First Course on Partial Differential Equations-II
Prof. A.K. Nandakumaran
Department of Mathematics
Indian Institute of Science-Bangaluru

Prof. P.S Datti
Former Faculty
Tata Institute of Fundamental Research Centre for Applicable Mathematics-Bangaluru

Lecture-28
Heat Equation-4

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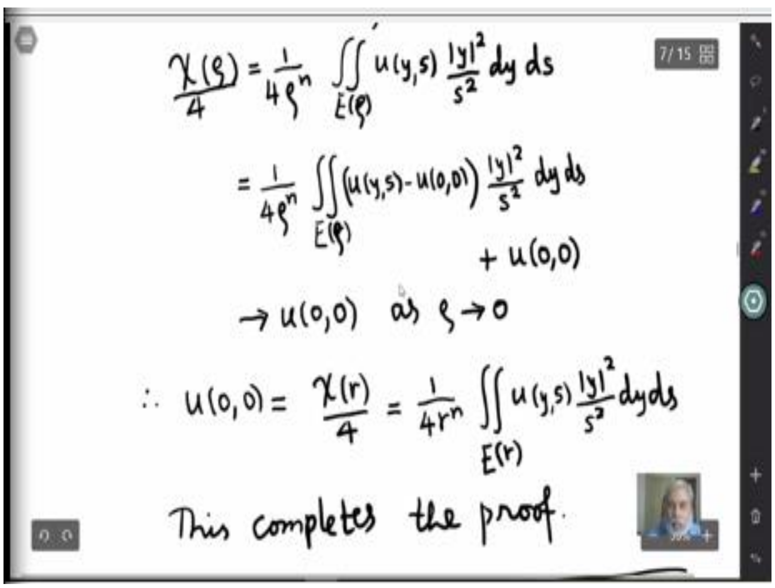

In the previous class we were discussing mean value property for the solution of heat equation. Let me again state that and we almost completed the proof.

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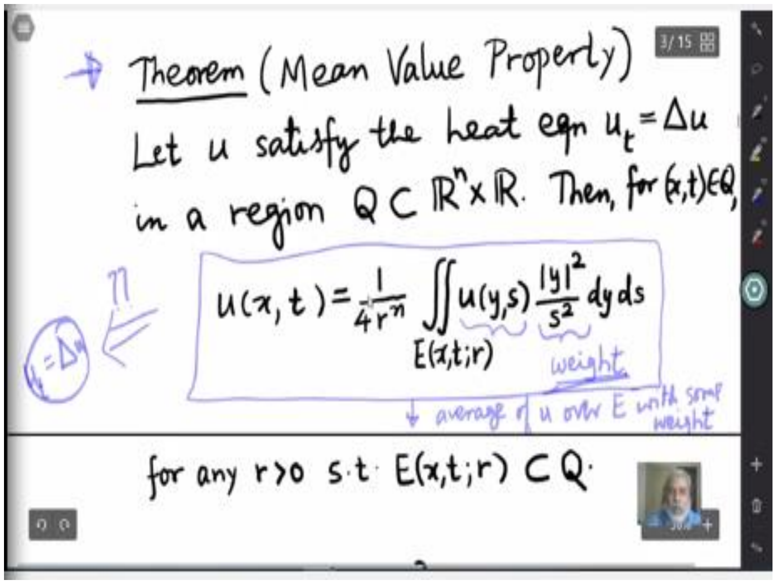

So, here is the statement of this mean value theorem. So, if u is solution of the heat equation in a region in R n cross R, then for all x, t in Q u x, t is equal to this double integral. In fact it is n + 1 dimension integral and we discussed why it is called a mean value property because of this reason.

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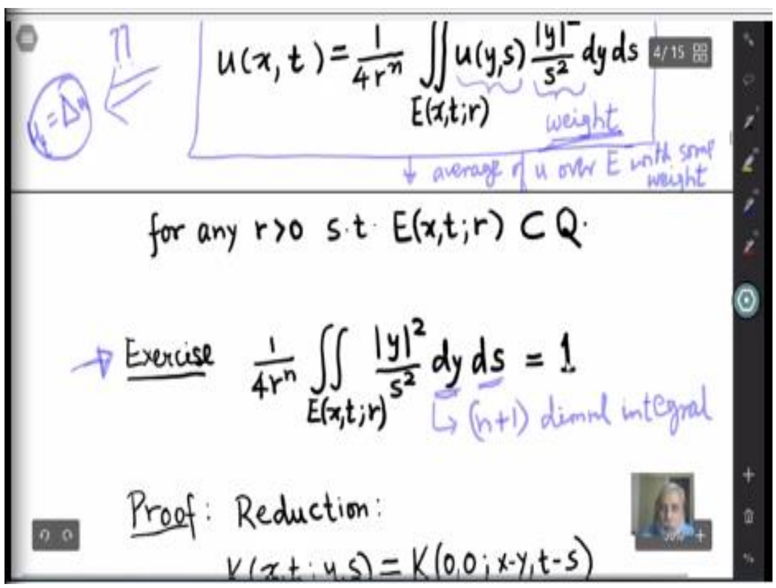

This integral = 1, so this weight is 1, so this u x, t is the mean value of u y, s over this E x, t, r that is heat ball with this weight.

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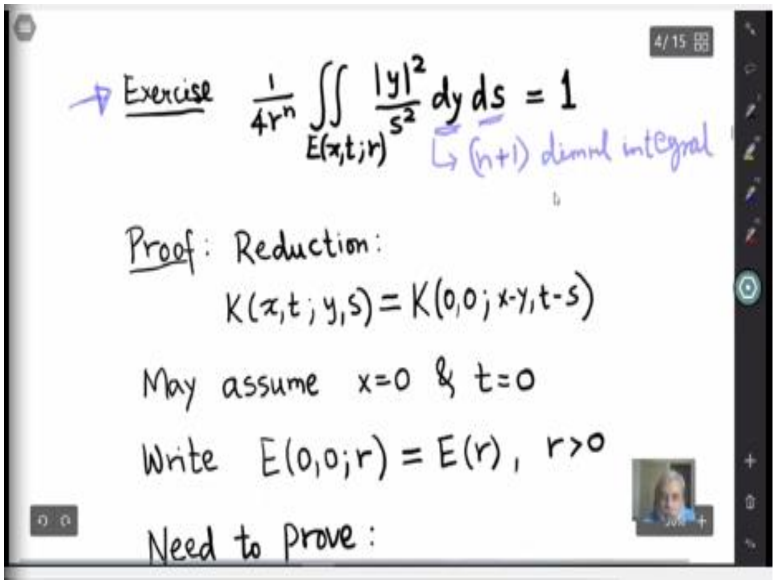

So, I almost completed the proof just one last line.

