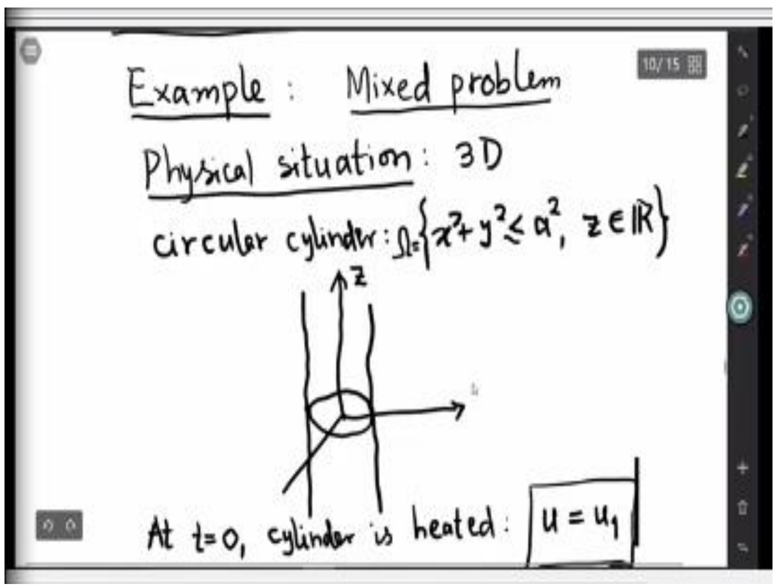
First Course on Partial Differential Equations-II
Prof. A.K. Nandakumaran
Department of Mathematics
Indian Institute of Science-Bangaluru

Prof. P.S Datti
Former Faculty
Tata Institute of Fundamental Research Centre for Applicable Mathematics-Bangaluru

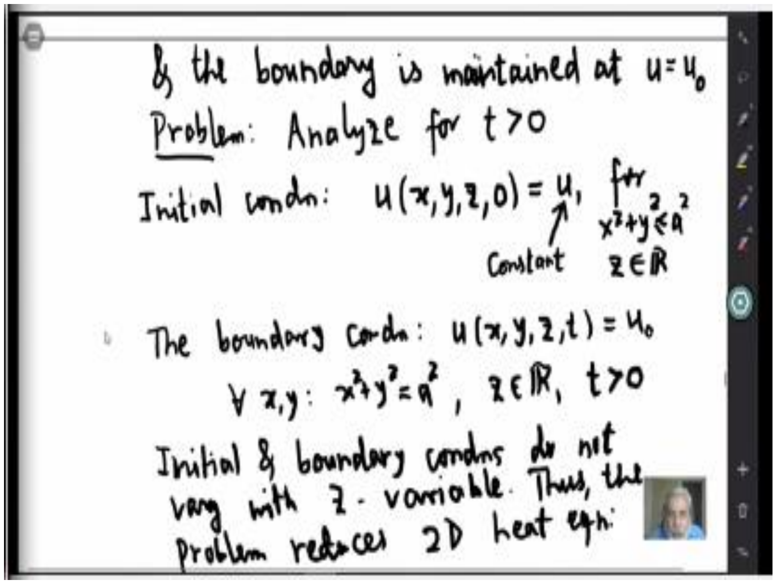
Lecture-29
Heat Equation-5

(Refer Slide Time: 00:20)



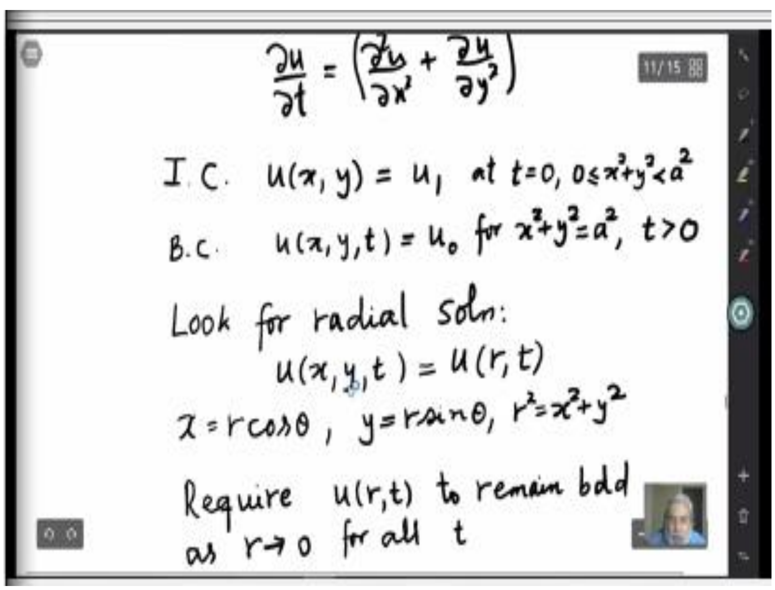
We will continue the discussion on the heat equation in more than one variable. And in the last class we started discussing an example which is a mixed initial board value problem involving and infinite circular cylinder and this was the physical situation.

