First Course on Partial Differential Equations-II
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Lecture-30
Wave Equation-1

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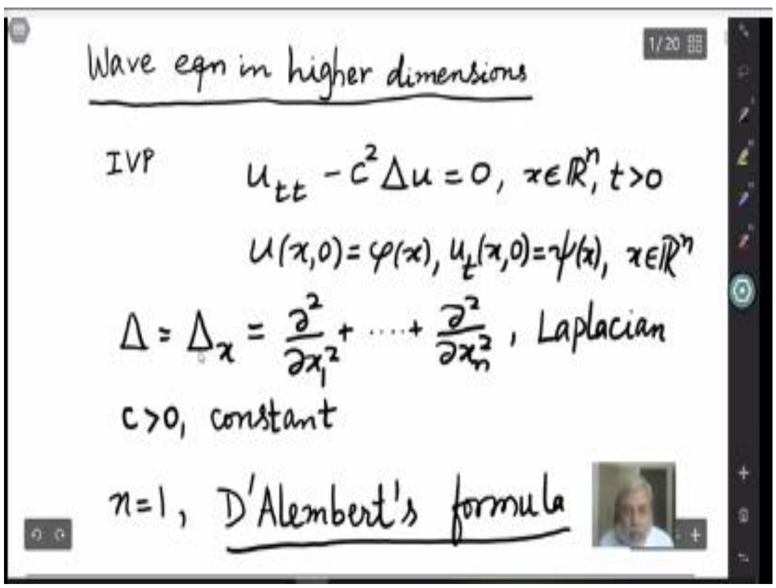

Hello everyone, welcome back again. In this lecture we will begin discussion on wave equations in higher dimensions. So, in the next few classes this will be the main discussion and also discussion will be mainly on the initial value problem. So, let me again state it, so this is the wave equation in higher dimensions. So, again a second order equation, so u tt - c square Laplacian u = 0, for x in R n and t positive and at t = 0 in that stage that is x space.

So, we provide the initial conditions, so that is the initial position and initial velocity. So, this equation also arises many physical processes, so as we go on we will discuss some of them. And as usual, so this Laplacian, so sometimes to stress the dependence on the variable we also write that as Laplacian sub x and that is the second or operator Laplacian and c is a given constant. So, before we go further in the discussion of the essential value problem, let us recall what we have done for the case n = 1, so that was in the first part of this course. So, the important formula we derived for the wave equation in one dimension and that is D'Alembert’s formula.

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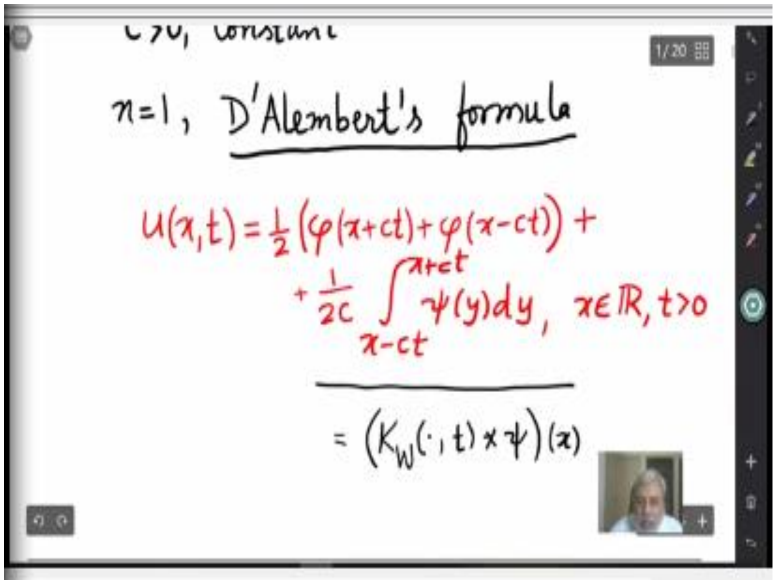

So, let me write it, so this is given by u of x, t = half of phi of x + ct + phi of x - ct + 1 by 2c this integral x - ct to x + ct psi y dy, so phi and psi they come from the initial conditions. So, the one difference I would like to remark here between this formula and the formula for the solution of the heat equation and also Poisson equation we have discussed. In that, so any solution of this wave equation in one dimension satisfies this D’Alembert’s formula.

And conversely any function u defined by this D’Alembert’s formula satisfy the wave equation with these initial conditions, so that is the difference. So, there is, so any solution automatically satisfied this D’Alembert’s formula and that is not the case with for example heat equation. Though we derived the Fourier Poisson formula for a solution, but in general that is not the only solution, there we have seen that uniqueness is not there.

But that is not the case with this wave equation and there is uniqueness here. So, at this stage, I also want to remark regarding a fundamental solution. So, in the first part of this course and even in the second part we have been talking of fundamental solutions for the Laplace operator and also heat operator. But we are never talked a fundamental solution for the wave operator. It is not that the wave operator does not have a fundamental solution, it has.