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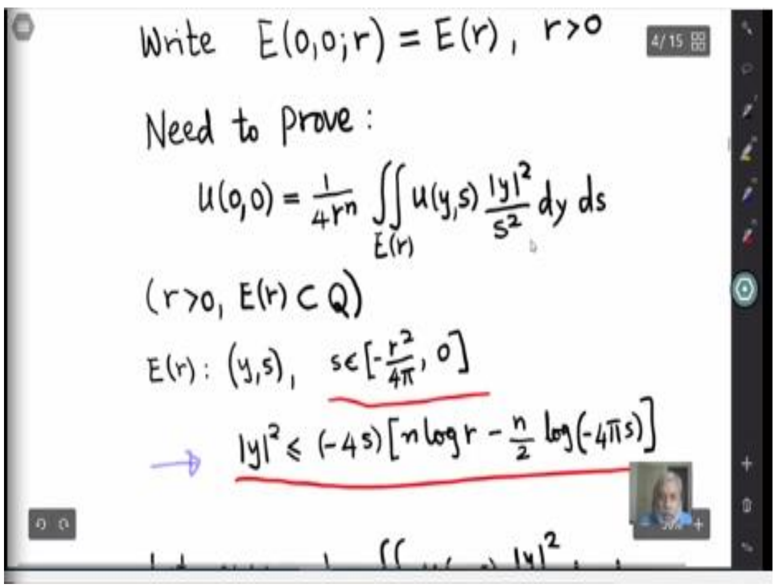

And the proof started with first some reductions, so we can assume that x = 0 in r n and t = 0 in r and that is just by translation and this integral does not change. So, we have to prove that u 0, 0 is equal to this integral.

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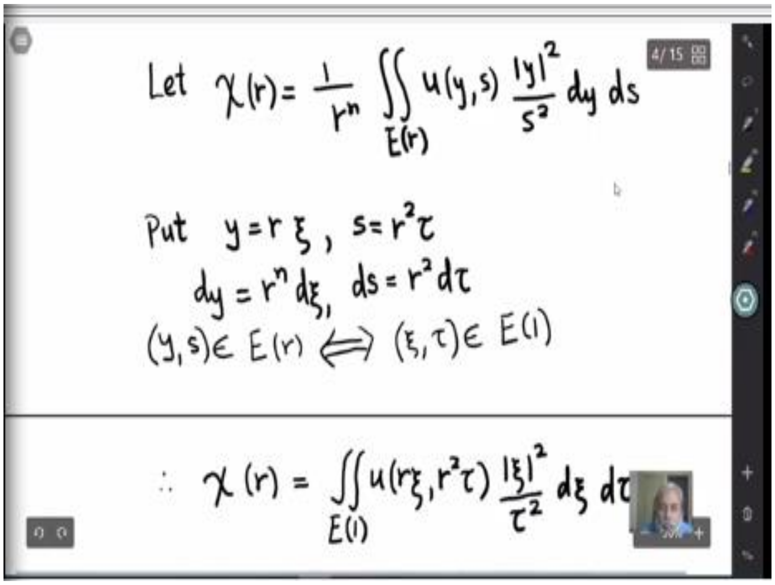

So, for that we just forget this factor 4 consider this function chi of r defined by this integral and we showed that chi is a constant function of r.

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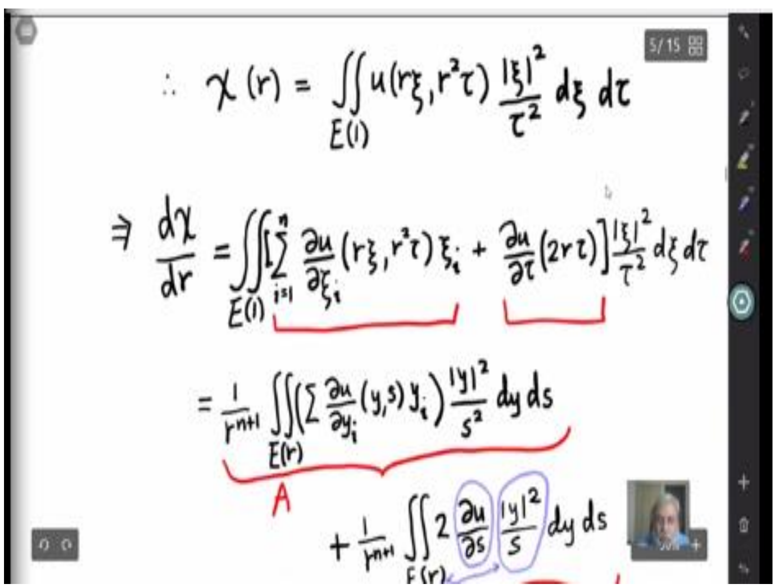

So, that involves some computation.

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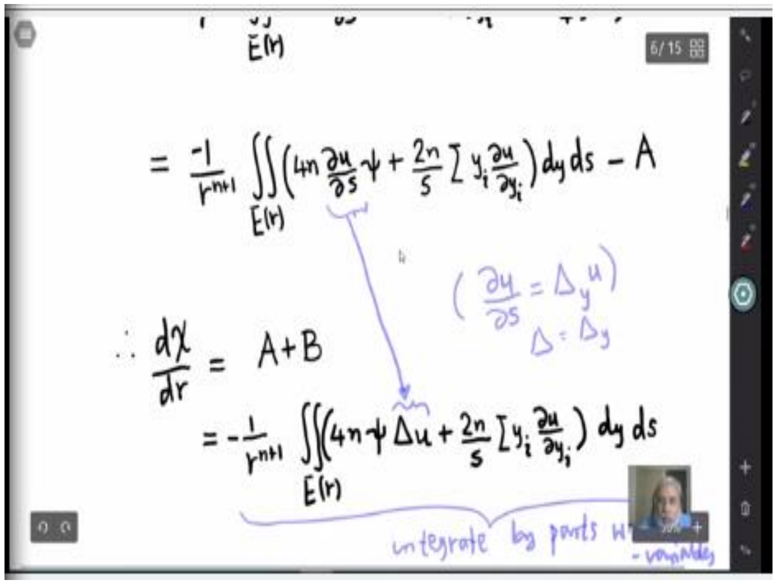

And only in the last part of the proof we use that u satisfies the heat equation. So, this del u by del s is replaced by Laplacian u.