(Refer Slide Time: 01:03)



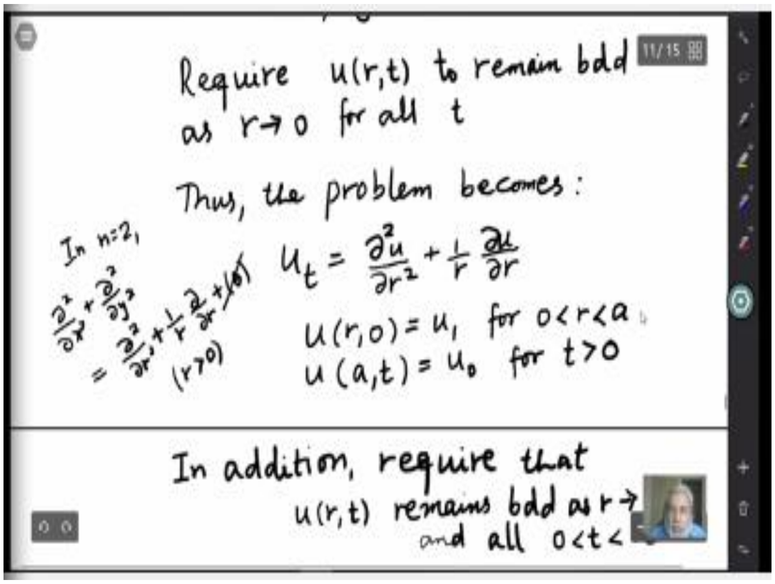
From the initial conditions at t = 0 and the boundary condition and the geometry of the problem. We see that there is no variation in the z direction. So, the problem essentially reduces to a 2 dimensional heat equation.

(Refer Slide Time: 01:31)



So, that is what we deduced from the initial boundary conditions and the geometry of the problem. And again looking at the geometry, so this is just a circular domain and a is the radius of the circle in the x, y plane centered at the origin, we can look for again radius options. So, u x, y, t = U of r, t, so**,** x = r cos theta y = r sine theta and r square = x square + y square, so these are the polar coordinates in 2 dimensions.

(Refer Slide Time: 02:37)

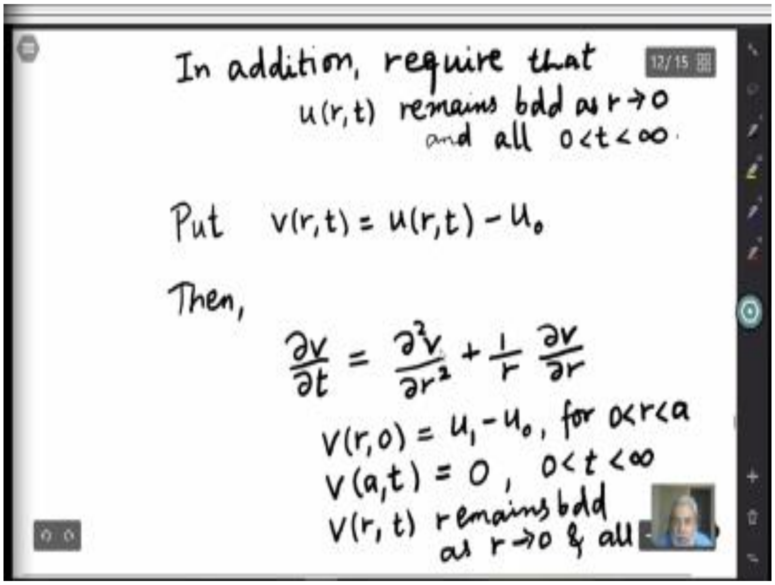


One word of caution here this change of variables from rectangular coordinates to polar coordinates it is singular at r = 0, so we should avoid that r = 0. So, while seeking a solution in the radial form we also require this whatever solution we get should remain bounded as r tends to 0 and for r, t. Because in the original problem there is no problem at r = 0, so with this change of variable there may appear a singularity at r = 0 but then we should look for solutions which remain bounded as r dash to j.

