First Course on Partial Differential Equations - II
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Lecture - 03
Hamilton Jacobi Equations

Welcome back to the lecture of Hamilton Jacobi equations. In the first lecture of Hamilton Jacobi equations have basically introduced the equation and we indicated it look comes from Hamilton calculus of variations and there are more generally equations like Hamilton Jacobi Bellman equations and Hamilton Jacobi Isaacs equations coming from optimal control theory and differential theory problems.

And we also see the, it is an equations in n + m variable when you start time variable t and then n spatial variables x 1 etcetera x n. And hence the characteristic equation say if you go to the first order theory or method of characteristics, it will ahead to n + 3 equations, but due to the special nature of this equation, it can be reduced to rather it can be separated and decoupled to get 2n equations for x i NPI variable.

That is du / dx i variable and these are called Hamilton's ODE. We will see its connections to classical mechanics a little later maybe in the next lecture or lecture after that. So and then we have seen 2 examples in the last class one is the very simple equation, but there is no solution classical solutions, but we have seen a lot of Lipschitz cardinial solutions, though. So, if you demand solutions except on find at many points, then you get solutions. But then you lose uniqueness, you also seen another example 1 with multiple solutions.

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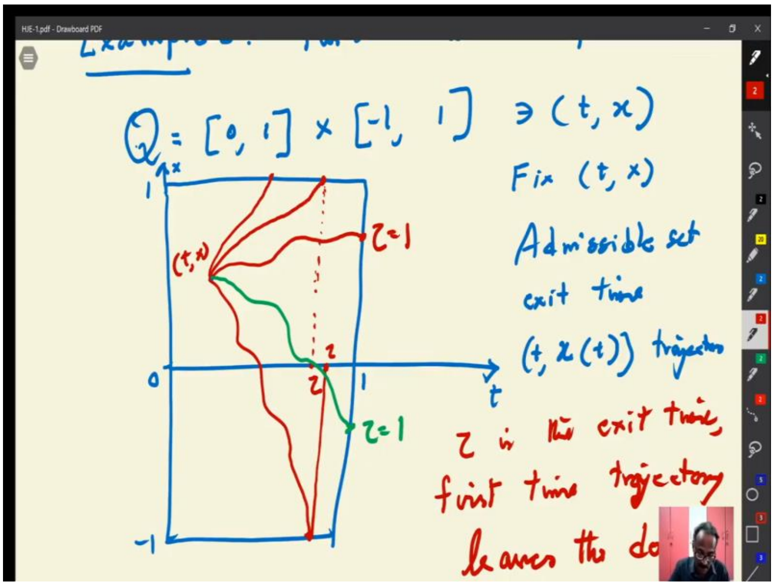

And the last example, which we were discussing, is about a time exit problem more, closer to our theory, which we were trying to do it. So, let me quickly recall. So, you have a domain and this domain is row 1 cross - 1 to 1 as you see here and then you fixed point t x in this domain and then look for all trajectories starting from t x and then if you take the time, so, the maximum time is 1.

So, look for the first time it exits this domain, so, if it exits before the time t = 1, whatever time it is the things and something like this. So, you look at here, so, if it exits like this and then this is the time it given on the other hand it exits in the right end of this thing and then tau = 1 that is called the tau exit type of this problem and what is your admissible trajectory. The set of all trajectories starting from t x and exit it definitely will exit at the latest time t = 1. So, the maximum value of tau the exit time is 1 and it can be before that it can exit immediately if you want it for example, you can connects it like that.

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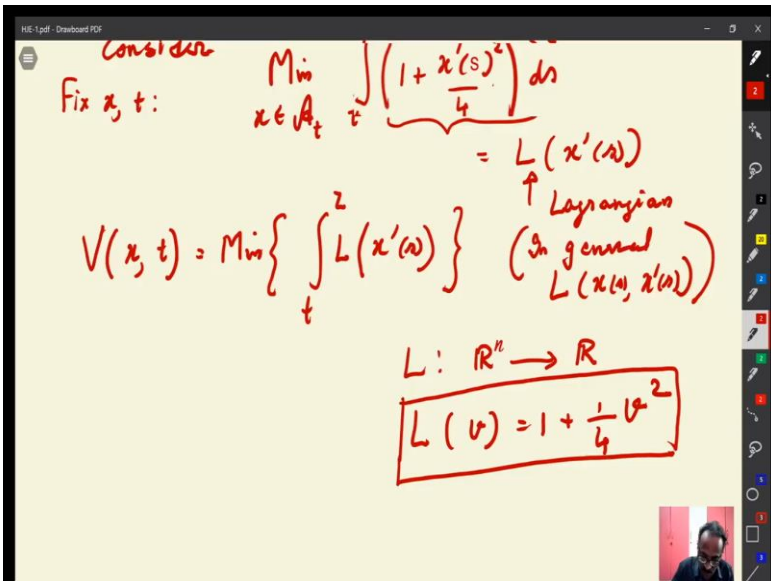

So, top can be anything between when it starts to that path. So, now, let us consider the minimization problem. So, consider the minimization problem. Consider the minimization problem what we consider is a minimum of you want to study at the exit time t to tau 1+ x prime of x square 2 / 4 these, this is the minimum problem, where x is in the minimizing trajectory.

You want to understand what is the minimum value? And you want to study the minimum problem? Of course, such kind of functional, you will see later this is a functional defend on L of x prime of s where eventually we will call L is the Lagrangian you will see why we call it that one. More general things we will call it but let us look at the typical example of the Lagrangian.

L can be in general it can be a function of x of s x prime and may be more s you can call it think more general x s these things you will see here. So, it depends of course it depends on x and t it because the x and t is fixed fix x and t and then you are trying to minimize this, so let us since the mean is then definitely the minimum will depend on that one. So you so let us call this to be V x t.