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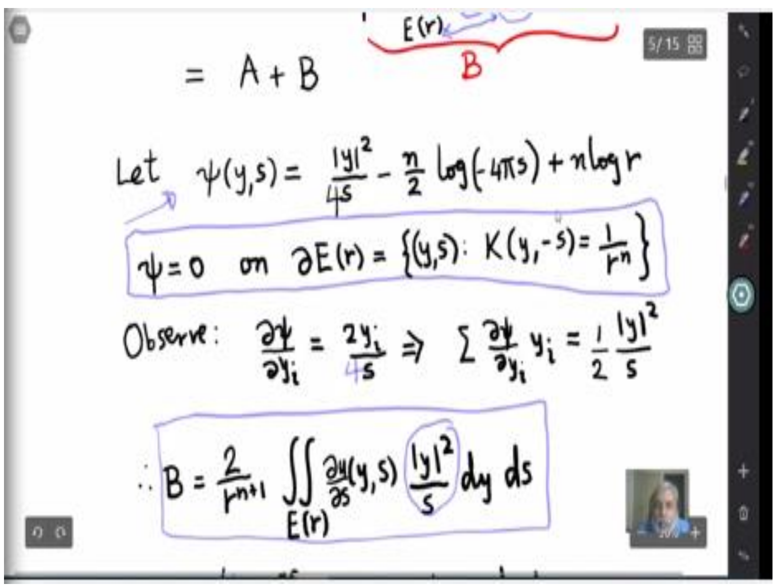

And proof mainly relied on divergence theorem and this important function coming from the nature of the set a heat ball E x, t, r.

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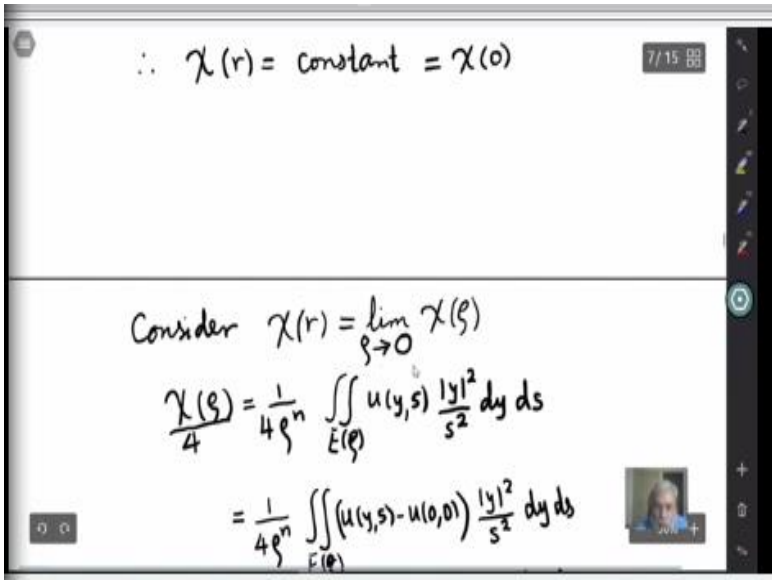

So, once we know that chi is a constant function, so this chi of r = limit chi rho 0 tends to 0.

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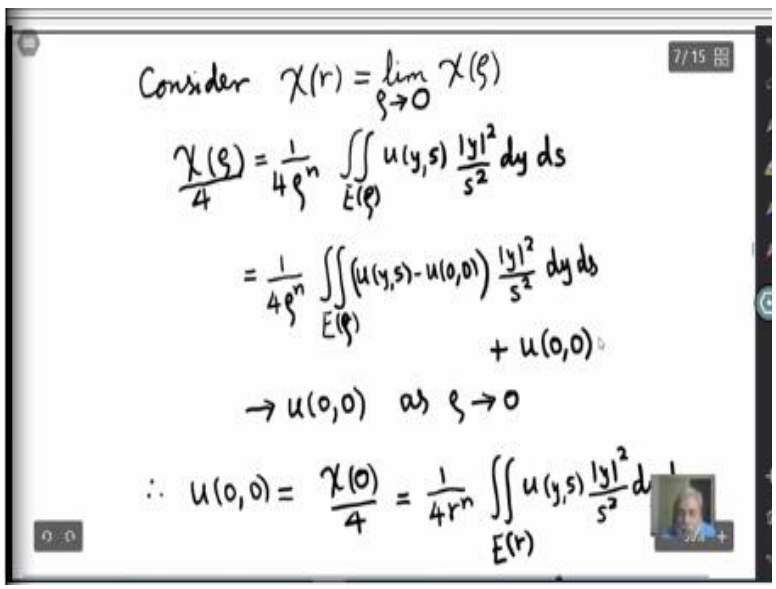

But then we can write this chi rho by 4 = 1 by 4 to the n this integral over E rho and then you add and subtract this u 0, 0 and because of that integral is 1, so this is just 1. And by continuity of u the first integral goes to 0 as rho tends to 0, so we are just left with u 0, 0 and that is what we wanted to prove.