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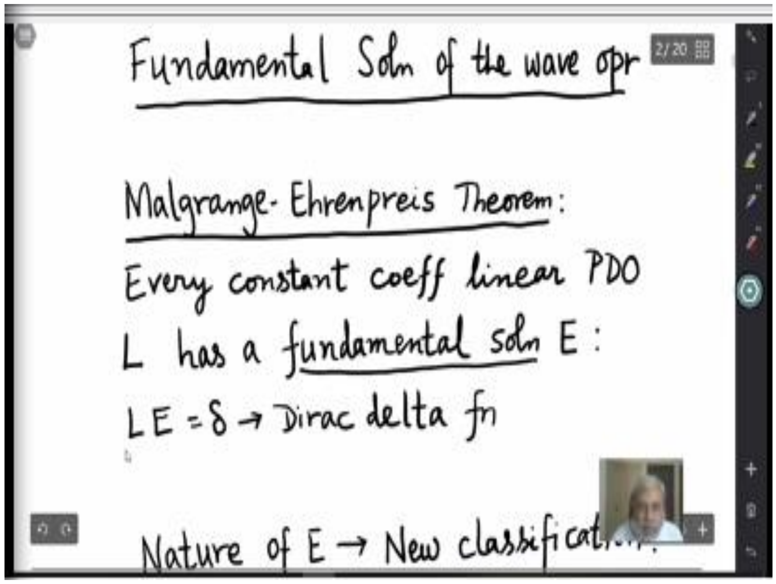

In fact it is a celebrated theorem of Malgrange and Ehrenpreis. So, just let me state for the record who is states that every constant coefficient linear partial differential operator PDO L has a fundamental solution E. So, the meaning of the statement is that this when you operate E when you operate the given partial differential operator on this fundamental solution, so that is L of E who should get this delta function.

And in the case of Laplace operator and also heat operator we have seen. In general situation this fundamental solution is not necessarily a function. So, we have to enhance, we have to go outside the realm of functions. So, in case of Laplace operator and heat operator the fundamental solutions are c infinity function you except a singularity at one point.

But in general the fundamental solution given by this theorem need not be a function and it is can be a distribution and that is where the trouble starts, so we have not developed any concept regarding distributions and their operations, their algebra, their calculus. So, that is very modern part of this subject this VDE subject.

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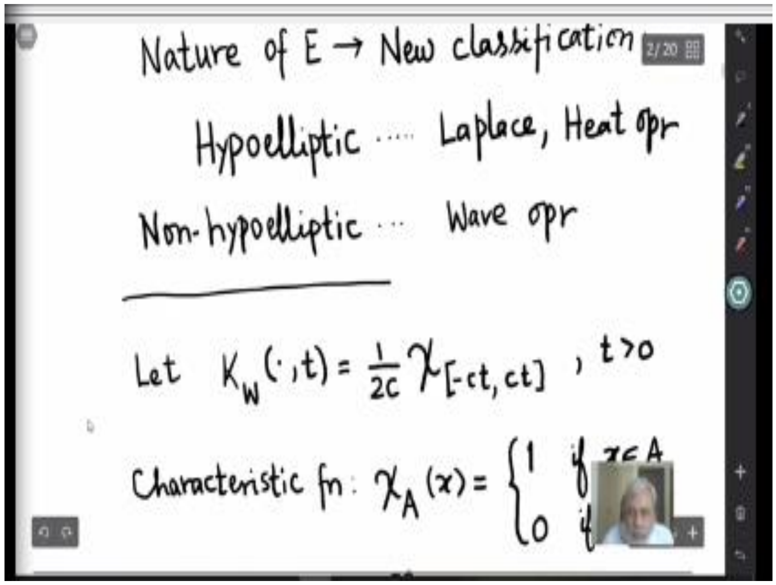

So, that is why you are not discussed in detail about the fundamental solution of the wave operator, I just tried to indicate what that is in case of n = 1. And more importantly this Malgrange theorem, Ehrenpreis theorem depending on the nature of this fundamental solution, a new classification arose. And now we classify the class of linear partial differential operator of any order, not necessarily second order of any order depending on the nature of this E. So, they are now classified as hypoelliptic operators and non-hypoelliptic operators.

So, this Laplace and heat operator they come in the class of hypoelliptic operators and wave operator is a typical non-hypoelliptic operator. So, let me just indicate what this fundamental solution for the wave equation in n = 1 and do we try to rate this D'Alembert's formula in terms of that fundamental solution. So, for that purpose I define this, so it is fundamental solution of the wave operator but again I will not go into the details how this comes and how one had do it in a general case?

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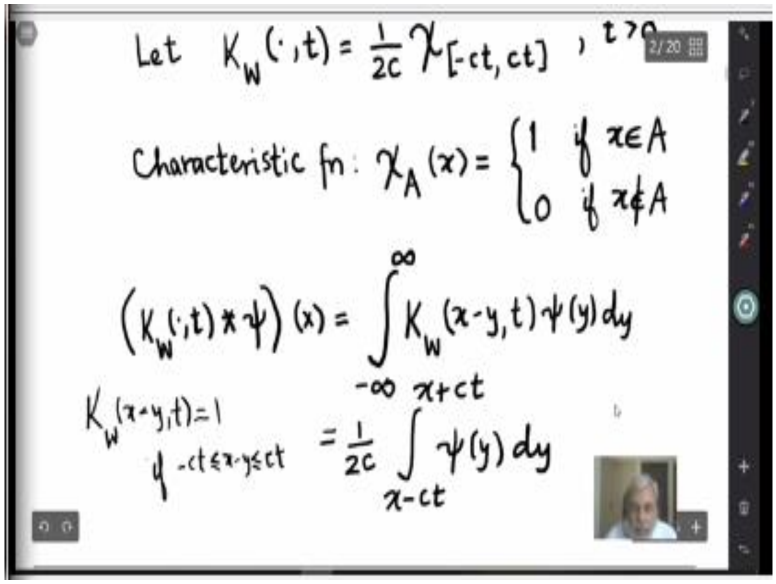

