First Course on Partial Differential Equations-II
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Lecture-27
Heat Equation-3

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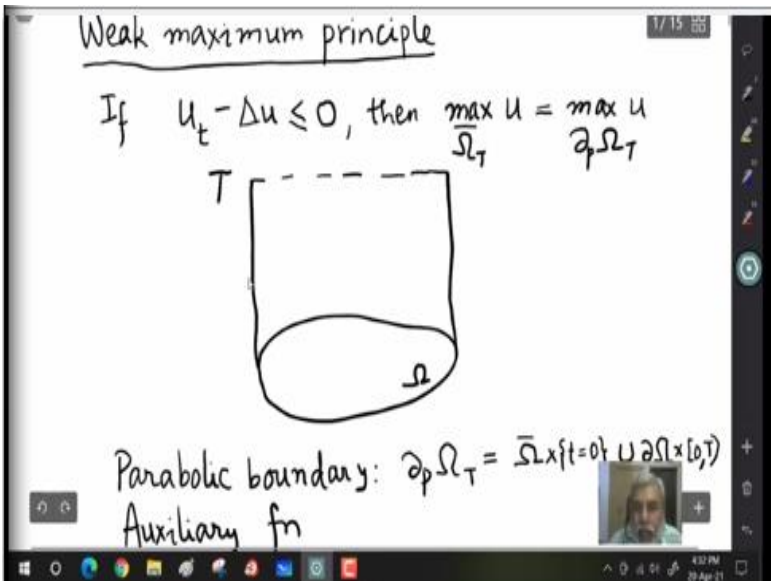

In the previous lecture we are discussing weak maximum principle for the solutions of the heat equation; we were in the just finishing part of that proof. So, let me just recall again, so, what did they mean weak maximum principle? So, if u sub t - Laplacian u is less than or equal to 0 in a domain in R n + 1, so that domain we are taking. So, omega is a bounded region in R n and T is any positive number given.

And we define what is meant by parabolic boundary in the situation. And then the statement of the weak maximum principle is if u t - Laplacian u is less than or equal to 0 then the maximum of u or omega T bar is same as maximum of u taken over the parabolic boundary.

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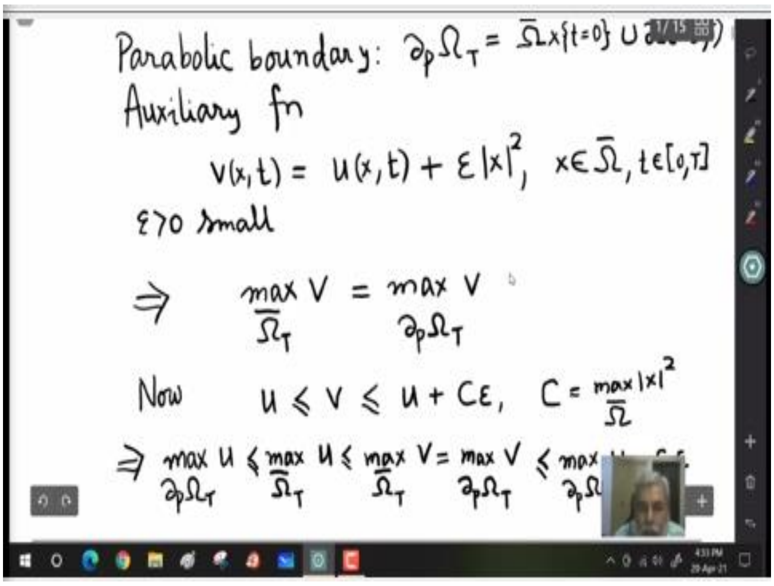

Of course this does not rules out that the maximum of u can occur somewhere in omega T bar other than the parabolic boundary. And the strong maximum principle rules out that in certain situations. So, this proof of this statement follows by taking this auxiliary function v of x, t which is equal to u of x, t + epsilon mod x square, where epsilon is a small positive number. And we showed that the maximum of v cannot talk outside the parabolic boundary and that in particular implies that maximum of v over omega T closer is same as maximum v for what the parabolic boundary.