So V x t is equal to minimum, I use the notation M, because we will do integral t to tau L of so in general, it depends on all the numbers is, so let me write it L, in this case, L of x prime of s. But so what is L here, so L is a function from R n to R. In this case, you can think it has a one dimensional function also. So you can think it does L is from in case in a more general things you can consider, L is from R to R, where L of v = 1 + 1 by 4 v square, so that is the

Lagrangian, which you can there is a special case of Lagrangian. So this, you want to minimize it, we are going to throw something in s t first of R t.

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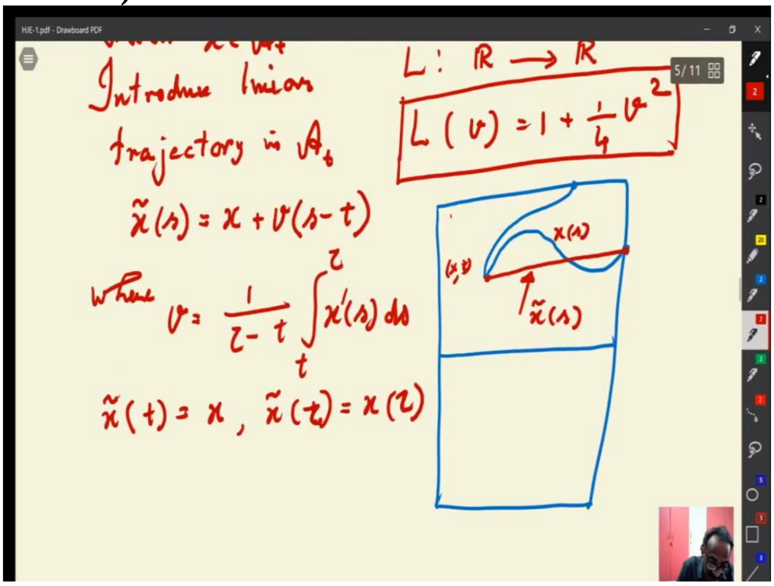

So you are looking for all the trajectories here. So let me maybe I will draw the picture once again here in a different colour, so I have a picture here. So I am going to observe which is a claim; which we can prove it here. So you have your trajectory and this trajectory may be something like that, of course this trajectory we can be something like this. So what I am going to claim here introduce a new trajectory connecting these 2 points.

So, this is your x t. So, I will connect these 2 trajectories. So, this is a linear trajectory I call the so this is your x t, x trajectory x of s and I call this linear trajectory x tilde of s, which is linear. So introduce the linear trajectory, linear trajectory, I am going to be this minimization problem to a minimization problem in R n R, you are going to minimize that problem that is what is going to do it.

So introduce the linear trajectories x tilde of s is equal to so it has to start from x plus some velocity I will write precisely the velocity into s - t you see and what is your v going to be where your v = 1 / tau - t integral from t to tau and x of s ds x is your given trajectory. So give so you are given x in A t I introduced a trajectory in A t. So, this you have to verify that is indeed true right what is your x tilde of t when x tilde at t it this vanishes it is x.

So this trajectory starts from x and what is your x tilde at tau it is tau - t and you will reach this top point. So you will get the same x tilde of t what is it is a trajectory it is the same thing same point you will get it exit. X tilde at the tau you have to calculate, so x tilde it will be tau - t so you will get the same value so you will it is the same trajectory connecting these 2 points that is how you have taken as v in this way because x tilde into tau will be tau - t and that tau - t when you calculate the tau – t, tau - t cancel and you will get x L.

So you I think you are to take x prime of x here to take x prime x. So, if you take that one, you will get exactly that x prime tau into x L and so it is a similar trajectory. So what is my immediate claim this is an admissible trajectory.

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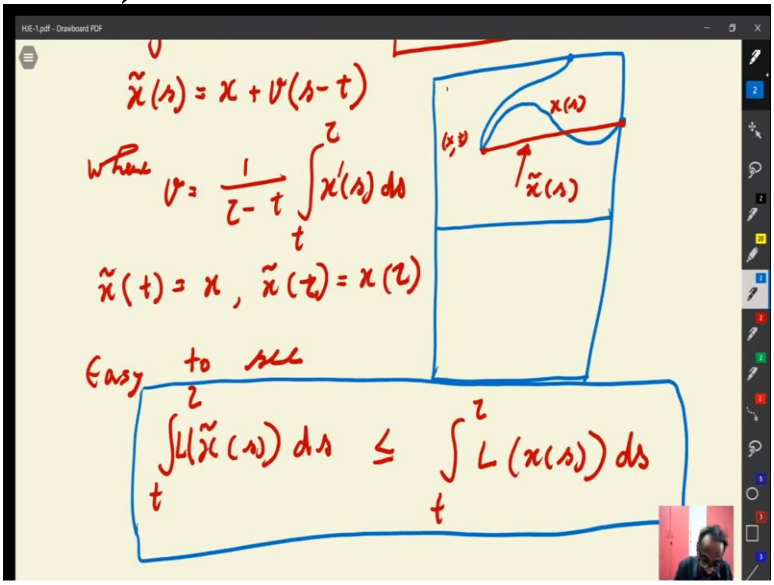

So it is we are looking at the minimization problem and then you can see this one. So it is easy to see integral t to tau x tilde of L of x tilde the problem L of x tilde of s ds less than or equal to integral t to tau L of x of s ds. So, this is an important thing you can see this you can immediately compute this trajectory. So, given any trajectory x you construct a trajectory x tilde of s and for that x tilde of your energy is smaller.