So, there is just take this as a kind of (()) (09:22) procedure. So, define this function k w? It is a function of 2 variable section t, for t positive, so that as a function of x is nothing but this 1 by 2c, so that constant comes. So, this is the characteristic function of this closed interval -ct to ct. So, taking t positive and c the positive constant data appears in t wave equation. So, characteristic function for a general subset is defined by A of x = 1.

If x belonged to A and 0 if x does not belong to A. And now with this that w is for wave, so this characteristic function fundamental solution for the wave equation. And now you compute this convolution. So, take any nice function psi a function of x only. And you compute this convolution and that convolution by definition is given by this integral. But now that k w, just so one more step here.

So, K w x - y, t is 1 - ct less than equal to x - y less than equal to c otherwise it is J. So, the syntax is restricted only to this interval -ct to +ct and then you make change a variable. So, from, so you are integrating with respect to y variable, so you make change of variable and finally you arrive at this integral, so integral 1 by 2c, so that 1 by 2c is factored in this fundamental solution and that is 1 by 2c to x - ct to x + ct.

And that is precisely if you look at the D'Alembert's formula, it appears here. So, this part now equal to K w of t x psi. So, what about the first part? That first part, so again you take the convolution of K w with phi.

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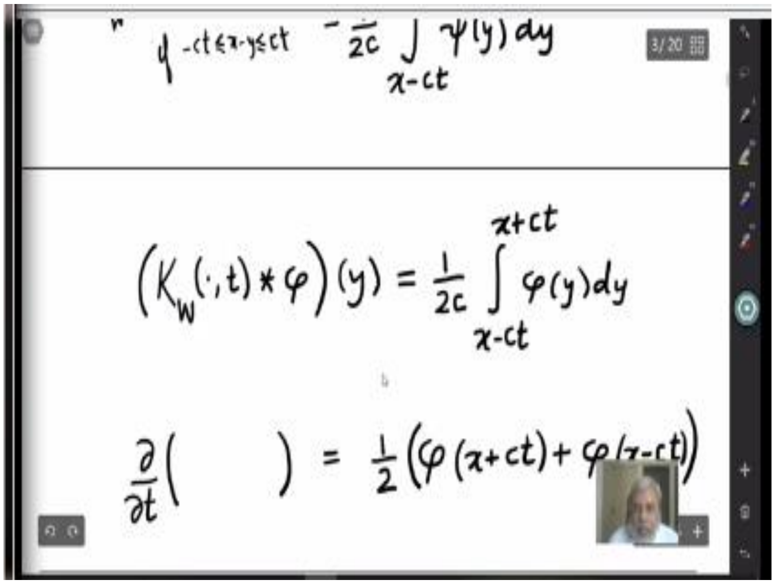

So, we get that, so just now we computed that convolution, so instead of psi we are taken on phi. And now you differentiate with respect to t, so if we differentiate this expression we get precisely, so there is a c there that c cancels here when you differentiate with respect to t and we get half phi of x + ct and + phi of x - ct and that is the part in the D'Alembert's formula. So, therefore, we rewrite the D'Alembert's formula, just recall that.

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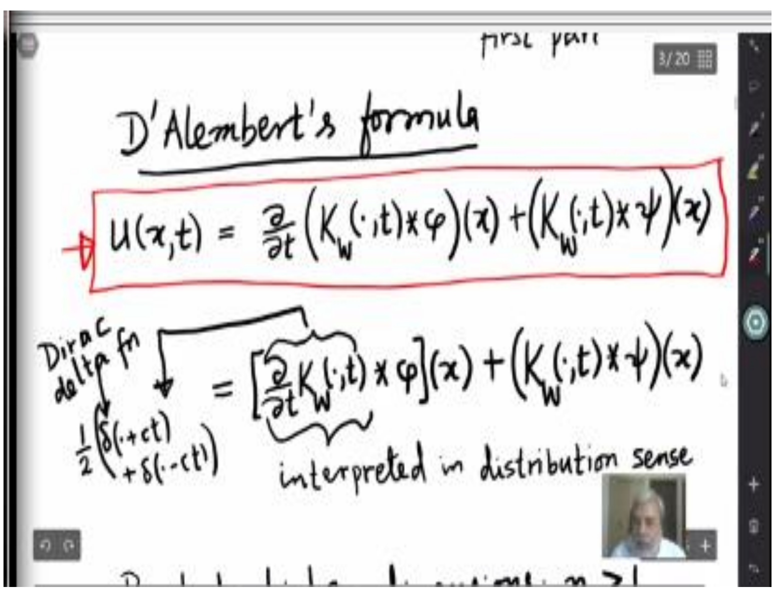

So, in terms of this K w, so that is d by dt of this convolution K w with respect to and the function phi and then there is no differentiation, so only convolution with respect to psi variable. And this is written as this d by dt of this but this has to be interpreted in distributions**.** Now this is just for the sake of information, we are not going to develop distribution theory in this course. And so when we learn that distribution theory, so this turns out to be half del z by +ct –ct.