To assert the same thing for the function u we have to just observe the following facts. So, by the definition of v, we have this relation between u and v. So, u is obviously less than or equal to v and v in term is less than or equal to u + some constant times epsilon, where this constant is the maximum of this mod x square taken over omega bar. So, there is no T here, so it is for weak maximum principle this auxiliary function or test function is a very simple one. But for strong maximum principle we need construction of some special test functions which are not trivial.

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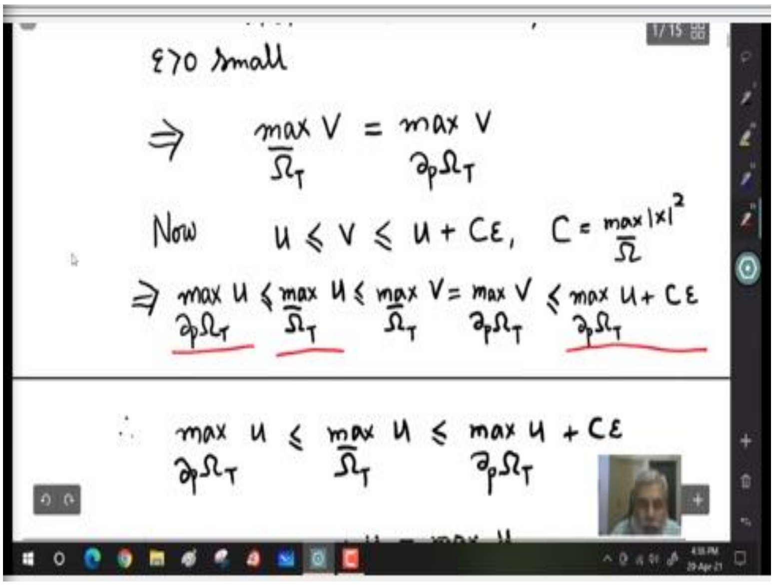

So, using this relation, now we take the maximum of u over the parabolic boundary and parabolic boundary is subset of omega T bar. So, maximum of u over the parabolic boundary is obviously less than or equal to maximum of u over omega T bar. And since u is less than or equal to v, so we have this maximum of u over closer of omega T is less than or equal to maximum of v over the closer out omega T.

And just now we have shown that this maximum is same as maximum of v over the parabolic boundary but then v is less than or equal to u + C epsilon. So, that is less than or equal to maximum of u over the parabolic boundary plus this constant C epsilon. Just you pay attention to these 3 terms that is this one, this one and the last.

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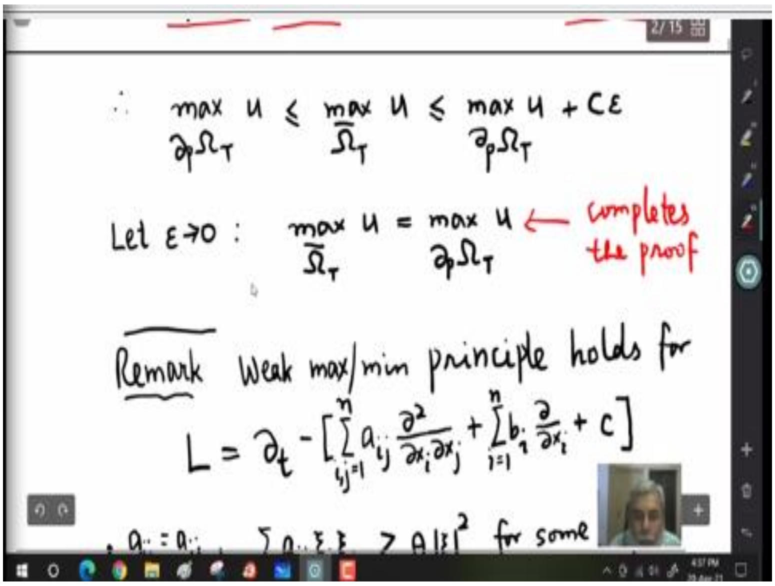