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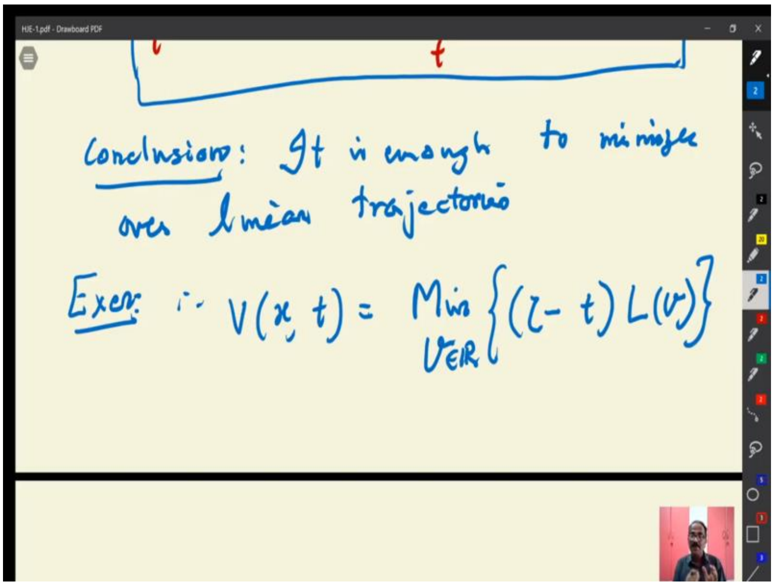

So, it is enough so this conclusion immediately the conclusion is it is enough to minimize because whenever you have a trajectory there is a linear trajectory which is a straight line trajectory and that the energy is less than the given condition. So, it is enough to minimize over linear trajectories that is an important thing. So, you have a small exercise to do it immediately exercise therefore, the problem minimization problem.

So, therefore the minimization problem can be reduced to a minimization problem over the reals it is enough to minimize over v integral of tau - t L of v. So this is a minimization over v over r. So, the problem of minimizing over trajectories reduced to a minimization over v EIR and you know how to whenever to minimize over v EIR your line, you look at it is derivative and then you can see that the minimum is achieved.

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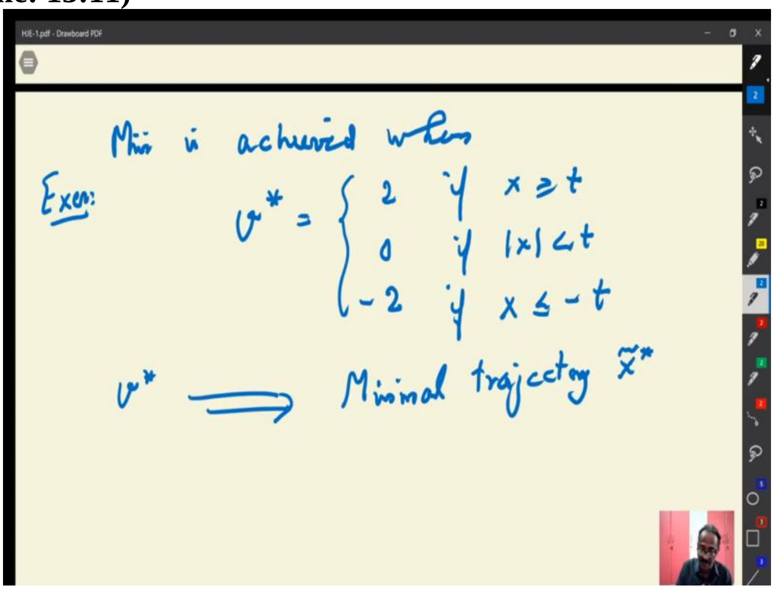

So, minimum is achieved when v star equal to you can prove at the minimum. So, you solve this problem these are all part exercise because we will not have time to do all the exercise minimum achievement, you compute the derivative of this one with respect to v and equate and that depends on x and t and then you have to do that. So, minimum will be 2 if x greater than or equal to t and 0 of if so mod x less than t and - 2 if x less than or equal to - t you will see.

So, you get the minimum is achieved when v star is equal to at that point corresponding to that v star, this v star will give you the minimal trajectory x tilde corresponding to that, because this is the slope corresponding to that you can with respect to that slope v star, you can construct the your linear trajectory and that is trajectory where your minimum is achieved and you can also compute the things you should do to learn these things.

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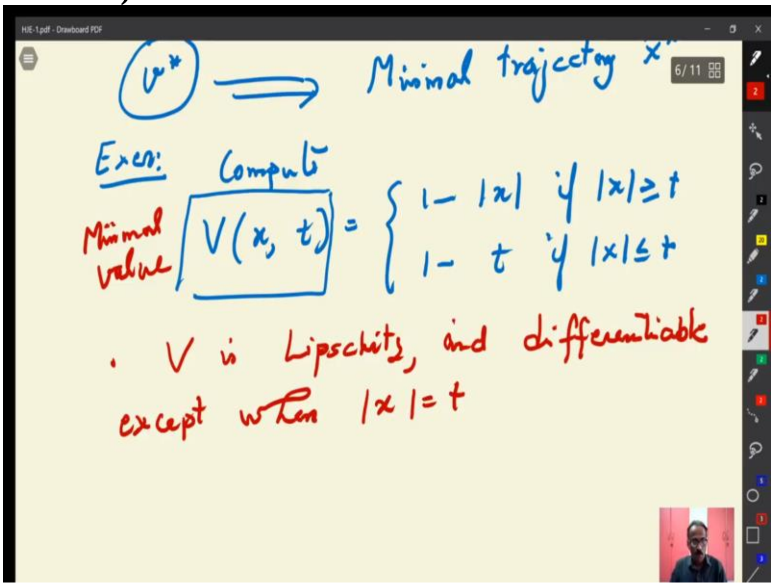

You are to keep on doing exercises we will learn so, you will do the exercise compute then compute the minimum value we V x, t = 1- mod x if mod x greater than or equal to t everything there depends on t you see and 1-t if mod x less than or equal to t. So, this is the minimum point and this is the minimum value this is the minimum point this is the minimal value so you get it. So, these function so the next step you can show that V is a Lipschitz function.

But and differentiable not all the points all these you can verify differentiable except when mod x = t except on these line segments what mod x = t you get to differentiability.