So, this is Dirac del, so this is the reason we did not have any discussion on the fundamental solar (()) (15:33). Because we have to develop these new tools and then the calculus related to that, so that will be really an advanced course. What I would like to remark is, so this is what we derived for the case n = 1 D'Alembert's formula and now we wrote that in a different fashion using this fundamental solution. And this form of the solution is retained for all dimensions, so that is the important thing.

So, there will be a K w, so that depends on the dimension, so we will have one convolution here and then we take the time derivative and to get another convolution there. So, the solution even in any dimension is given by such a representation, so that is one important remark I want to make. And nature of this, though I am not explicitly be stating that but when you write the formula for the solution of the wave equation in higher dimensions, you will recognize the form of this fundamental solution.

And that is very much dependent on the dimension n and in all dimensions, it will have a particular qualitative property which is very, very different from fundamental solution even dimensions. And, so even in one dimension, so we will see it later that there is no resemblance of this fundamental solution in one dimension and higher dimensions.

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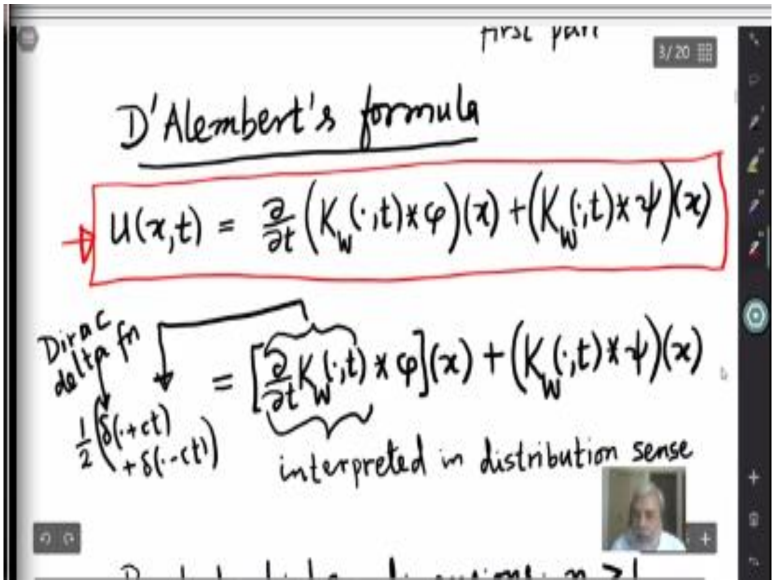

So, now we are going to derive a formula for the solution of the wave equation in higher dimensions, so that is our next agenda. So, unlike the case n = 1, we immediately do not have any idea how to proceed. So, to get some ideas and motivation, so let us start with some simple solutions. So, begin by looking at radial sources, so if there are possibility of finding a radial solution to the wave equation.

And this is certainly a possible way when the initial conditions phi and psi themselves are radial functions. And then in that case certainly you can look for a radial source and so this dependence on of u on x is only in the radial direction, so it is just a function of mod x.

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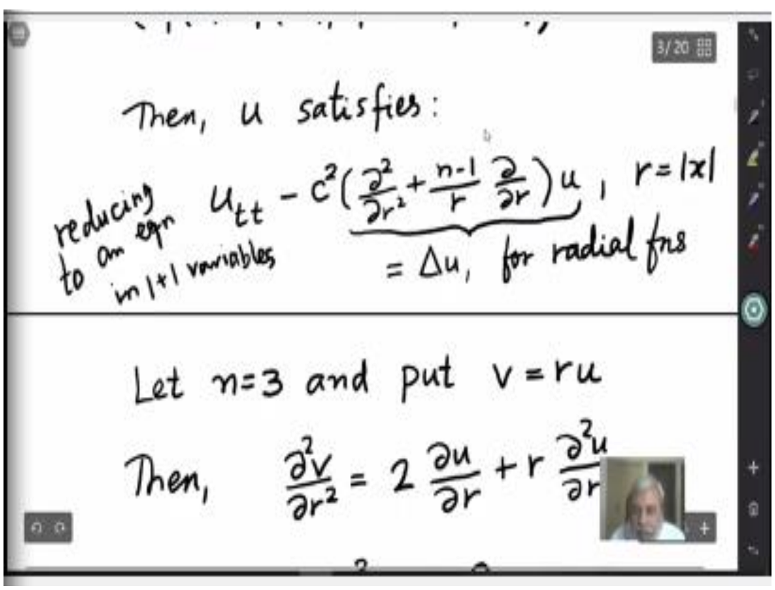

And in that case, the Laplacian you already seen this, so the Laplace operator in the, so this is r = ok that so r =1 x. So, in this way we have reduced the given wave equation mean n + 1 variables into an equation which is 1 + 1 variable, so this is essentially reducing equation. So, let me write that 1 + 1, so there is already a t variable and now we have only the r variable. And if this factor were not there it simply wave equation in one dimensions. So, in this presence of this a first order derivative with respect to r it is no more an wave equation. But in some cases it can be reduced to one dimensional wave equation by some suitable transformation.