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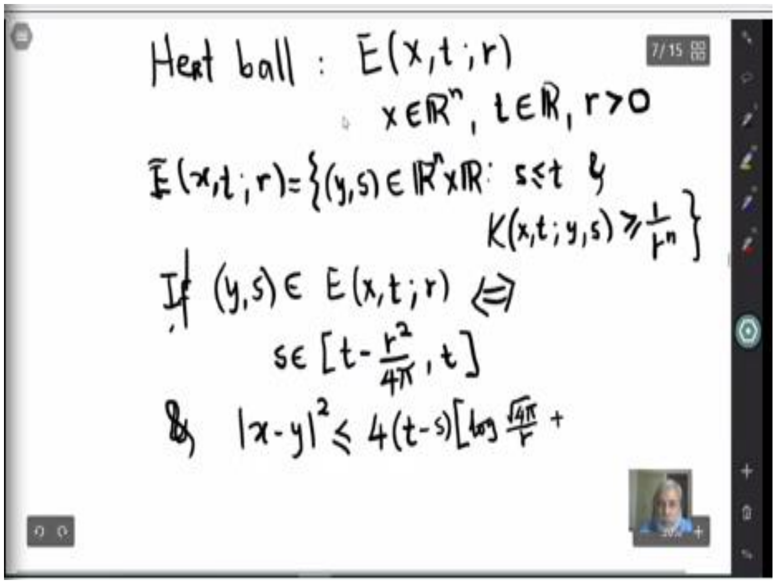

So, again let me just recall this heat ball E x, t, r. So, x is in R n, t is a real number and r is a positive real number. So, if what is E x, t, r? So, let me just again recall that, so this is set of all y, s in R n cross R n R such that s is less than or equal to t and this is the heat kernel, so x, t, y, s bigger than or equal to 1 by r to the power n. So, if y, s belongs to E x, t, r, so using the expression for the heat kernel we saw that this is equivalent to, so s lies in the closed interval t - r square by 4 pi t.

And then y it lies in the ball and x - y square, so y lies in the ball with centre x and radius square is given by this number log root of 4 pi, you can check that, this is oh some end is missing, so let me somehow is missing, so it is here only, let me just go there and stop rewriting it, it is here.

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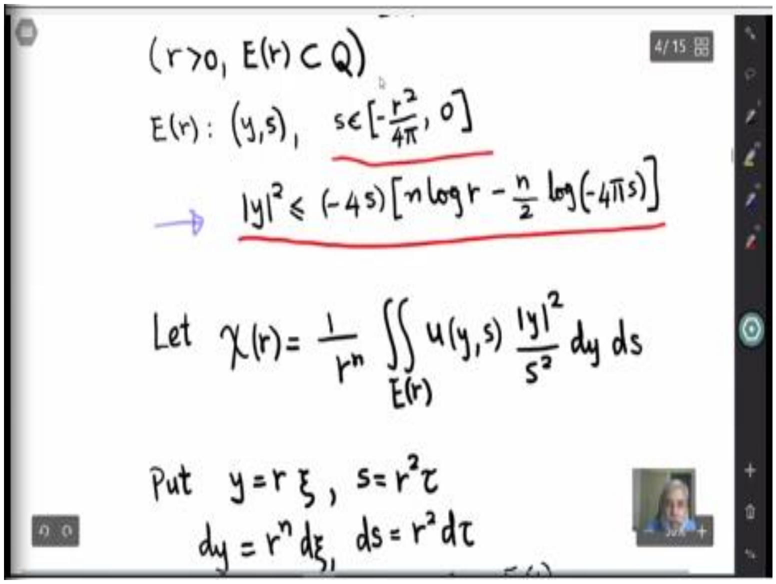

So, this is for 0 otherwise you will get x - y square and so here I have taken this x = 0 and t = 0 but that just translates.

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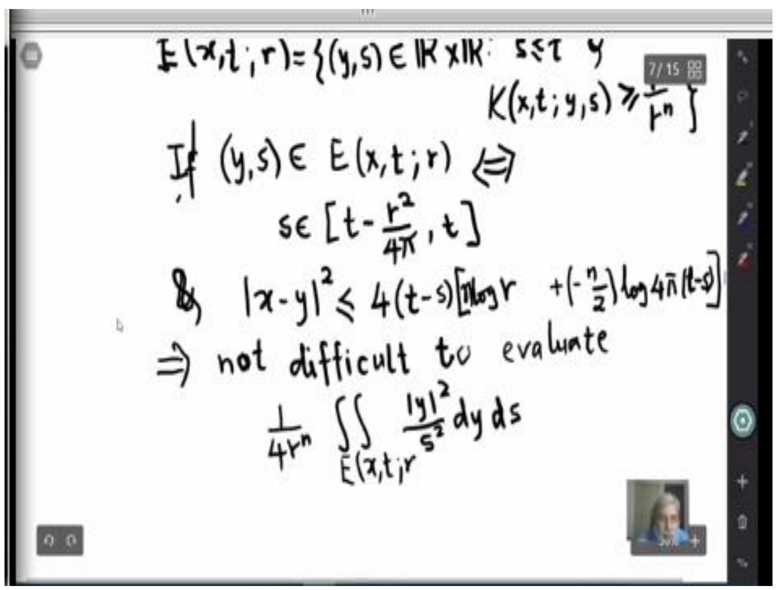

So, this is n log r + - let me write it here n by 2 log 4 pi t - s. So, using this domains for s and y, so it then easy to, so not difficult to evaluate, if this integral 1 by 4 into r with n which is n + one dimensional integral mod y square + S square dy by ds. So, since the integrant is, now negative we can integrate as an iterated integral. So, first one integrates with respect to y using this bound and then you integrate with respect to S. And even integration with respect to y, so since this is a radial function, so we can use spherical coordinates.

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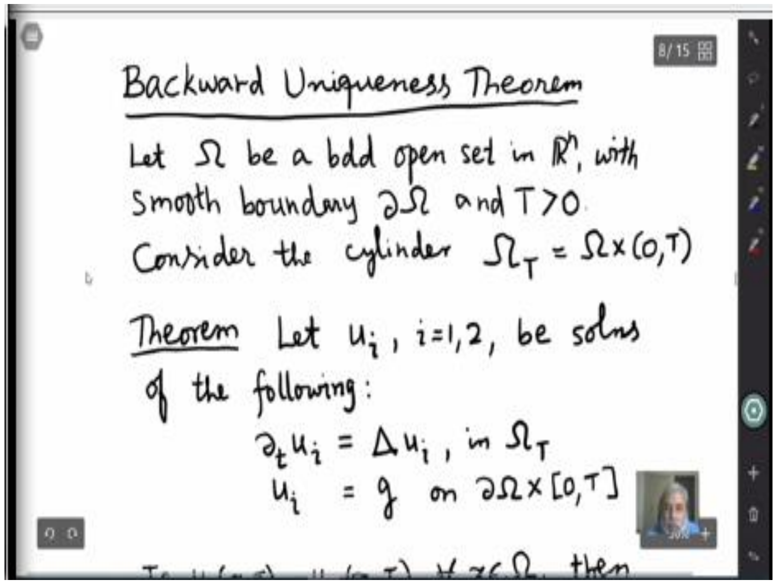

So, with this we move to another interesting property of the solution of the heat equation and this is referred to as backward uniqueness theorem. So, let me first take the theorem and then I will remark on it is interesting properties. So, again let omega be a bounded open set in R n with smooth boundary del omega and t is any positive number and consider this cylinder. So, this smooth boundary is required just because we want to integrate the parts and the divergence theorem should be valid, only for that reason we are assuming that smooth boundary.