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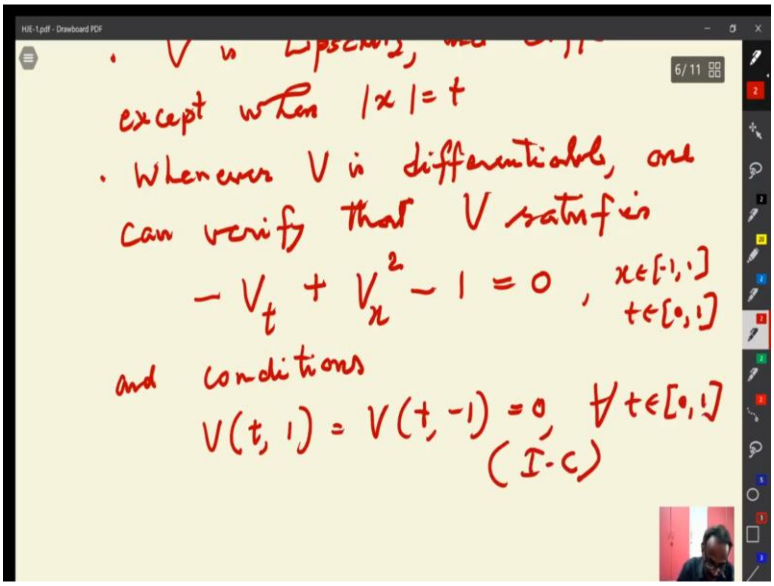

And whenever it is differentiable whenever V is differentiable from when can verify that we satisfies Hamilton Jacobi equation and satisfies I did like let me not call it the Hamilton Jacobi equation V t + V x square - 1 = 0 and it also satisfies the conditions boundary and initial conditions. So, this is you know that in that interval only x is in - 1 to 1 t is 0 2 and so, you have your condition the initial and n boundary conditions and this is the boundary conditions when V t 1 = V t at -1 is = 0 for all t in 0 to 1 initial condition.

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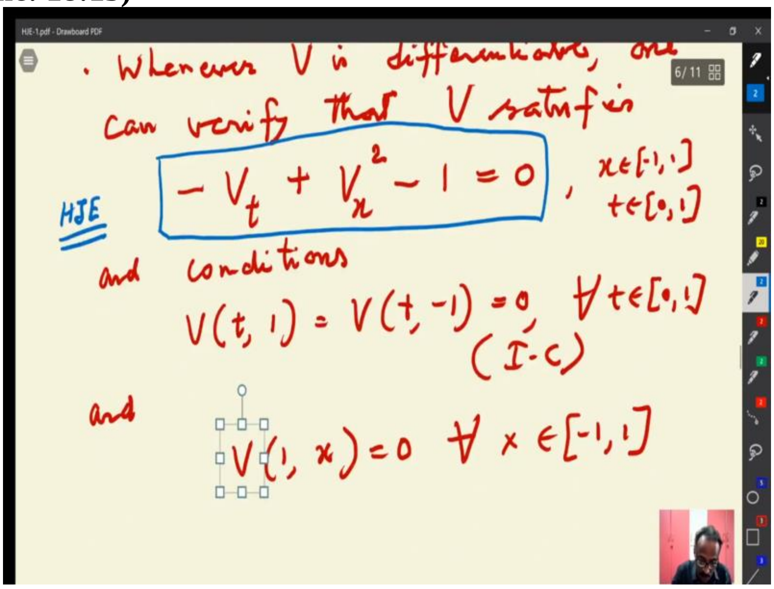

So, this is the initial condition and V at 1, x = 0 for all x when t = 1 so it satisfies them condition x is in -1,1. So, you will see so you have minimum value satisfies a differential equation this is something like Hamilton Jacobi equation so this is what we are going to expecting to study. So, you have your so recall that V t, x is a minimal value given by L a kind of Lagrangian.

So, in this course we want to understand with more than one Lagrangian and its minimization problem and then we want to see whether we satisfies the any equations and if this is a partial differential equations and partially differential equations has a rich theory and that is what we will be going to do it. So, before that what we will be doing planning to do in this right now is what is called another important aspects related to the minimization problem or the calculus of variation problem. So, calculus of variation Hopf-Lax formula so, in this now, before we complete todays lecture.

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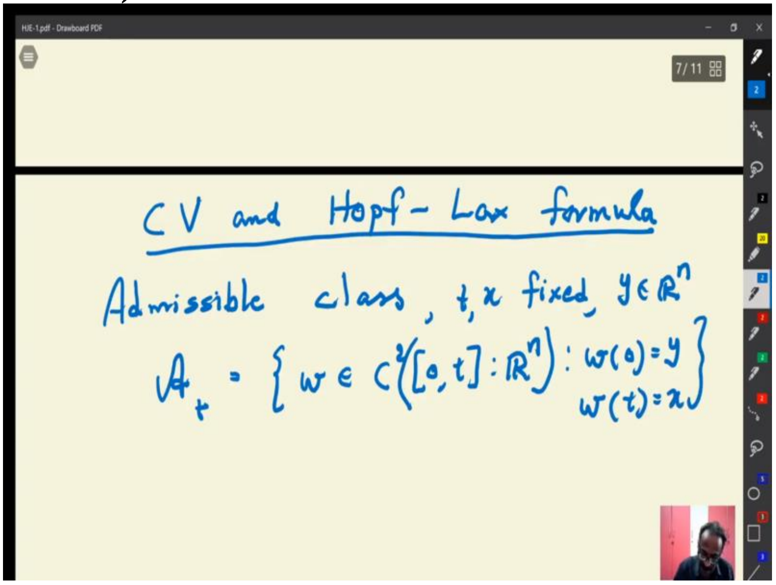

So we want to do that one. Calculus of variations and Hopf Lax formula in some general setup Hopf-Lax formula. We will try to derive it let us see whether we will have enough time to prove it today so, we will so let me do the setup admissible class. So, first we want to do a minimization problem admissible class was so, let me do it in a very systematic way so you have A t.