So, again just writing the Laplacian in 2 dimensions, so in n = 2, so we have this del square by del x square + the Laplace operator in rectangular coordinates converted to this 1 by r del by del r, you see that is division by r, so this always supremum greater than 0. So, in general there will also be a theta bar, so theta variable but in this case it is not there because we are assuming u is only a function of r.

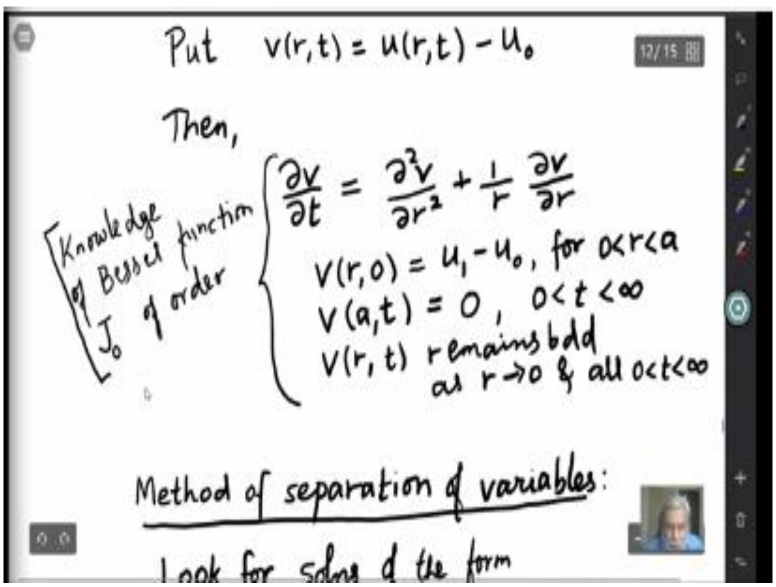
So, then the problem reduces to, so this heat equation becomes in r, t coordinates u t = del square u by del r square + 1 by r del u by del r and the initial condition become u r 0 = u 1 for 0 less than r less than a. So, again I am awaiting that r = 0, but you should look for a solution which remains bounded as r tends to 0. And the boundary condition that on the circumference of the circle that u is u 0 for all t positive becomes u of a, t = u 0.

(Refer Slide Time: 05:42)



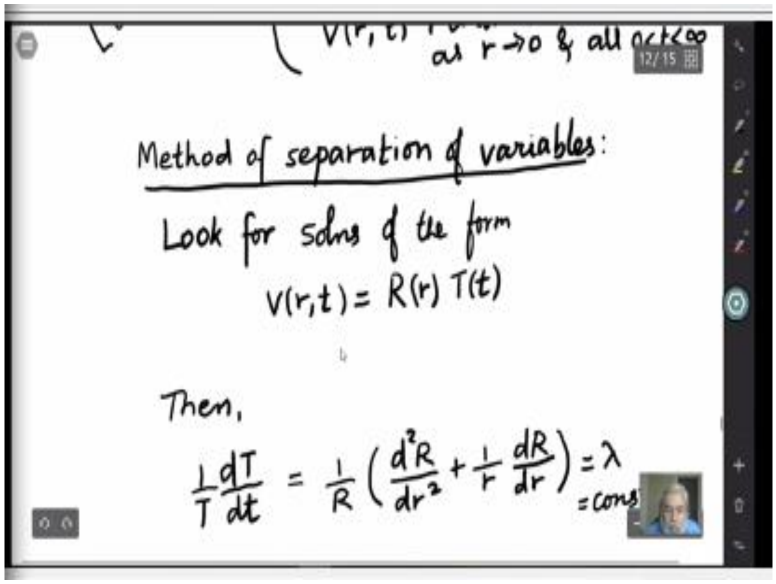
So, what I have said, I have just written there. So, we required the solution to remain bounded as r tends to 0 and also for all t 0 to infinity. So, we make again one more reduction, so put v of r, t = u of r t – u 0, so u 0 is the boundary condition, so that is the boundary of the cylinder is maintained at that temperature, so just subtract that.