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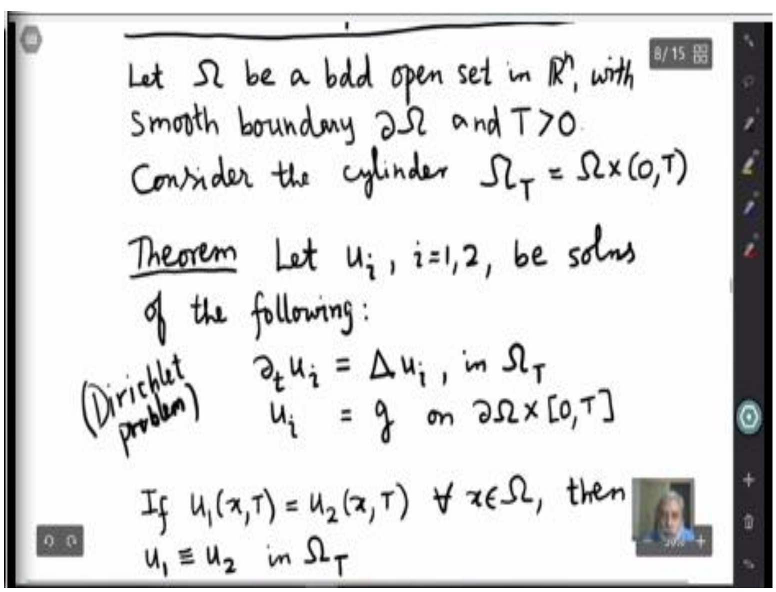

Let u 1 and u 2 the solutions of the following heat equations, so del t u i = Laplacian u i in omega t and u i = g on del omega cross 0, T. So, this is a Dirichlet boundary value problem. So, the important thing here is both u 1 and u 2 satisfy the same boundary condition g, so g is not changed.

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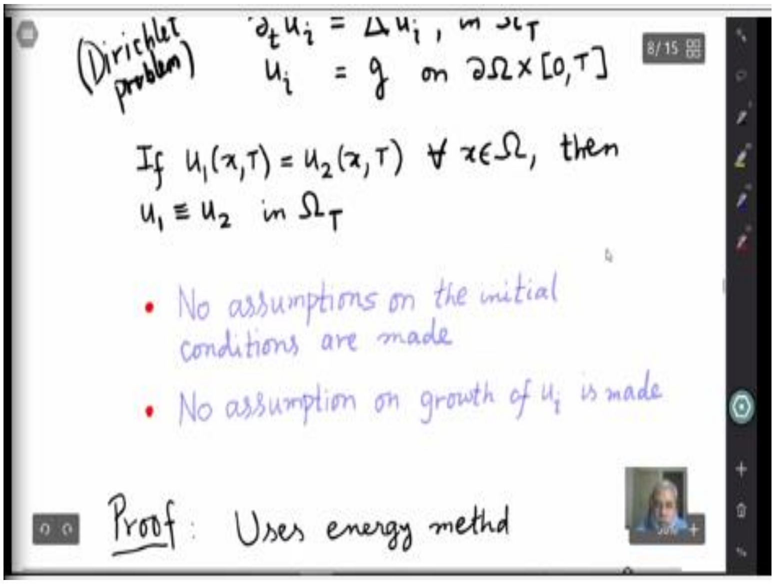

Then the statement of the theorem is if u 1 x, T = u 2 x, T for all x in omega then u 1 is identically equal to in omega T. So, that is why it is called uniqueness theorem, it is backward because at a positive time the condition is given, so namely that u 1 and u 2 are equal. And the conclusion is that u 1 and u 2 remain same for all T less than T. So, the importance of this theorem is that no assumptions on the initial conditions on u 1 and u 2 are made in the statement of the theorem.

So, there is no statement regarding the initial conditions on u 1 and u 2. And we have already seen for the initial value problem to obtain a uniqueness result we have to make some growth assumptions on the solution. And in this theorem no such assumption on the growth of u 1 and u 2 is made. And yet this uniqueness statement is made in the results, so that is why it is interesting and also important with all site minimal data we are claiming the uniqueness of the problem.

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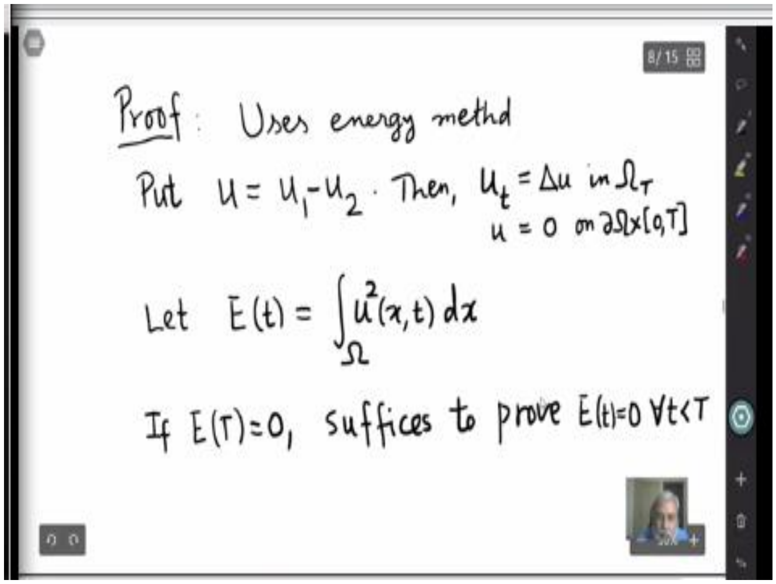

So, the proof uses energy method and this energy method we also use in wave equations. And this heat equation also enjoys some energies, energy estimate we can derive on them. So, for some again reduction since the problem is linear, so linear equation and linear boundary condition, so put u = u 1 - u 2 then u satisfy the heat equation in omega T. And since g is same so u = 0 on the boundary del omega cross 0, this is the boundary of the cylinder.