So, you x is fixed for the time being t, x fixed and then y is R n, x is R n and y is R n, y eventually you will be varying so you are considering because y is the recovery of your initial condition. So set of all twice differentiable functions C 2 from 0 to t R n. So set of all twice differentiable functions and that the initial value which will eventually vary is y and w at x and t = x.

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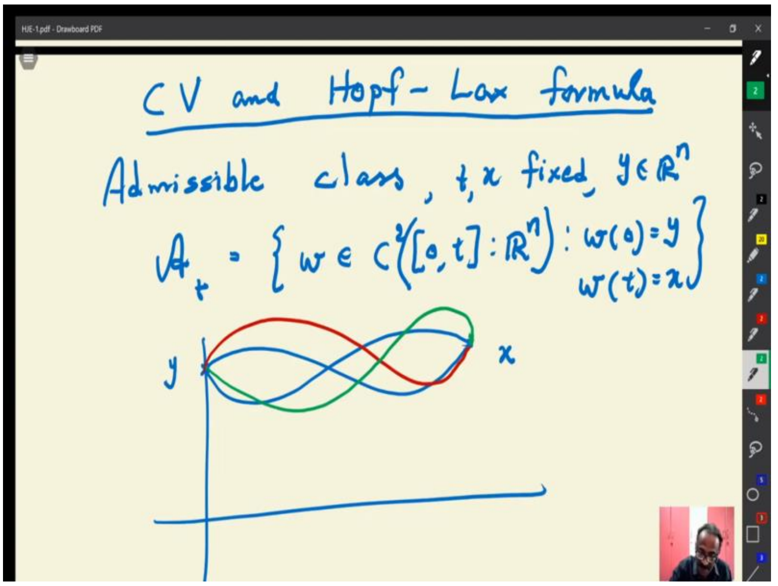

So these are the boundary conditions so basically if you look at the figure so you have a domain here this is your t so at t equal to so you have also you are having 2 positions. So let me mark it here if you want it and you do not need this picture. So you have y x so we are looking for all possible paths. So looking for all possible trajectories that is what you are take what is your minimization.

So L is a Lagrangian and given let. So that is admissible class you are considering or possible twice differentiable kind of functions continuous and differentiable functions starting from V to x and eventually first you look at it for all points reaching there and then all points reaching from all the points eventually we will vary x, y as well, but P and x are fixed according to that and undefined.

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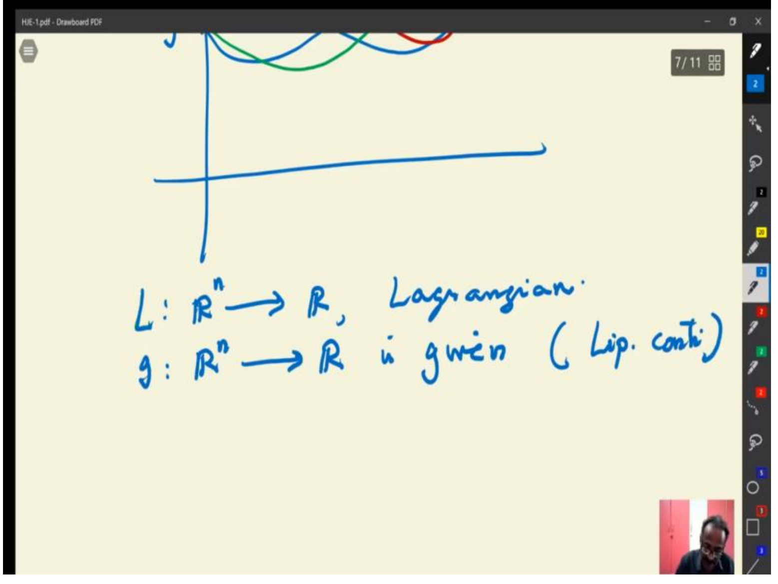

So, you have an Lagrangian and so you have what are all we are going to take away at L is from let me do this notation, L is from eventually we will have conditions Lagrangian you will see why it is called Lagrangian later in classical mechanics, this is nothing but the difference in kinetic and potential energy and g from R n to R is given these 2 are given we will have conditions. So, basically you will have conditions of Lipschitz continuous for the given function, but here you need more conditions which will come to that.

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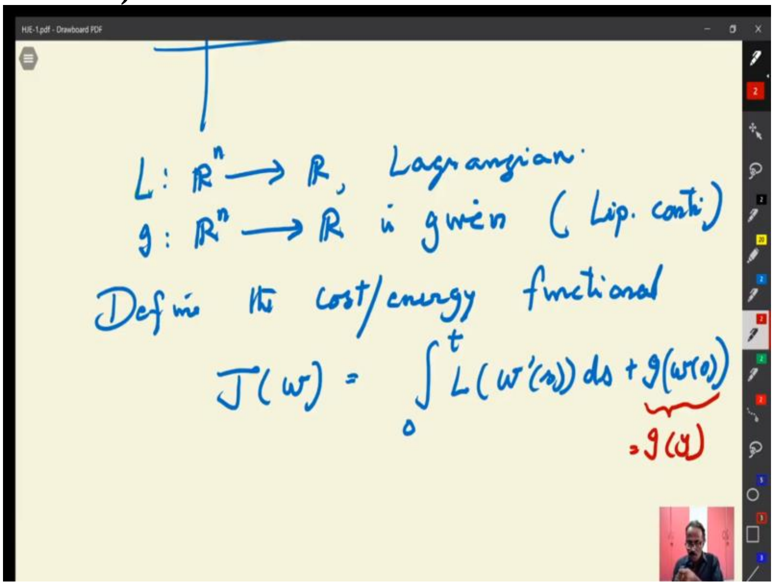

When we put our assumptions. So, this is given to you. So, define the cost or energy functional you call it cost or energy depends on the physical applications energy functional J of w. Given w is that to an integral 0 to t is the interval L of w prime of s ds + g of w at 0 w 0 this term if you look at it, this term is g of y. So, you can call base you will see that this is the initial cost.