(Refer Slide Time: 06:19)



And again by linearity, so the equation for v is same as the equation for u, but only there is some changes in the initial condition becomes u 1 - u 0 and boundary condition become 0 because we are subtracting u 0. So, this is the problem we are going to obtain solution in an explicit form. So, we are going to use a method of separation of variables and in the process we do require some in this case knowledge of Bessel function for the 0, so this is usually denoted by J 0. So, this is one occasion where we do need this knowledge of Bessel functions and their properties.

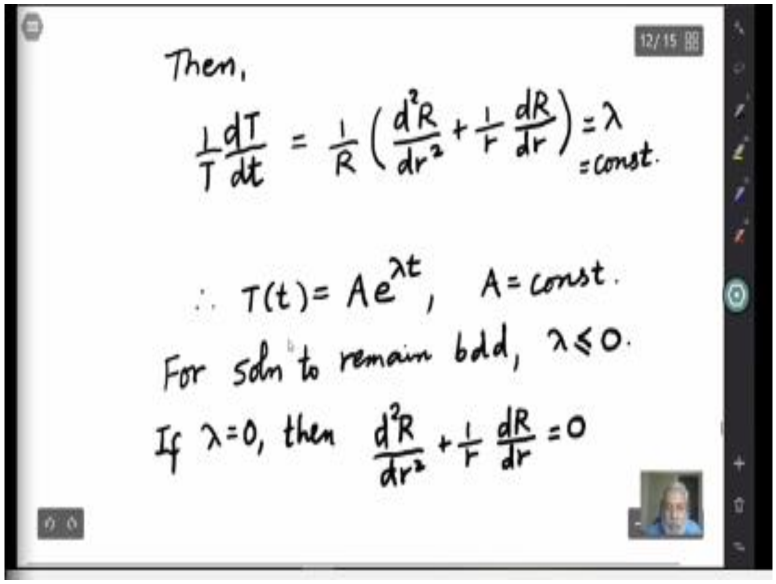
(Refer Slide Time: 07:47)



And at some other places in this course, we do need the knowledge of this Bessel function and it is properties. So, you can refer to any good book on spatial functions including these special functions. And for your reference, we are going to prepare some notes on the various properties of the Bessel function; we are going to use in this course in particular this example. But we are not providing any proofs by no means the property we are going to use are easy ones, they need proof.

So, you should learn from some good book on Bessel functions. So, if we look for solutions in the separation of variables form, so we write v r, t = R of r and T of t, so R is a function of r only and T is a function of t only. And when you substitute this form in the given equation namely dv by dt = d square v by dr square + 1 by r dv by dr, this is the user process.

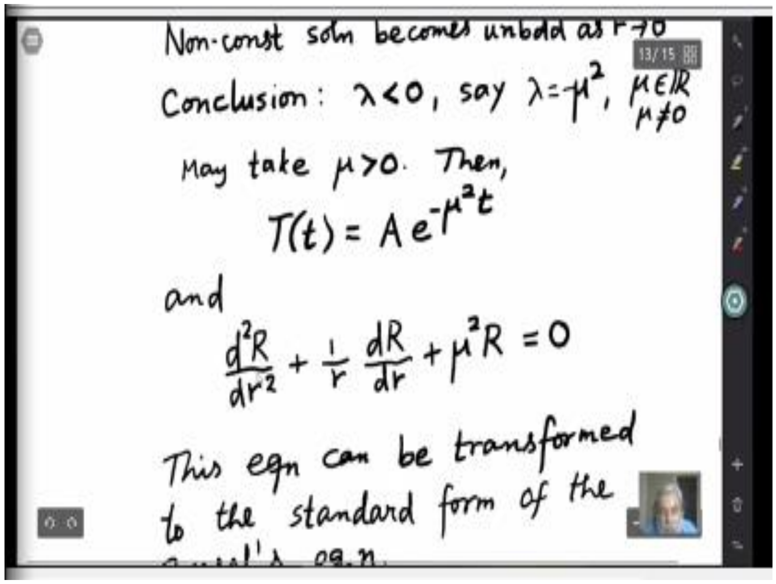
(Refer Slide Time: 09:26)