So, if you write that thing, so get maximum of u or the parabolic boundary is less than or equal to maximum of u over the closer of omega T which is again less than or equal to maximum of u over the parabolic boundary plus some constant times epsilon. And now letting epsilon = 0 gives a result. So, they may leave maximum of u over the closer of the whole domain is same as maximum over the parabolic boundary, so and that completes the proof.

And that is what we wanted to prove and for **v** the minimum principle, so you have to just replace u by -u and for it is equality occur here then we apply this region to both u and -u. So, then you have even u satisfy the heat equation then maximum of mod u. So, you can replace that by mod u over the whole boundary is same as maximum of mod u over the parabolic boundary. So, as a closing remark, so this is not just special for the heat operator, so you can just extend this weak maximum and minimum principles for a general parabolic operator which is of this for L is equal to this del t.

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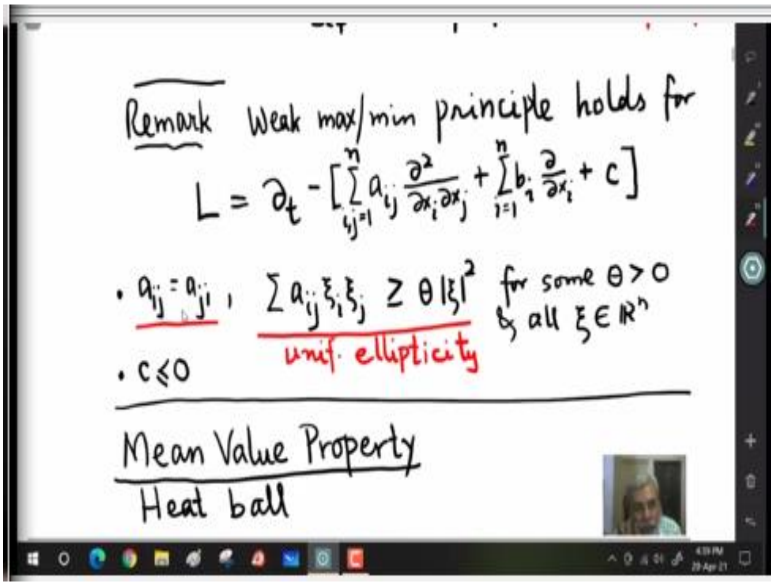

So, that is partial differential operator with respect to t. And then you replace the Laplacian by a general elliptic operator. So, a ij or smooth for a ij and b i and C are smooth functions defined in some domain in R n + 1. The important thing the assumptions on a ij, so they are symmetric and this is referred to as uniform ellipticity is nothing but the symmetric matrix with entities is a ij is uniformly positive (()) (07:46).

But in this context is also referred to as uniform ellipticity and there is the same condition on C. With these assumptions again one can prove weak maximum and minimum principle for solutions of L u = 0. So, that is not difficult and just it uses some simple property from linear algebra, so that we will discuss in assignments.

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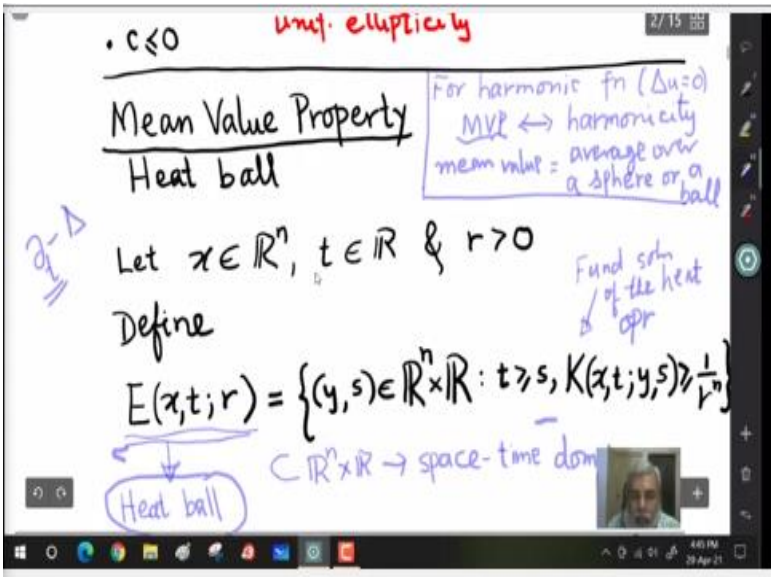