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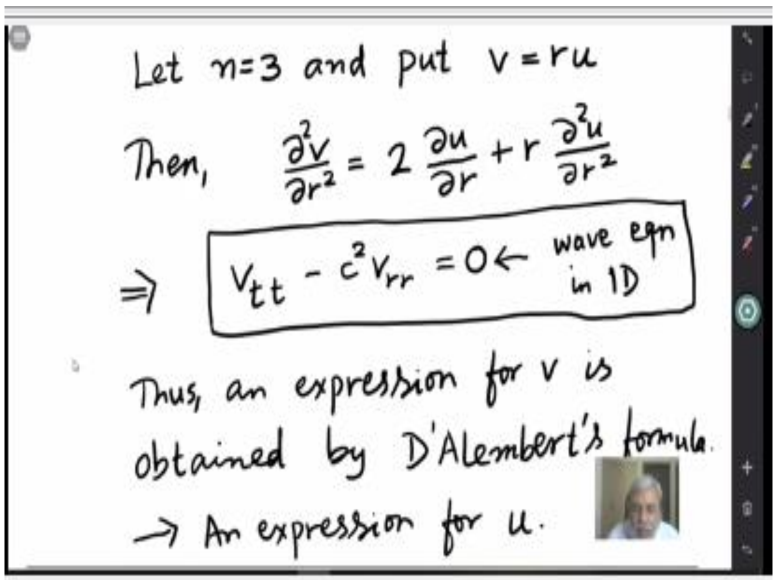

So, for a (()) (21:19) we will see later, but for the time being just consider the case n = 3 and so this coefficient will be 2 by r. And this simple transformation, now you just put this v = ru and then use the Leibniz rule. So, I want to compute the second derivative of v, so just leave the Leibniz rule with respect to r. So, there is no problem with t because this is again multiplying by r, so t derivatives of v are same as t derivatives of u that multiplied by r.

So, this del square v by del r square = 2 del u by del r + r del square u by del r square. And now when you substitute these computations here, you immediately see that v satisfies this first order, so this is wave equation in 1, now the variable r and t. So, we can immediately use the D'Alembert's formula to write down the solution for this v. So, obviously the initial conditions are t = 0 for u and u sub t are transformed to initial conditions on v.

So, using those initial conditions, we can immediately write down the solution v. And once we know that solution v, so simply divide by r and we get an expression for this solution of u in this particular case. And this gives us good motivation, so somehow bring in an extra variable r in which case the given wave equation is reduced to an equation in 1 + 1 variables and by suitable transformation eventually to 1D wave equation.

So, our next target is how to transform the given wave equation into a one dimensional wave equation. Because we know a lot about one dimensional wave equation and that will help us in deriving a formula for the solution even in higher dimensions. And such a procedure of transforming the given wave equation in n dimensions to an equation in just 2 variables is provided by this method of spherical means.

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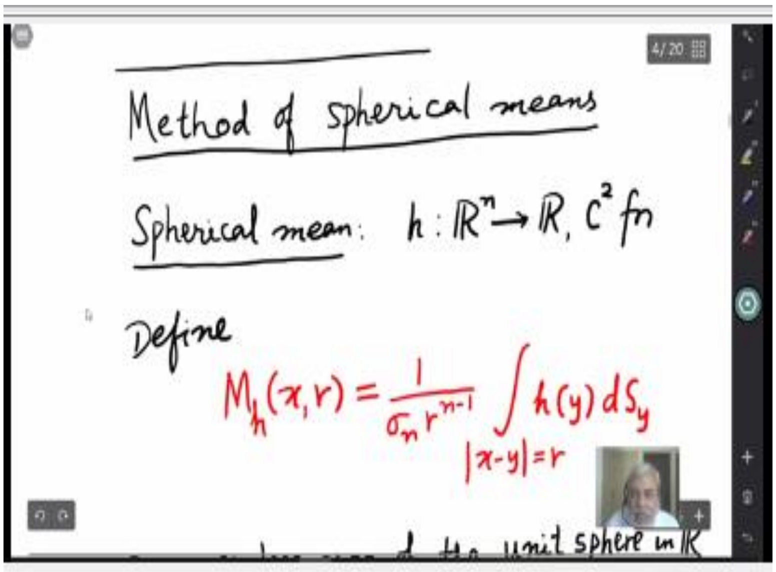

So, let me again recall what is spherical mean of a function? This we have seen in a lot in the study of Laplace equation and Poisson equation and very fruitful one. And we will also see that it is also fruitful in dealing with this wave equation in multi dimensions.

