And this r is coming because of this V, V is r M u, so M u is V by r, so V there is no r, so we just divide by r, so we get this here.

So, we have obtained a formula for the spherical mean function of the solution and from there would like to get a formula for u itself. And this is where again this usefulness of the spherical mean functions come into play. So, if we take the limit as r goes to 0 we get back the original function, so this is what we have to do. So, we have to take limit of this right hand side as r goes to 0, so there is r in the denominator, so we have to exercise little here there. So, let me just do one by one**.**

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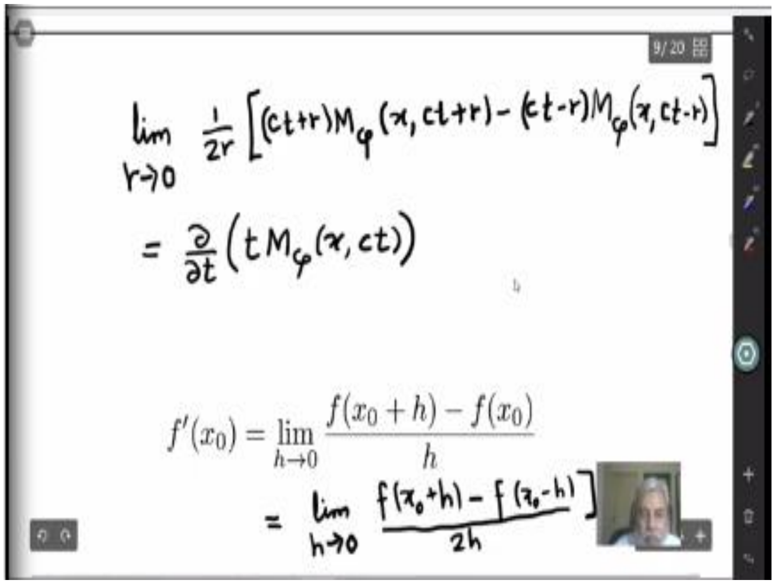

So, first let me take that first term, what happened to the limit 1 by 2 r of this function ct + r M phi x ct + r - ct – r like that M phi of x ct - r. So, what is this? As r, just we want to convert this into a derivative, so how we can do that? So, if you have any functions, so this is just one variable function. So, we know that the derivative, we simply limit you have found (()) (23:42). So, this is plain definition function of one variable.

So, this is also equal to limit f of x 0 + h – f of x 0 - h divided by 2h. And we are in such a situation, so here is our function if you take this ct into M phi x ct, you take that as the function. And then you apply this formula for the derivative, so what we get is derivative of t into simply derivative. But, so this is simply just work it out, so this is d by dt of t M phi x, ct. So, just use this definition of the derivative you will get that. The second one is much easier, so that is the integral. So, here it is just integral, so you get just the mean value of this integral, so this let me write that.

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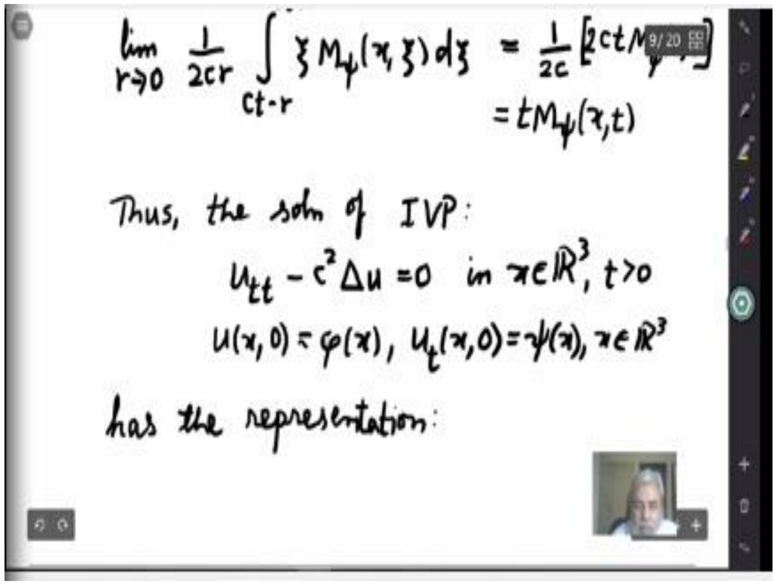

So, 1 by this only (()) (26:18) as r tends to 0, 1 by 2cr integral ct - r ct + r Xi M psi x, Xi, d Xi = 1 by, so that there is a c there, we write that c, ct M Xi x, t maybe there is a 2 there and that just simply comes to t M Xi x, t. So, as I said computations are bit lengthy but they are straightforward. So, thus the solution of IVP, so finally we can write that u tt - c square Laplacian u = 0, remember we are dealing only in n = 3, so let me just stress that. Because this is not the formula for generally n, so this with initial conditions u of x 0 = pi x u sub t x 0 = psi x in x in R 3 has the representation.