With this thing now we move on to another qualitative property of the solution of the heat equation. And this is the mean value property and in this context, so we are taking average over (()) (08:47) which is known as heat ball, so we will define that thing. So, for just fix an element in R n and t in a real number, so it need not be one positive and r is positive.

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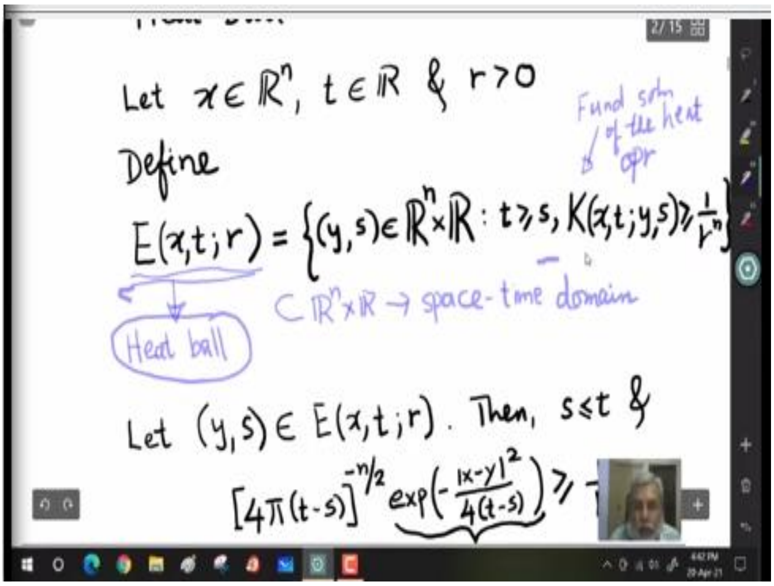

Then this defines the set E x, t, r, so this is a subset of R n cross R. So, for this reason, so since t plays the role of time also referred to as space time domain. It is bit looks complicated but we will just unfold what that is. So, this E x, t, r consists of all elements y, s, y is in R n and s is a real number, where s is less than or equal to t, so t is given to us. And this is a fundamental solution of the heat operator, so this is. So, this sect is defined in terms of the heat kernel with fundamental solution and this is referred to as to heat ball. It is not exactly a ball we will see the geometry of that but it looks like a ball. So, this set E x, t, r, so for given the x in R n and t in R and r positive.

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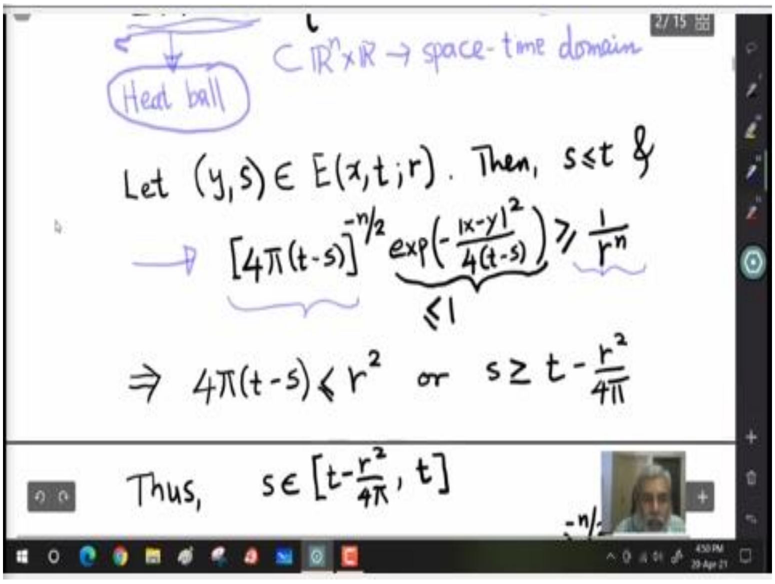