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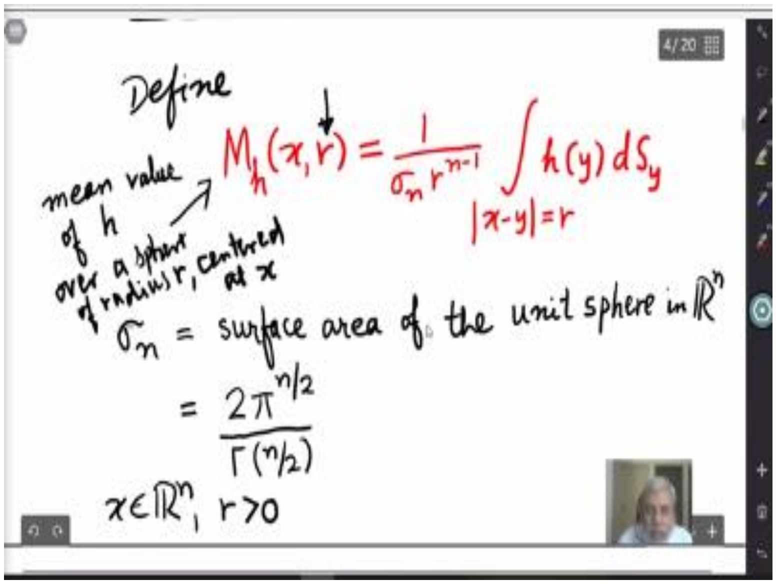

So, let me just again recall that what is the mean value of a given function h? So, h is a given function from R n to R, a c 2 functions. For the definition we do not need c 2 but we want to do some computations, so that is where I need c 2 function. So, even just with continuity we can define the spherical mean function. So, we denote spherical mean function of h by M h, so it is a function of 2 variables x and r.

So, this is nothing but the mean value of the given function h over a sphere of radius r centered at x. So, this is mean value of h over a sphere of radius r centered as x. And as we proceed further, it will be clear that, so this is the one new variable we are looking for. So, this is just to begin with this a positive real number, so we soon extended it to all real numbers. So, x, essentially plays the role of a parameter, so in the discussion of the equation satisfied by this M h, x hardly plays any role as we can see further.

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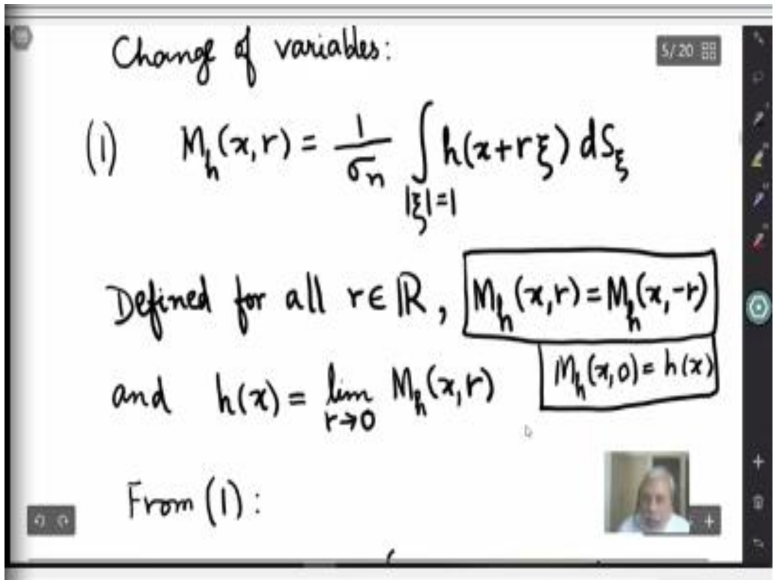

And sigma n is the surface area of the unit sphere and it is numerical value is given by this number 2 pi n by 2 divided by gamma n by 2 and gamma is the Euler gamma function. So, in this definition this M h is defined only for r positive but by simple change of variable will immediately see that is defined for all r. So, you just change the variable here, so you put I replace this y by x + r Xi and Xi varies over the unit sphere, change of variables this r to the n - 1 just vanishes.

In this form M h is now define for all r and by changing r to -r and then we can change the variable Xi to -Xi and this is your sphere, so there is no change there. So, you immediately see that as a function of R this mean value function M h is even function. And the good thing about this mean values, so we can also recover the function we started with namely h by taking limit as r goes to 0. So, these 2 properties just, so it is an even function of R and we recover h by taking limit r = 0, so these 2 are important properties.

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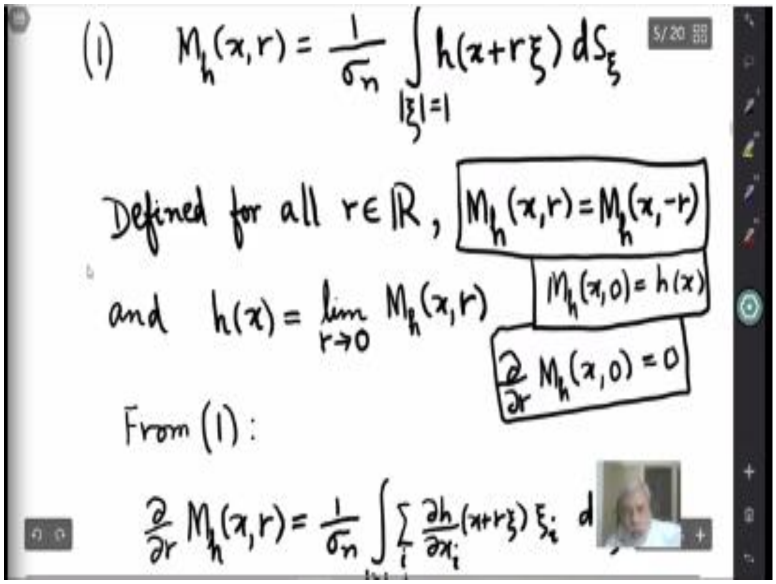

And because this is an even function, so we will also see that, so del by del r, just work out it is 0.