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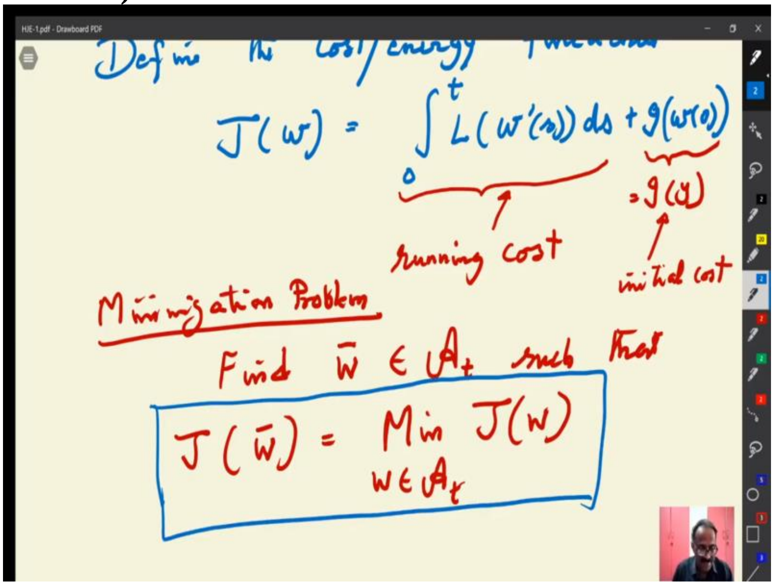

So, in physical applications this is the initial cost. So, your trajectory you want to read so you will do start at dynamics or whatever it is you will have an initial cost and these are the terminologies from optimal control we call and this is called the running cost. Because this is each time value that the total running cost basically. So what is your minimization problem? Find W bar in A t such that J of w bar is equal to minimum of J of w but w is A t this is what your minimization sum total for me this is a calculus of variation problem, where you want to minimize over the trajectories which you want to do.

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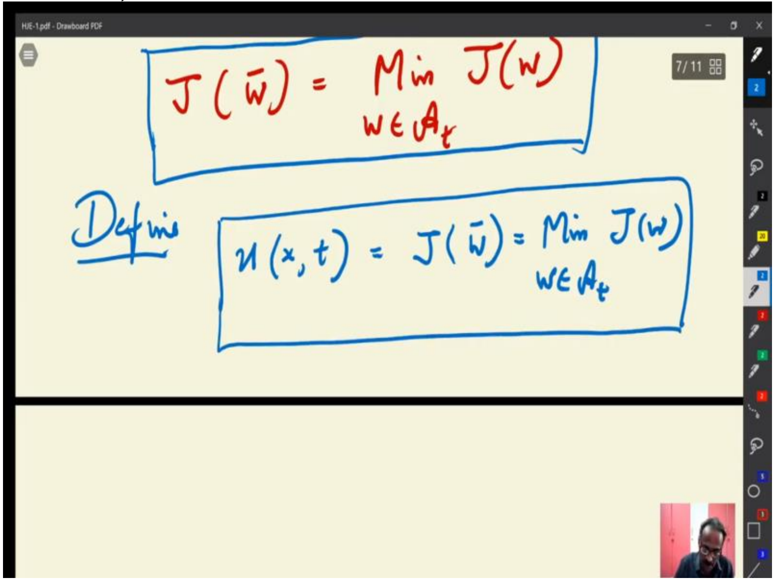

So and this depends on x and t so we call this minimum value as u x, t. So define u x t is your minimum u x t is nothing but because it depends J w bar equal to minimum of J w, of w A t. So, eventually you will see u is a solution to your Hamilton Jacobi equation but what is the corresponding L these are all the issues. So, we are started with an L which is a Lagrangian which is a minimizing cost and you will see a very specific examples later of this L.

At least some 2 to 3 examples of which you are familiar probably in calculus of variation we will be giving so we are trying to give it in a slightly in general setup. So, this is your u x t. So, you have your definition now.

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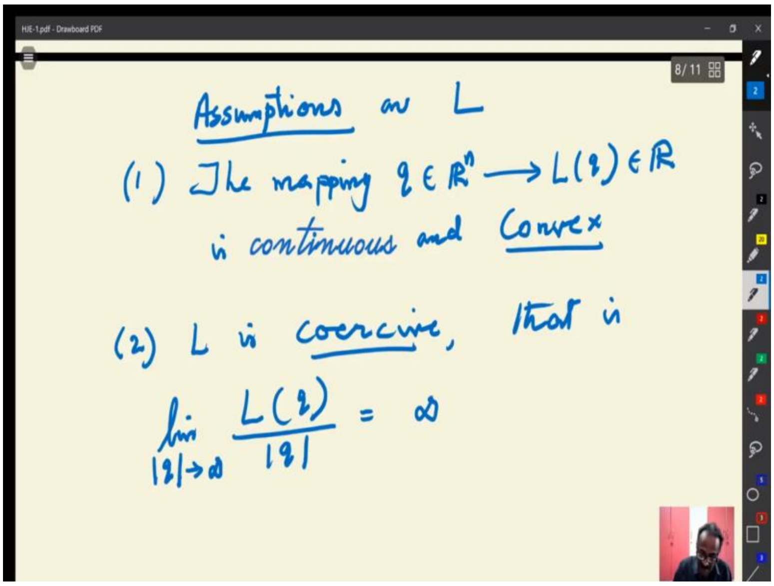

So, now, let me make my assumptions these are all important this you are to understand and should not remember assumptions that are the main assumptions of L. So, there are assumptions on L some assumptions and g basically we take it as a Lipschitz continuous equations. So there are 2 assumptions one the mapping q, q is in R us or you will be using these notations throughout my lectures eventually enough.