It is refer to as heat ball. So, let us just examine, before we examining what the set is?

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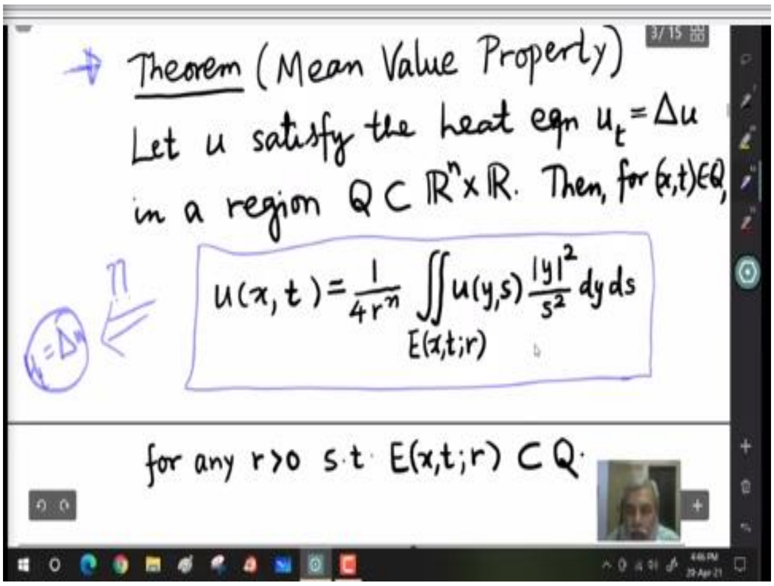

So, let me just state the theorem, so here it is. So, let u satisfy the heat equation in a region to Q in R n cross R, then for x, t in Q u, x, t is equal to this integral. And why it is called a mean value theorem or mean value property? And where are we taking the average? So, we will discuss all those things then you will understand why it is called mean value property.

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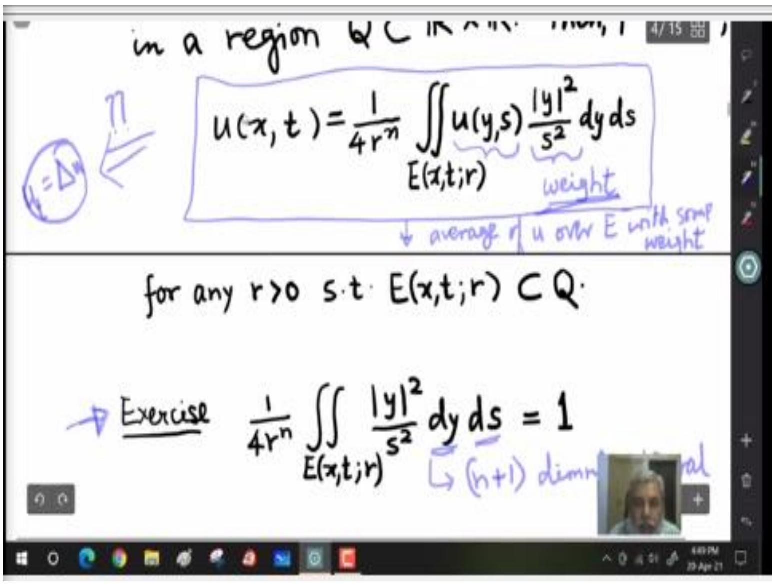

So, this is proof for any r positive such that this heat ball is contained in this given region Q. Before going into that, so let me just recall again. So, this just as a, so for harmonic functions, so we saw, so this is Laplacian u = 0, this MV is mean value property is equivalent to harmonicity. And for mean value here, so we used, so mean value mean is here either average over a sphere or a ball, so this we studied in detail.