So, we get this equation 1 by T = dT by dt = 1 by r d square by dr square + 1 by r dR by dr. And right hand side is only a function of r and left hand side is a function of t only, so this must be a constant. So, if you just use the equation for T the solution is general solution that is a first order ODE. So, T of t will be equal to Ae to the lambda t, where A is a constant. So, since, so these are the important requirements, so we want this V to remain bounded as r tends to 0 and all t up to infinity and that forces this constant lambda to be only less than or equal to 0.

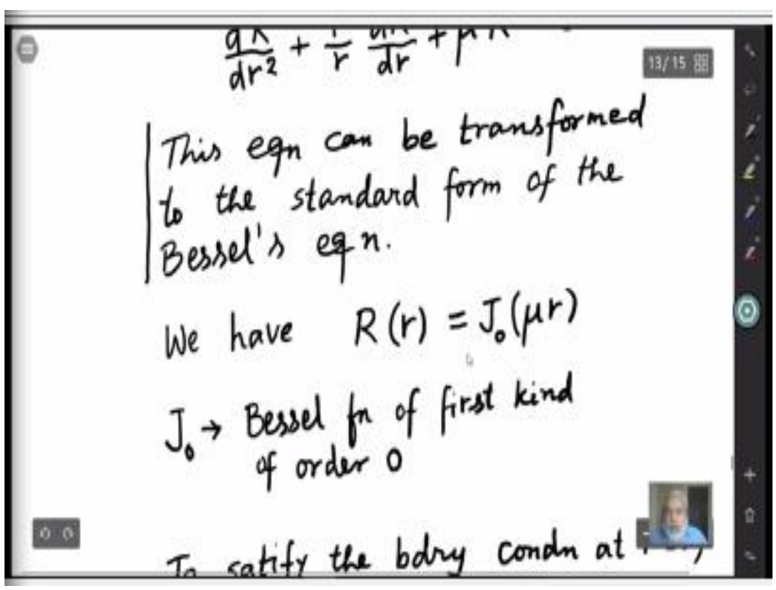
So, if lambda is positive, then as t goes to infinity, this T becomes unbounded and so that is not a required solution, so we want bounded solutions. Again when lambda = 0, you consider the other equation. So, then we get d square by dr square + 1 by r dR by dr. And if you look for non constant solution of this equation, when lambda is 0, so that is 0.

(Refer Slide Time: 11:10)



The non constant solution becomes unbounded as r tends to 0. Again, we do not want that, we want the solution remain bounded as r tends to 0 and that forces lambda to take volume negative values. So, we put lambda = -mu square and we may assume that mu is first and then, so this solution for T you get T of t = A e to the -mu square and what about the equation for R? Now we will put lambda = -mu square and then the equation for R is d square by dr square + 1 by r dR by dr + mu square R = 0.

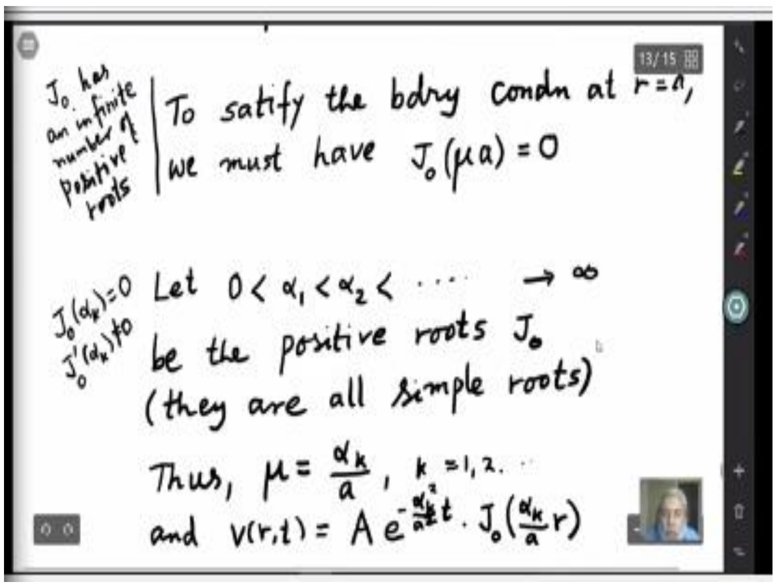
(Refer Slide Time: 12:11)



From here we start the journey to Bessel's function. So, this equation can be transformed with a simple transformation to the standard form of Bessel equation. So, what this equation will write in the notes? And for the time being just you if your study well and good, so the solution of this equation, design the solution of this equation is given by R of r equal to, so I can always put a constant multiple.