Because these are all again coming notations used in the classical mechanics to L of q this is in R is continuous and convex convexity plays a very big role. So, you can study problems without convexity. So, not that in applications all the time L is convex then the problems are more generally complicated as I told you we are trying to derive very specific formula. So obtaining such specific formulas are not easy all the time.

So, in this setup, we are going to derive a very specific formula for L that what I thought I will prove the complete proof here, but I will not but a continuity and convex. So, this condition is very important and the second condition is the L is coercive, this is another thing you have to understand very well that is L of q by mod q the limit is infinity limit mod q tends to infinity is equal to infinity.

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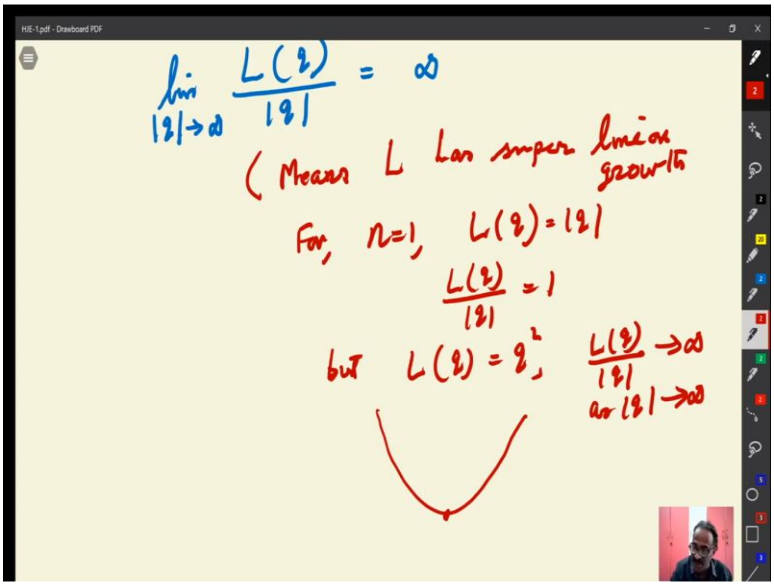

So, this condition means that means L has means see mod q tends to infinity the denominator goes to infinity, but then L of q should be more than the linear mod q is something like linear and if a L of q is also q, then it will be q / mod q it will not go to infinity. So, that means, L has super linear growth typically quadratic growth, super linear growth. So, for example, for when n = 1 in one dimension if L of q is equal to say mod q then that limit L of q / mod q = 1 it will not go to mod q = 1.

But you see when L of q = q square, this is a quadratic growth, then L of q / mod q goes to infinity as not q tends to infinity. So, you need something more than the linear growth in fact, you do not need q square all the time, you can take qe + 1 + alpha something like that little more than that linear growth or some other growth it should go to infinity and this is something like that, if you have a quadratic growth something like you have got convexity and quadratic growth and you know that for things that the minimum x is even in one dimension.

You know such figure for you have a function which is convex and continuous and type of which go into that one quadratically then it will have a unique global minimum. So, there are such results in one dimensions so we want to understand a little more about it.

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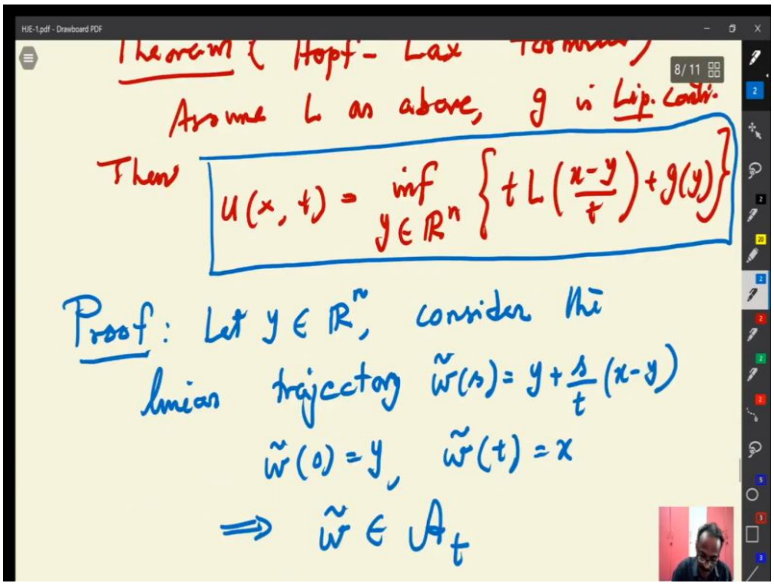

So, let me state a theorem, I thought I will prove the theorem also here but I will not be able to complete the proof here. So a theorem that is what the Hopf-Lax formula of like formula so maybe we will try to prove maybe a little take little extra time. So assume L as above with the above conditions maybe as in g is Lipschitz continuous with you are anyway going to do it Lipschitz is continuous.

Then u can be represented given by a formula u x, t = infimum of y in R n I represent g Lipschitz continuity may not misery just continuity will do but since we are working with the Lipschitz continuity and we are assuming that really may not be necessary for this particular representation y in R n of t L of x - y / t + g of y you see. So, you have your solution and this is called the Hopf-Lax formula, maybe I will try to give its proof so that I do not have to recall here the other properties I will recall it this thing.

So let me try to give a proof of this some ideas we already introduced. So we will do this thing. So for so let y in R n x in by we are already fixed. So you see this is a minimization in R n go to understand the difference here you are having the minimization over a class of trajectories the minimization is for a class of trajectory, but then that minimization you are reduced to your minimization over the you see the minimization over R n.

So here is your minimization. You see so the minimization stage to R n and this is exactly what we have seen it the earlier example, we minimize that one. So it is a representation with a general L is not that one particular L. So what you require is this property of the convexity and coercivity and both plays a crucial role in this **(()) (33:58)** representation. So that is why let me give you a proof of it.