So, obviously here we cannot take a surface or a ball in R n + 1 because there is no ellipticity here. So, there is lack of one derivative here, so this last one, so this only first order, so we cannot expect that. So, in essence this mean value property for the solution of heat equation may not look as elegant as in the case of harmonic functions and also this equivalence. So, for harmonic functions we saw the mean value property is equivalent to harmonicity that is also not clear in the present case.

In the sense that suppose u satisfies this mean value property for all E x, t, r in Q does this implied a big question mark? u t = del n, so that is not clear, so there are lots of things to be still analyzed, found out. And whether this kind of mean value is the best possible thing or are there other ways of finding the mean value? So, in this case let me quickly say this thing. So, there is we are not just integrating the function but there is a weight, we are multiplying by some function.

So, these are generally referred to as weights, the property of this weight is, so if you remove this u, y, s, so this integral over, so writing here double integral just to indicate that this is n + 1 dimensional integral is in, this is one integral is use with n dimensions and another one is one dimension. So, if you integrate this weight function namely mod y square by S square over this heat ball and then you divide by 4 to the R n this is equal to 1.

So, this is a good exercise in computation of some integrals. So, that means we are indeed taking average of u y with the function u over this heat ball or the right hand side in the case, so average of u over E with some weight, so this is indeed average. So, and the mean value property say that this u, x, t is equal to this. Proof is simple but lengthy and involves as many computations.

So, first let us understand what it is E x, t, r the heat ball. So, let us analyze that, so y, s belongs to E x, t, r, then by definition s is less than or equal to t and this K x, t, y, s bigger than or equal to R n and that you express. So, I am just expanding what is K, the fundamental solution heat operator and since this is exponential with the negative exponents, so this is always less than or equal to 1.

So, that further implies that, so if we just consider this factor and that factor, so this is less than or equal to 1, so you find that 4pi t - s is less than or equal to r square or s is bigger than or equal to t - r square by 4pi. So, the s variable for any point in this heat ball is a bounded interval, so that is the first observation.

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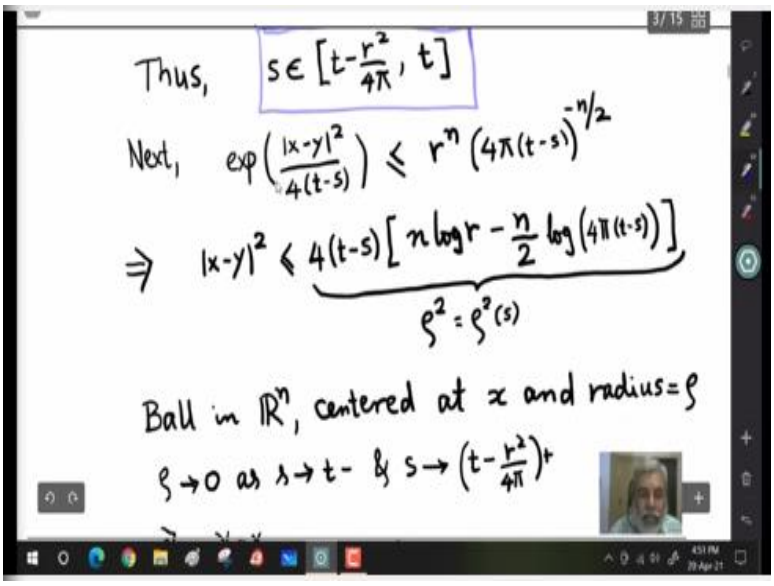

From the definition it is not even clear whether the heat ball is bonded or not. So, at least the s variable is easily seen to be bounded and it lies in this input. Next you rewrite this inequality as exponential x - y square by 4 t - less than or equal to r to the n. And then you take the logarithm on both sides and you get x - y square is less than or equal to this quantity. And this quantity for s lying in this interval is always non negative of course it depends on S. So, what it gives us?