But that we already put in t itself, so we can just write R of r = J 0 mu of r, so where J 0 Bessel function of first kind of order 0. So, then the solution is given by this multiplication of T of t and R of r. And now we have to satisfy this initial condition and boundary condition. So, let us take the first boundary condition, so see this is given only at r = Ae, so that R function we have to worry about that.

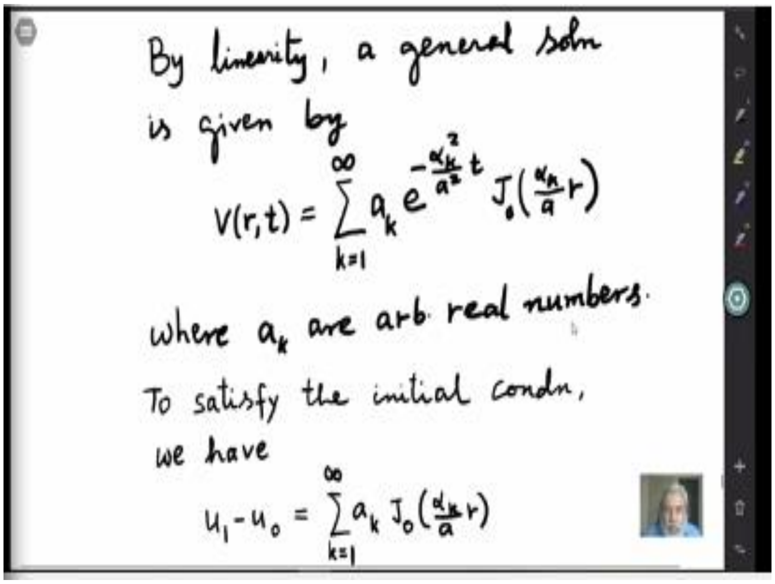
(Refer Slide Time: 13:40)



So, in order to satisfy the boundary condition at r = a, we might have this J 0 mu of a = 0. So, it is an important property of this Bessel function 0. So, J 0 has an infinite number of positive zeros, positive roots are zeros, so that is an important property of this J 0. So, this also has close connection with trigonometric functions, sine and cosine. So, you just put them in some order, so let 0 less than alpha 1 alpha 2 be the positive roots of J 0, so you list them.

And another important property is that they are all simple roots. So, this, so J 0 alpha k is 0, J 0 prime alpha k is not 0, they are all simple roots. So, thus this mu was started with this contract mu has to take the form that alpha k by a, so mu a = 1 of the alpha case. So, that mu has to be alpha k by a for some k to 1, 2.

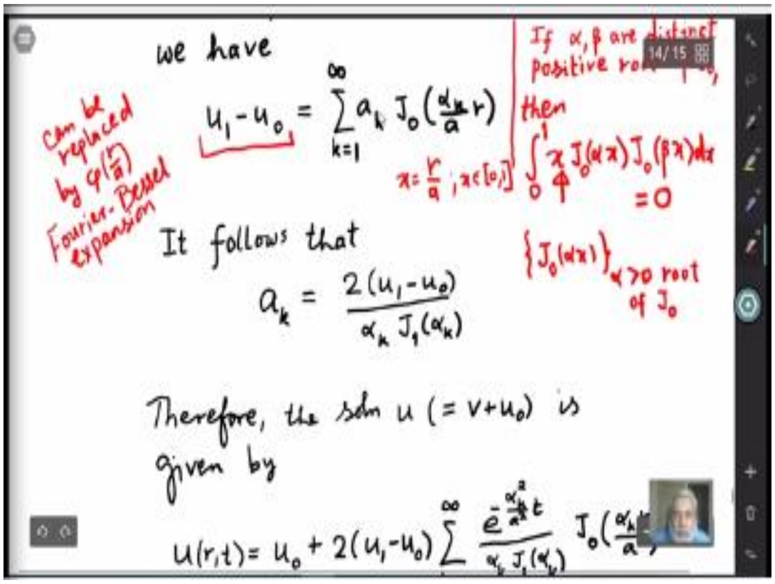
(Refer Slide Time: 16:04)