So you start with the y in R n and then look for the linear trajectory. Look consider the linear trajectory, consider the linear trajectory x tilde of maybe what is we are having using w, we are using the notation w. So, let me use w tilde r trajectory w tilde of s = y + s / t that is a slope I am taking into x - 5 this indeed W tilde of 0 = y. And W tilde of t s / t is 1 x y and y cancel this is x n. So, that implies w tilde of is an admissible trajectory and then u x t is the minimum. So, u x t is the minimum of that trajectory so any other trajectories the cost will increase.

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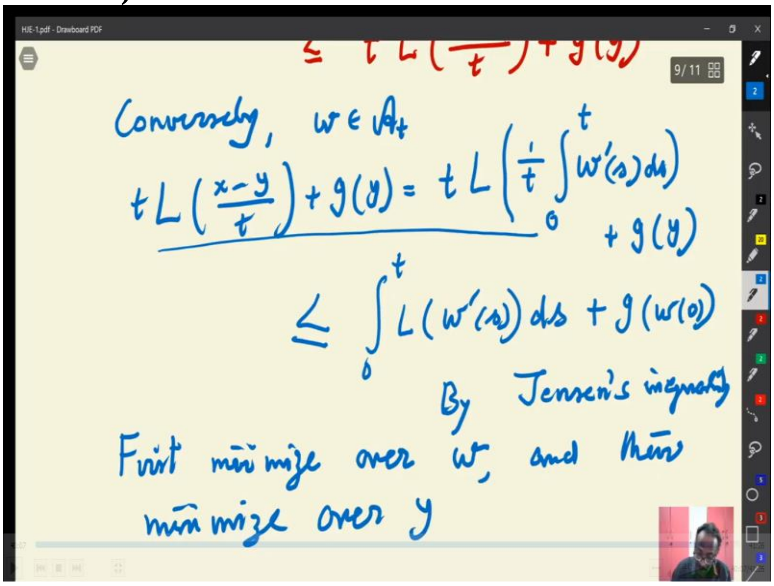

So, therefore, u x, t has to be less than equal to J of this w tilde because u x t is the minimum among all trajectories and that is nothing but integral 0 to t L of w prime of s ds + g of y but then w tilde because you are taking but w tilde of prime with respect to s is nothing but x - y by t. So, this w tilde of x - y is nothing but x - y by t there is no s there because this is a linear trajectory so, that is less than or equal to so, that will come out.

And then you integrate you get it t L of x - y / t + g of y. So, one way you have prove you want to prove this infimum. So for every u x t is less than or equal to this one, let us see what we have seen now, you have to prove the reverse inequality. So, one inequality you will get it and then you will prove for every other trajectory. So, now, conversely you compute t into L of x - y / t + g of y please go through the proof before reading that g of y this is equal to t L.

I do some trick here this is 1 / t and then integral of 0 to t w prime of s any w for any w conversely let w belongs to this is not linear trajectory. So, w prime of s ds + g of y but integral 0 to t w prime of s ds here this one is nothing but w t and that is what you want to this one it is w t and g of y dy and this is less than or equal to that is what you use it and this is writing it as less than equal to and you apply what is called Jensens inequality.

Once you have a Jensens that this is an average and you can put it inside and then you will you will have you can take L inside. So, this is just a computation because these will be w t - W 0 that will be x - y / t. So, I have not done anything here. But I have done important thing is to taking L inside when a L inside t and t cancels and it will be 0 to t L of w prime of s ds + g of y is nothing like W 0.

You see, so you have this inequality proving it. So, by Jensens inequality, this you should know when you can apply Jensens inequality, please refer some other book Jensens inequality. Once you have this one, what you do is that you why you are fix it minimize over first to minimize. There is no W here now first minimize over w and then minimize over y and then minimise over y.

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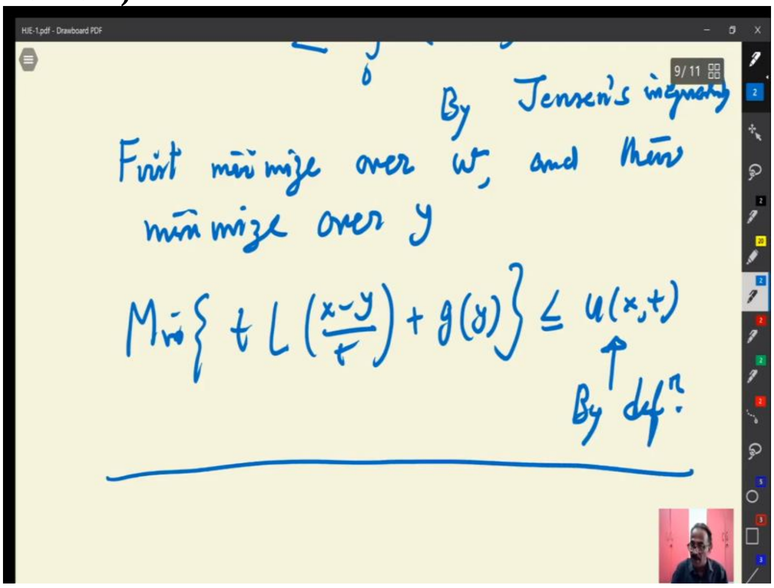

So, if you do minimise over W and then minimise over y you will exactly get your minimum integral of t L of x - y by t + g of y and the right hand side is minimum by definition, that is nothing the minimum is nothing but u x t this is by definition and that is exactly you want to prove it. So, I will stop here. So, what we have done is that maybe we will recall again that every solution the minimization problem over trajectory reduced to the minimization of the Euclidean space.

And that you can solve this trajectory with minimization reduce this formula, this will allow you to get the solution explicitly in a very specific way because you can differentiate and you can get explicit solutions by minimising over this one. So, I will stop at this stage and we will continue the lecture in the next class.