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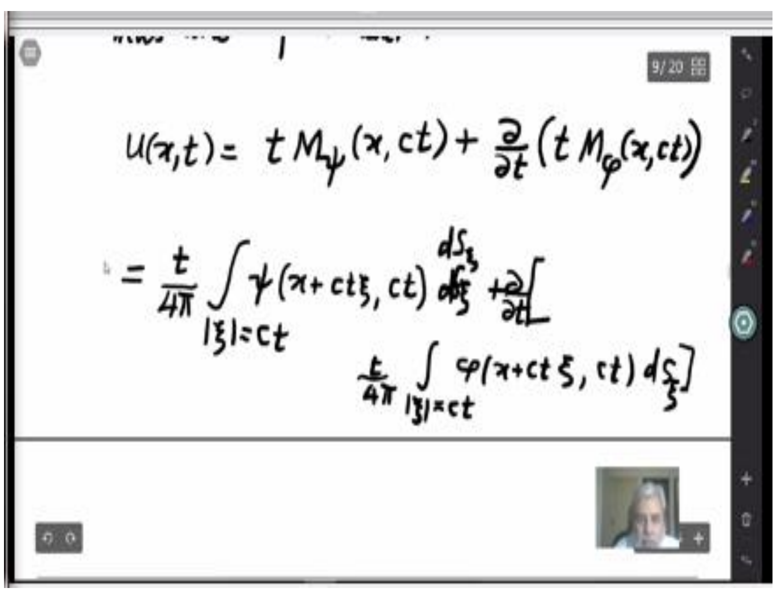

So, let me write that u of x, t = t M psi x ct + d by dt of t M phi x, ct, where M psi and M phi are the spherical mean functions of psi and phi. So, we can express directly in terms of that, so this is just t by, so in n = 3, sigma 3 is just 4 pi, so let me just write that integral mod ct psi of x + ct Xi ct d Xi and similarly this one. So, d by dt of this big expression t by 4 pi integral mod Xi = ct psi of x + ct Xi ct dS Xi. So, these are surface integrals, let us not forget that, so this is dS Xi.

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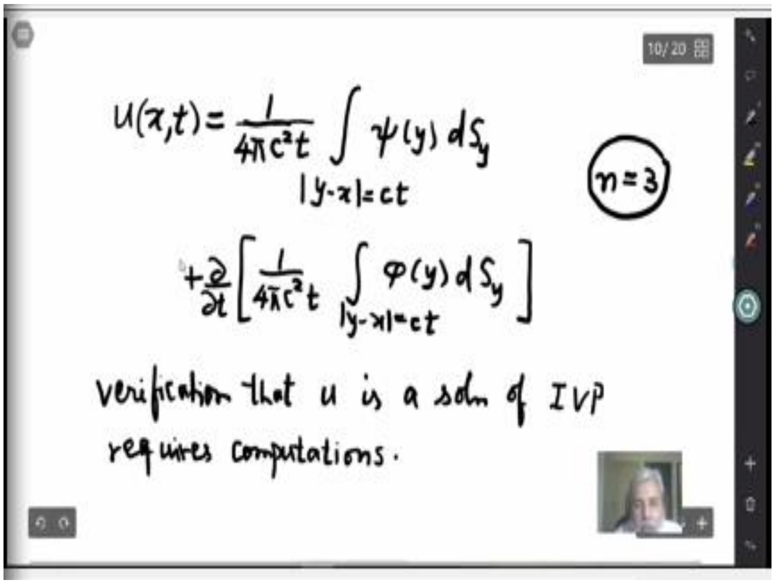

So, again by change of variable we can write. So, this is 1 by 4pi c square t surface integral y - x = ct psi of y dS y, so this is the surface measure on the surface, so y that variable + d by dt of 1 by 4pi c square t y - x = ct phi y dS y. So, again just I will stress, so this is the formula for n = 3 let remember that**.** And so even though this looks very simple one, so the verification that u satisfies u is a solution of IVP.

So, that is not straightforward, so it requires some work, require some competition. So, for example even verification of the initial conditions, so straight away we cannot put t = 0, so because there is a t in the denominator. So, we have to only take this limit as t go to 0, so that requires work.

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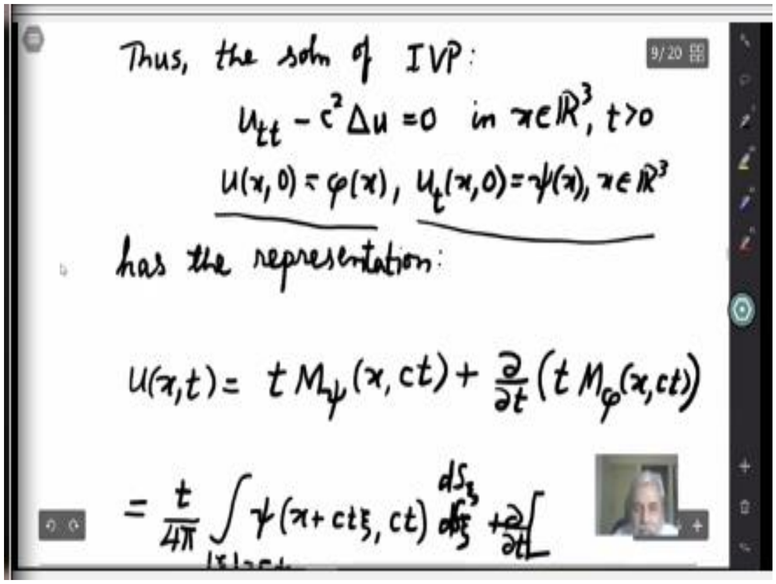

So, even just verification of the initial conditions is not straightforward in this case, so in the case of D'Alembert's formula that was very straightforward. But here we have to do some computation, so we have to take the limit. And in the next class we will further analyze what the solution and what are the qualitative properties of the solution? Again domain of dependence, range of influence and we will also come across the Huygens' principle and what does that mean? So, we will discuss all these things in the next class, starting from this representation of the solution in n = 3, thank you.