So, if you just pick one k, so the solution is given by V r, t = some constant into e to the -alpha square by a square t into J 0 alpha k by r. So, by linearity of the problem equation and boundary conditions, so we can write the general solution of the given problem. As v r, t = a k, a k's are sum arbitrary real constants into exponential alpha k square by a square t J 0 alpha k a by r. And now, so we have another condition to satisfy namely the initial condition. So, initial condition is V of r 0 = u 1 - u 0.

So, just substitute t = 0, so again there is no problem of convergence for t positive because of the properties of this Bessel function J 0, there is absolutely no problem. So, in order to determine this (()) (17:42) to satisfy the initial condition, so formally you just put t = 0 here, so we get u 1 - u 0 equal to this summation k = 1 to infinity a k J 0 alpha k by a r. And to in order to determine this a k, just like we do in Fourier series, we do require some orthogonality properties of this J 0.

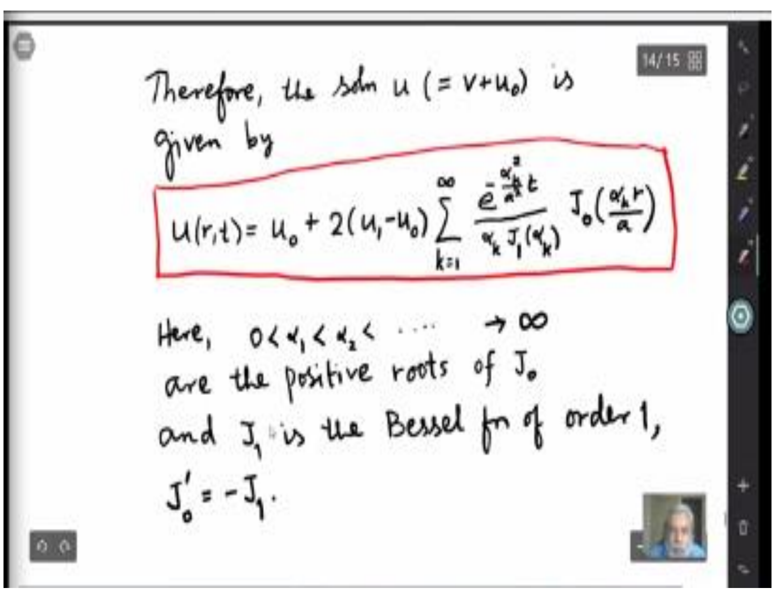
(Refer Slide Time: 18:20)



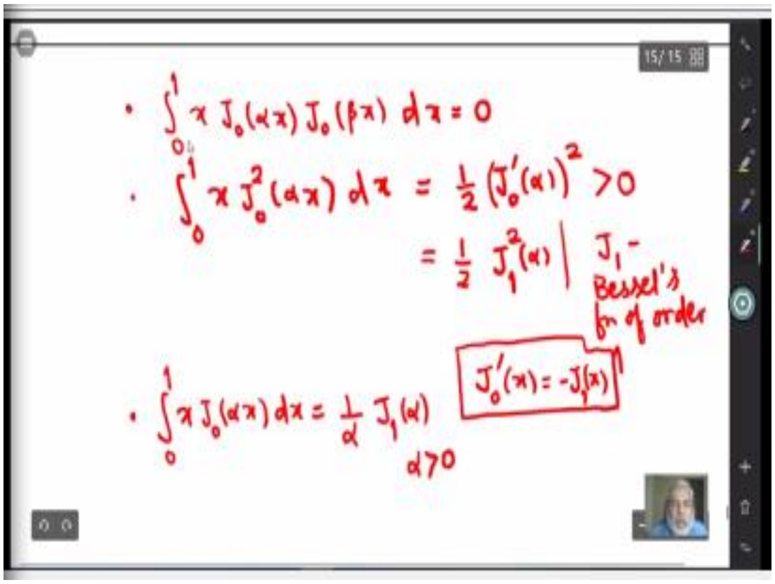
So, I will just write them here, let me use different template. So, these are the properties here, if alpha, beta are distinct positive roots of J 0, then so we have this orthogonality condition integral 0 to 1 x J 0 alpha x J 0 beta x dx = 0. So, if you consider this family, so you consider this (()) (20:02) J 0 alpha x alpha positive root of. So, this collection of functions or the interval 0, 1 they are orthogonal with this weight.