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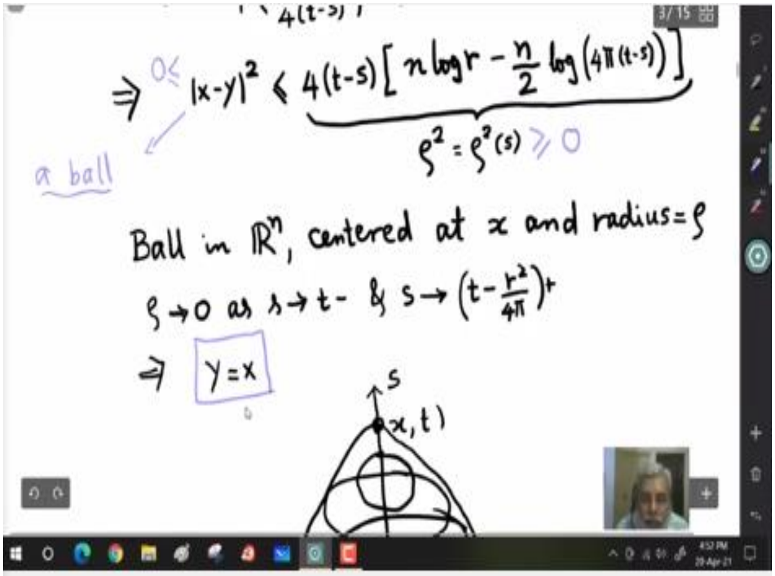

So, for each level s, so this is just defines a ball, here this represents your ball in R n centre at x that is the given point in R n and radius rho. And as s goes to t - S is always less than or equal to t. So, this radius becomes 0 and same thing as S score to the other limit t - r square by 4pi, so at both values of s obviously Y has to be equal to x because this is non negative thing, so let us stress that. So, when this is right hand side is 0, Y must be x.

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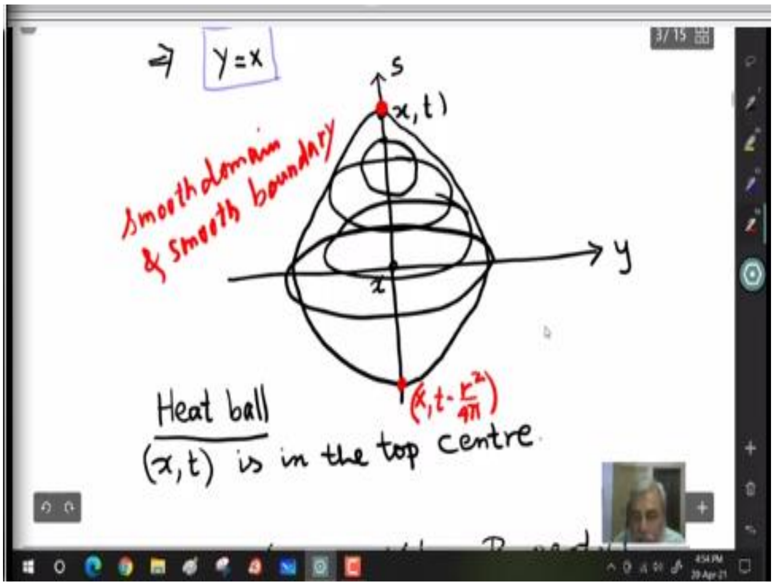

It is not a good figure but ok I drawn here, so for each S you get a sphere in R n and then we get this x, t at the top of this heat ball and it is in the centre. So, here also this is S quantities x and this one is r square by 4pi. So, exactly not ball but a very nice, so it is union of spheres for these values of S. So, heat ball is nothing but to get several balls and then you take the union, so it is a nice smooth domain smooth bound. In fact boundary zone we see that, it is given by a function 0 set of a function. So, x, t is not in the centre just like for the harmonic functions but the x, t here the point in question is the top of this heat ball.