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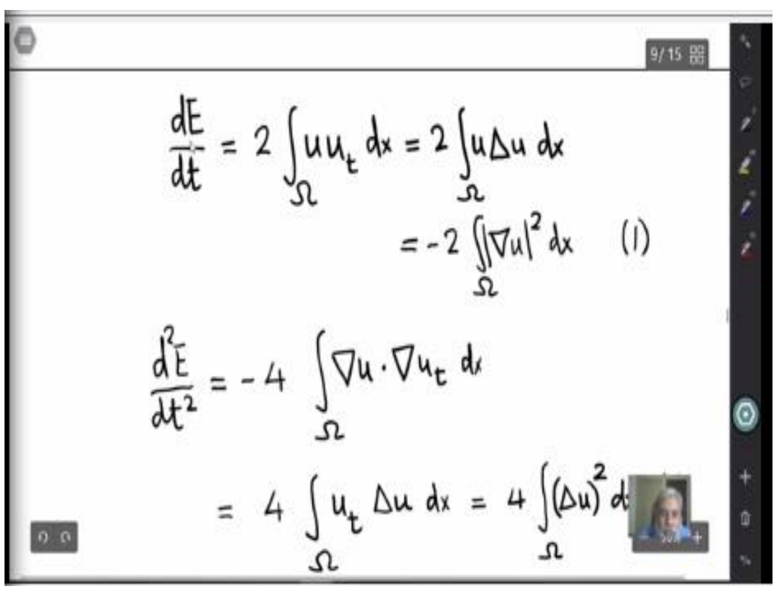

And here comes the energy, so E is the energy, so put E of T = integral omega u square x, t dx. So, you integrate only with respect to x, so this is a function of T alone. So, and this represents the total amount of heat at time t that is in the domain omega. And now we are given that E of t

0, if you see the hypothesis u 1 of x, T = u 2 of x, T that T. So, u = 0 t = capital T, that is given to us.

And we need to prove that E t = 0 for all t less than T. Because then this integral is positive, so if we prove that this is 0 then this is 0 everywhere and now we proceed to do that. And now we use this energy and so that is a function of t alone, so you differentiate with respect to t. So, just see how the energy varies with t and this very simple thing, so this you get 2 u t and u satisfy the heat equation.

So, u t is replaced by Laplacian u and then u integrates by parts, so you get -2 integral omega grad u square dx. And there are no boundary condition as u = 0 on the boundary of the cylinder, there are no boundary conditions. So, we differentiate one more time, so second derivative of E with respect to t d square E by dt square and again you use the last expression, so you get another 2 there, so -4 grad u dot grad u t.

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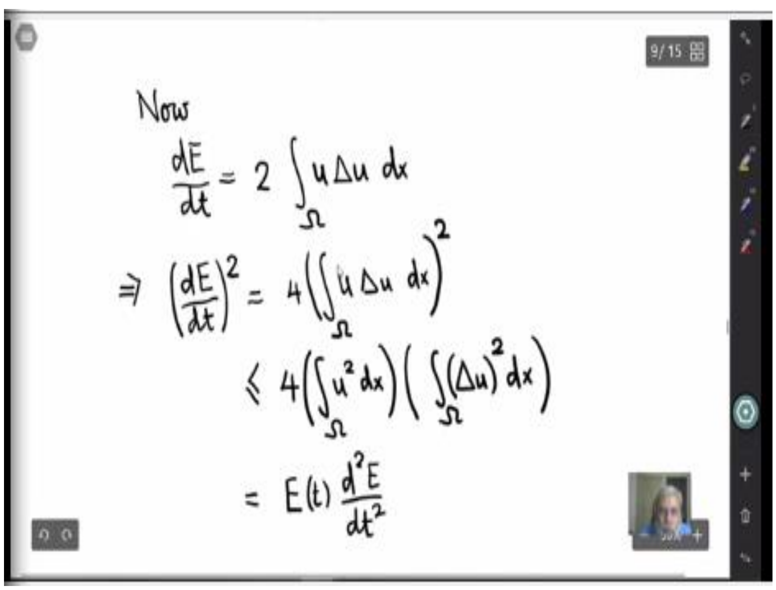

And again integrate by pars, so you get u t Laplacian u. And again there are no boundary condition as u is 0. And now again you replace u t by Laplacian u because u satisfy the heat equation, so we will get finally 4 times integral Laplacian u whole square dx. And now we compare this the first derivative of the energy with the second derivative, so some simple estimate. So, dE by dt is I am just using this expression dE by dt = 2 integral u Laplacian u and when I take square on both sides, so I get 4 this integral whole square.

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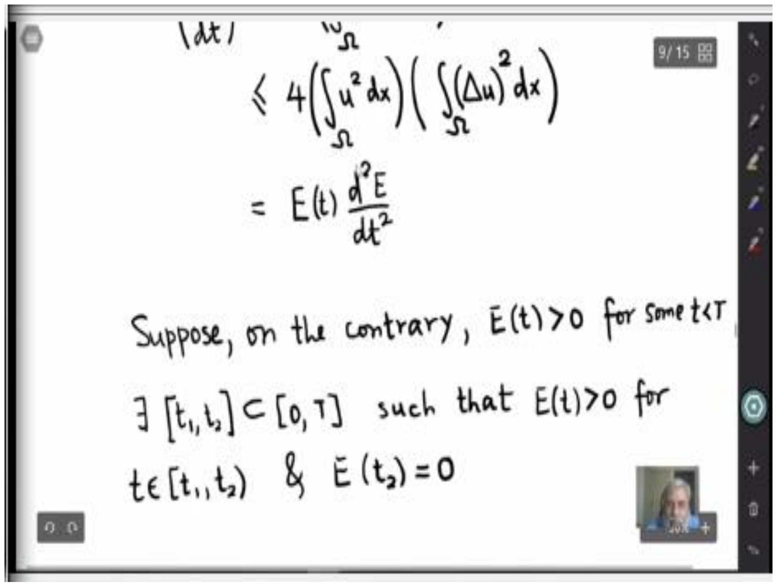

And here I use (()) (19:31) inequality and this integral is less than or equal to integral u square dx to the half but there is a 2 there, so it becomes 1 and similarly that 1. And now this first integral is nothing but the energy integral, so E t is just integral u square dx. So, if I take this 4 inside here then that is precisely d square E by dt square. So, we have this dE by dt whole square is less than or equal to E t into d square E by dt square.