So, this just remember there is a weight here, for trigonometric functions there is no weight there, weight is 1 but for this Bessel functions there is a weight there. So, this is one important property. And what happens if I take alpha = beta? So, let me just take that one.

(Refer Slide Time: 21:04)



(Refer Slide Time: 21:05)



So, 0 to 1 x J 0 alpha x J 0 beta x dx is 0 and x so I am take alpha = beta dx and that is half J 0 prime alpha square. So, as I have commented on the roots of the function J 0 are all simple, so this is strictly positive. So, in fact this can be expressed as J of 1 alpha square, so this J 1 is Bessel function of order 1. So, we have the relation J 0 prime x -J of x. So, all these properties can even be deduced from the (()) (23:20) theory, we study in ordinary differential ignitions to comparison theorems and so many other things.

So, there is another one, in this computation we require, so this is just 0 to 1 x J 0 alpha x dx, so here we require this alpha to be a root of J 0 but in this computation we do not require that, so this is just J 1 of alpha, so alpha positive. So, if you use these 3 properties, then you multiply this by x. So, see for example here these r by a by our choice, so if we call it x, then x belongs to 0, 1. So, it fits into this orthogonality conditions and normalizing conditions etcetera.

So, you multiply this both sides by r by a, so r by a you call it x and then again multiply by J 0 alpha k x and then integrate from 0 to 1, use this orthogonality conditions and then other 2 conditions here. So, only one term survives every time. So, whatever J 0 alpha k term we multiply here only that survives and there is a, a k there. But then there will be some integral and those integrals are evaluated by these 2 formulas. And when you do that computation you get this a k = twice u 1 - u 0 divided by alpha k J 1 alpha k.

So, this both terms in alpha k is positive by choice and this J 1 alpha k is nothing but J 0 prime alpha k with negative sign and since alpha k are all simple roots, so this is also non zero, so this a k is given by this number. And at this stage you also realize that this u 1 - u 0 this can be replaced by any function phi of r by a and that you express as a summation using this Bessel functions and that is termed as Fourier Bessel expansion. Once you prove this family of functions is an orthonormal family with suitable normalizing constants. So, orthogonality conditions are there and then you divide by some normalizing conditions, so it becomes an orthonormal family.

So, then there is a general theme, so you can expand any suitable function in terms of these orthonormal family and in this case since it involve Bessel functions, so it is termed as Fourier Bessel expansion. So, finally, ok we have the formula for all the a case and then the solution u, so remember from u went to v, so now it is u = v + u 0. So, we have got only v here, so we have to add u, so and that is given by this, so an explicit form.

So, to arrive at this explicit form, so we have made use of several things here, assuming that their validity or so that we will provide in some notes. But, so you have to really look into some literature in order to prove these relations, their important relations. And we also need this knowledge of Bessel functions in some other situations during this course. So, we will also give some similar problems in the assignments, so that you can work out.

So, in only few situations we are able to find a solution in explicit form. And there is, so here we assuming that the initial and boundary conditions do not vary in the z variable, we did use these things. So, there is also possible to make this everything depending on z, so still one can work out with separation of variables. But computations are more, lengthy and more involved and the expressions also look more complicated, but that is also possible.

So, you can include z variable in this analysis. So, with that remark, so we more or less finished this topic of heat ignition. So, again one more remark, so we have included so many topics in this VDE course. So, it is not possible to give all the details about any one particular topic. So, here our aim is to provide a general viewpoint and hope some of you may get interested into studying further topics in partial differential equations. In any of them each topic here we have chosen either material for one semester or even more. So, you can imagine, so we cannot provide all the details of your heat equation or wave equation or your Poisson equation. So, just a general viewpoint we are providing, hoping that some of you will take up further study in these topics, thank you.