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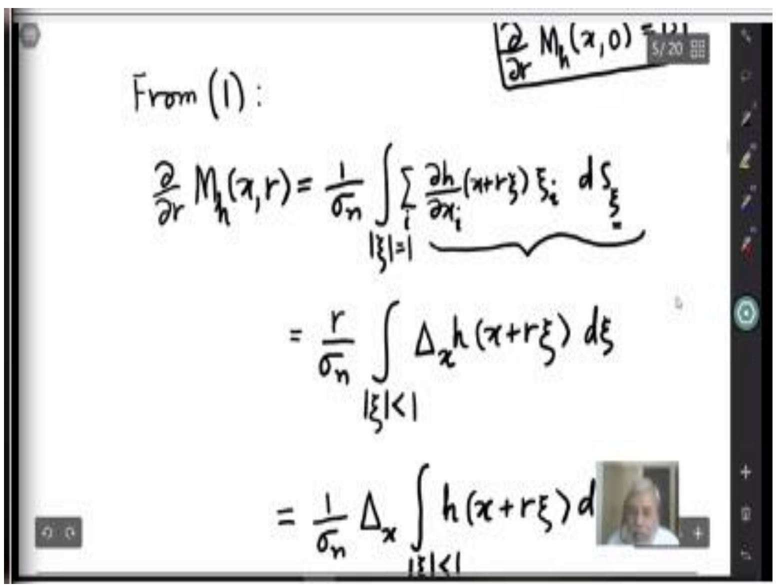

So, we are interested in finding the second derivative of this mean value function M h with respect to r, so that is what is. And you want to relate that how it is related to the second derivative of h? So, we will begin some computations, so if we use this form of the definition, then it is difficult to differentiate with respect to r. But this change of variable now r has entered the integrant, so there is no r on the domain, so it is easy to differentiate.

So, we just take the differentiation sign and to the integral sign and so that is del by del r is equal to. So, now r is here, just take the del h by del i and then when you differentiate this variable with respect to r you get a Xi i. And that certainly you can write it as, so remember, so this Xi the variable which we are integrating, so just exercise some.

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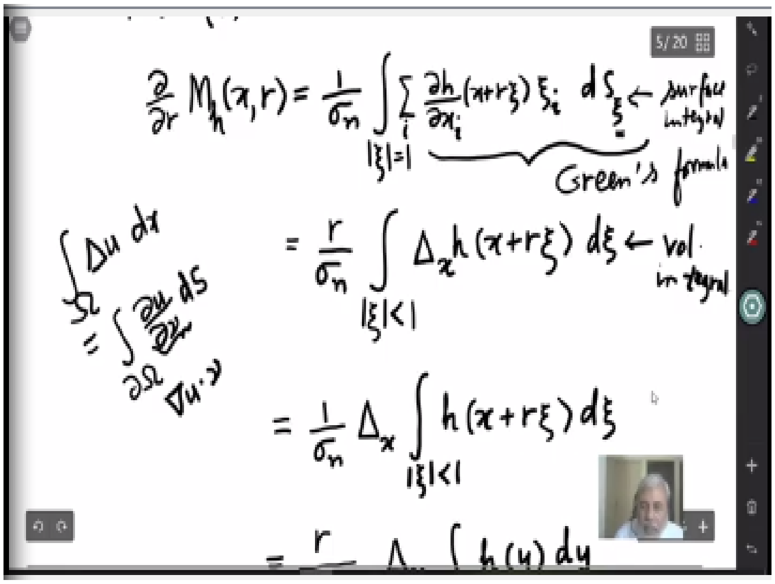

Here there and now we want to apply the Green's formula. So, let me again write it in the side here, so we have a Laplacian u, so this is volume integral and this is just, where u. So, this is the normal derivative, this is dS and this is just nothing but grad u tau. And since this is the unit sphere the unit normal is in the direction of Xi itself, so this is mu i in this case.

So, it is of this form but only thing is you have worry about this r coming there, so this produces an extra r here that is the only difference. And now this is volume integral, so this is surface integral and using Green's formula that we have converted into volume integral. And now there is r missing here.