And now to complete the proof we assume on the contrary that E of t is positive for E is always non negative, so E is positive for some t less than T. So, we will get a contradiction with that assumption and then by continuity so we try to find the first t less than T with this property and that just by continuity there exists a sub interval t 1, t 2 in 0, T. So, t 2 could be T such that E t is positive for all t in this semi open interval t 1 to t 2 and E to t is 0. So, this t 2 can be t itself that is given to us, so E of T is 0, so certainly there is one such thing.

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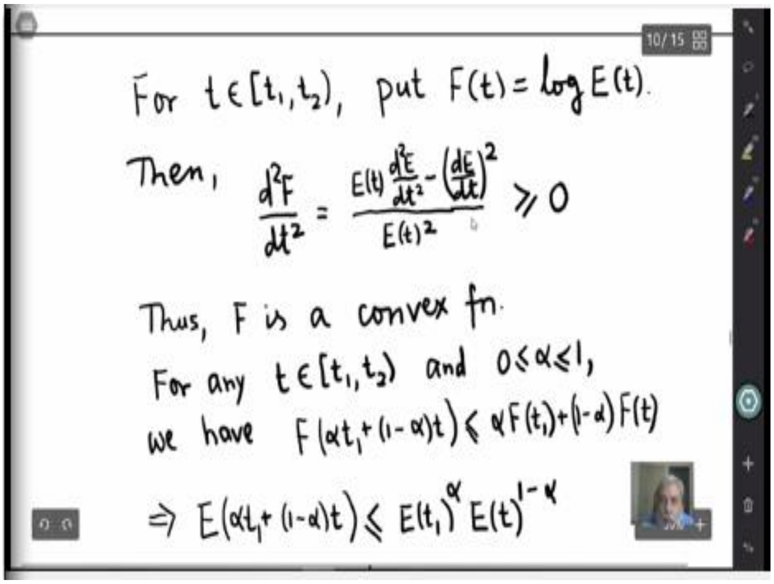

And now since E t is positive in this interval, we can take logarithm. So, for t in this semi open interval put F of t = log of E t and now we want to convert this second derivative of F, the second derivative of E t and that simple calculation yields us d square F by dt square is equal to, so there is numerator and denominator, denominator is E t square and numerator is E t into d square E by dt square - dE by dt whole square.

And that is non negative by the estimate we obtained for dE by dt square. So, F is a function of one variable such that it is second derivative is non negative in this interval. And that is equivalent to F being a convex function. So, then this because of this condition on the second derivative F is a convex function.

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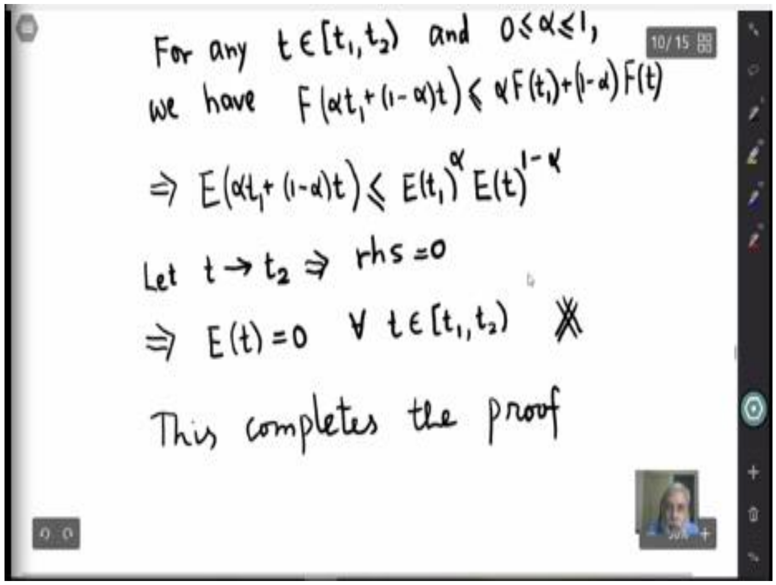

And by the definition of the convex function we have that for any t in this semi open interval and any alpha between 0 and 1, we have F of alpha t 1 + 1 - alpha t less than or equal to alpha F t 1 + 1 - alpha F of t. And if we translate again back to the energy function E of t, so we have to take exponentials, so we take the exponential both sides and that yields this inequality for the energy. So, E of alpha t 1 + 1 - alpha t is less than or equal to E of t 1 to the alpha E of t to the 1 – alpha and this is true for all t in this semi open interval.

And now on the right hand side you just let t to t 2, let t goes to t 2 and right hand side gives us 0 and that means we have that E of t = 0 for all t in this t 1, t 2 in that semi open interval. And that is a contradiction to our assumption, we have assumed that E t is strictly positive in this semi open interval and that contradiction completes the proof. So, these are the some interesting qualitative properties of the heat equation that is fun.

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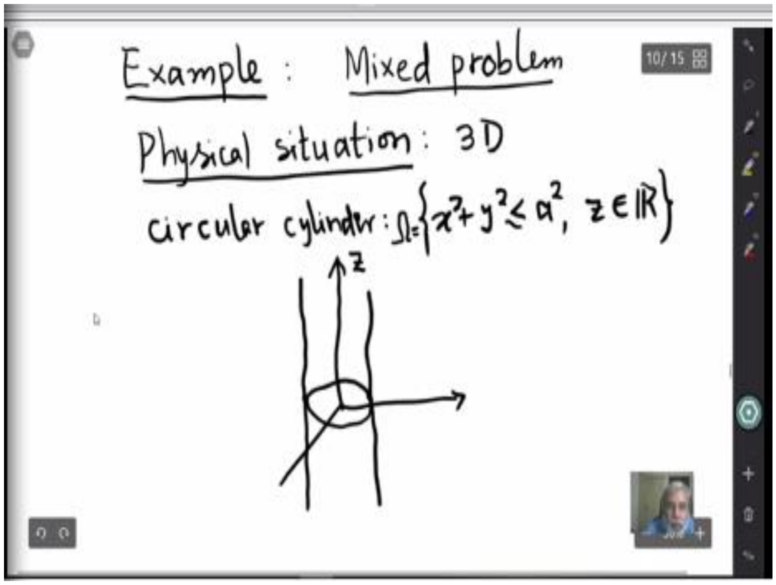

And next we discuss one example again regarding heat equations for this. Example, so let me state the problem and we will discuss the details perhaps in the next class. So, this is a mixed problem, so this is the physical situation, so we have a circular cylinder in three dimensional, 3D circular cylinder. So, x square + y square is less than or equal to a square and z is in R. So, ideal situation, so this is the z direction, so we have this so infinite slope. So, this is (()) (26:59) omega.