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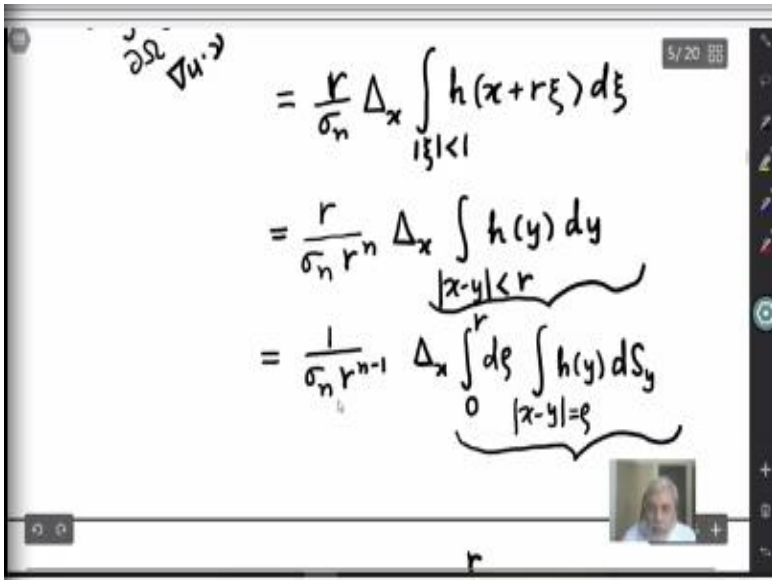

So, this is r and the variable of integration is Xi, so that is nothing to do with the x, so you just bring this Laplacian operator outside the integral and rest of the integral there. And again that you change the variable, so you just write that h y over surface, this is surface integral. So, let me, this is volume integral, no problem and this volume integral using spherical coordinates; we just write it like this.

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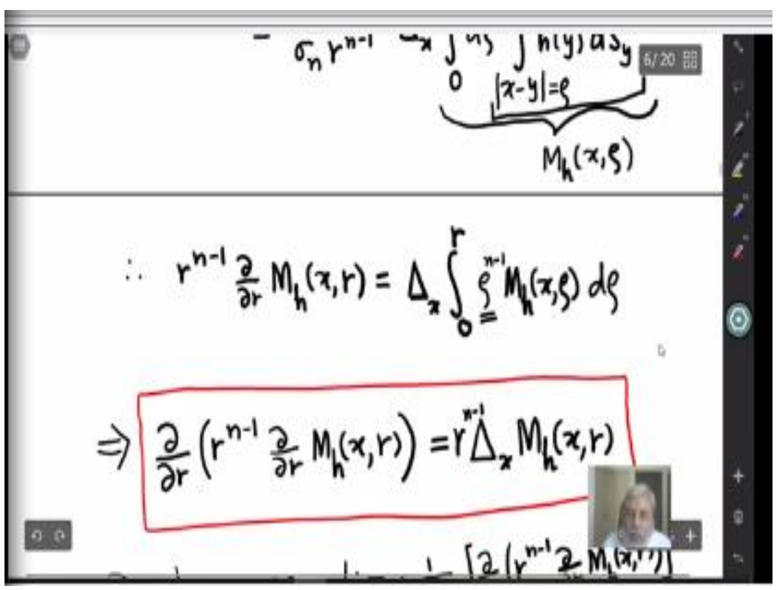

And when you do that change of variable here, you get a factor of r to the n, there is already r there, so this is just 1 by r to the n - 1. So, now multiply both sides by this r to the n - 1, so that is left hand side is already del by del r M h x, r and this one I want to write as this surface integral as M h x rho. So, the only thing is there is a factor 1 by rho to the n which is not there, so that is why you are multiplying by this extra rho to the n - 1.

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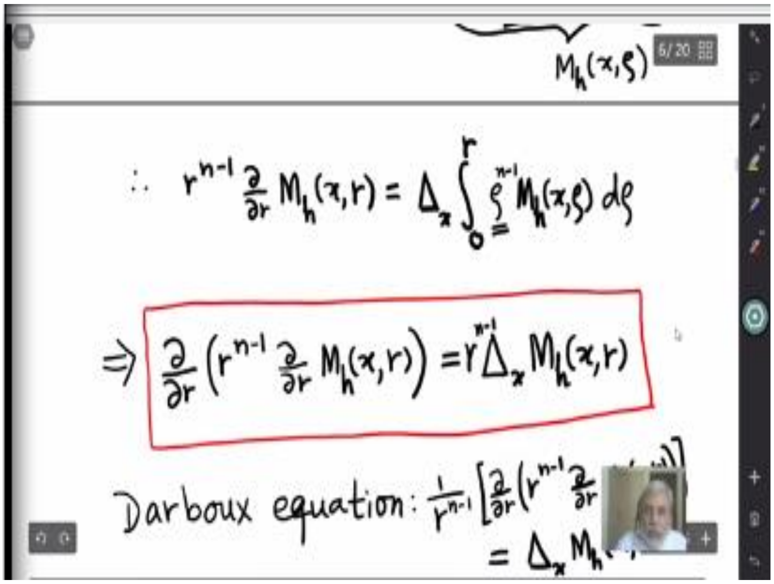

And now you just simply differentiate one more time, so we get this, so that when you differentiate with respect to r, so you just comes out to be r to the n M h x, r and again this r index is nothing they are independent. So, this Laplacian operator I can move anywhere. So, this is the relation between this second derivative of the spherical mean function with respect to r.

And the Laplacian of the given function h and this is called Darboux equation. And we will continue from here next time. And how this should be applied to the wave equation? So, ultimately it is for the purpose of getting solution of the wave equation and we will see how this Darboux equation is applied to the solution of the wave equation, thank you.