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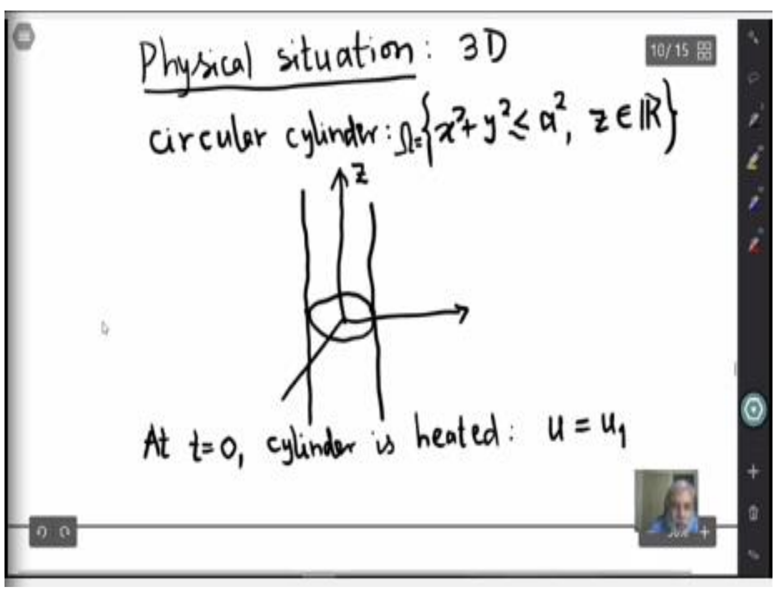

And initially so at t = 0, so this metallic cylinder is heated, so assume that u = 1.

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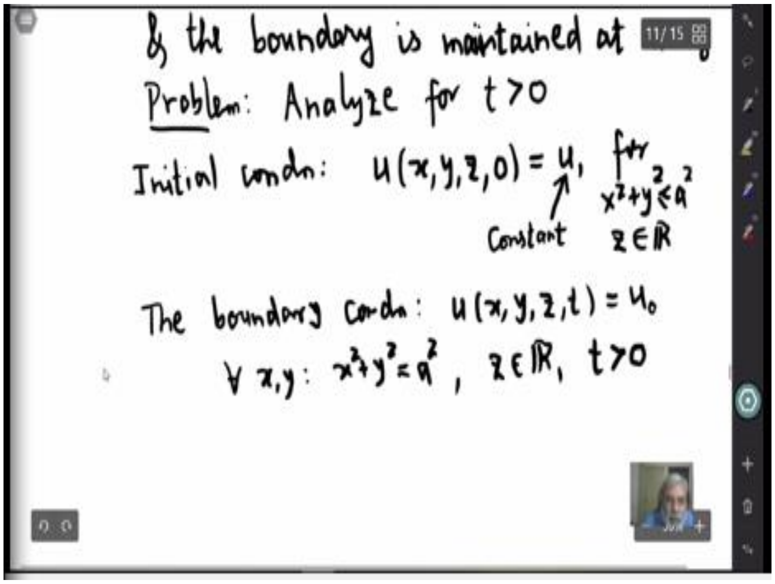

And the boundary is maintained at u = u 0. So, here I should clarify this one, this initial condition, so let me just say what that is. And then the problem is to analyze what happens for t positive. So, initial condition, so this is just u x, y, z at = 0 is u 1 for x square + y square less than or equal to a square and z R, u 1 is just a constant. So, we see that the initial condition and again what is the boundary condition? Boundary condition u of x, y, z, t = u 0 for all x, y such that x square + y square = a square and again z in R and t was 0.

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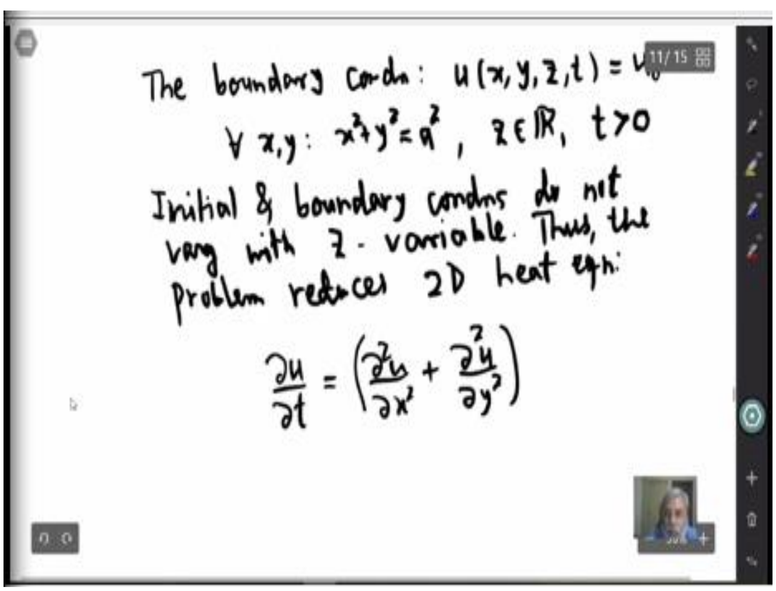

So, we see that this both initial and boundary conditions do not vary with z variable. Thus the problem reduces to two dimensional heating equation. So, du by dt d square u by del x square + del square u by del y square. So, here again I am assuming the diffusivity coefficient is 1, so one can put a number depending on that physical material. So, for simplicity again I am taking this diffusivity constant 1. So, we will continue the analysis of this initial boundary value problem in the next class, thank you.