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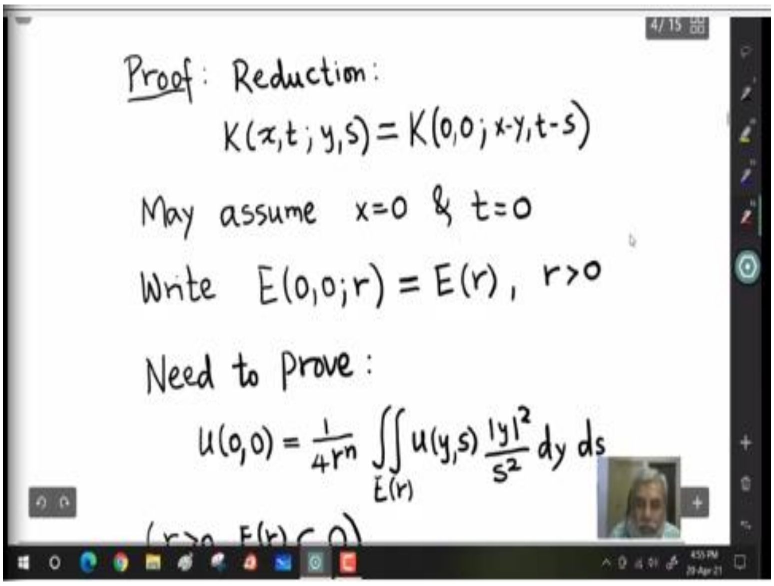

So, as I said proof involves several computations. So, we will start with first reduction, so observe that the heat operator heat kernel, heat kernel heat operator heat kernel K satisfy this K of x, t, y, s = K 0, 0, x – y, t - s. So, we can always do translations and assume that x is 0 in R n and t = 0, real number. And in that case, we write this E 0, 0, r as simply E of r at the 0, 0 is fixed.

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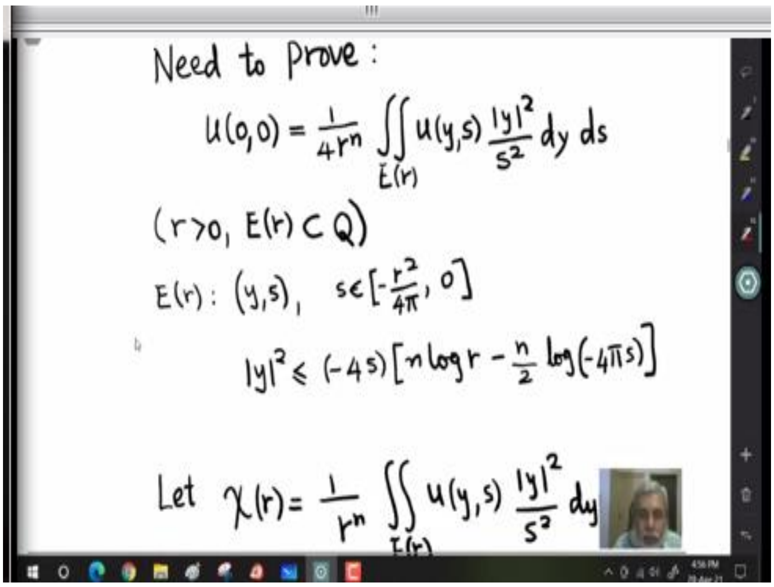

And then the statement of the theorem, we need to prove that u 0, 0 = 1 by 4 r n, u y, s etcetera integrated over E r. So, here r is any positive number such that E r is in the given domain Q. So, let us reexamine what are the points in E r?

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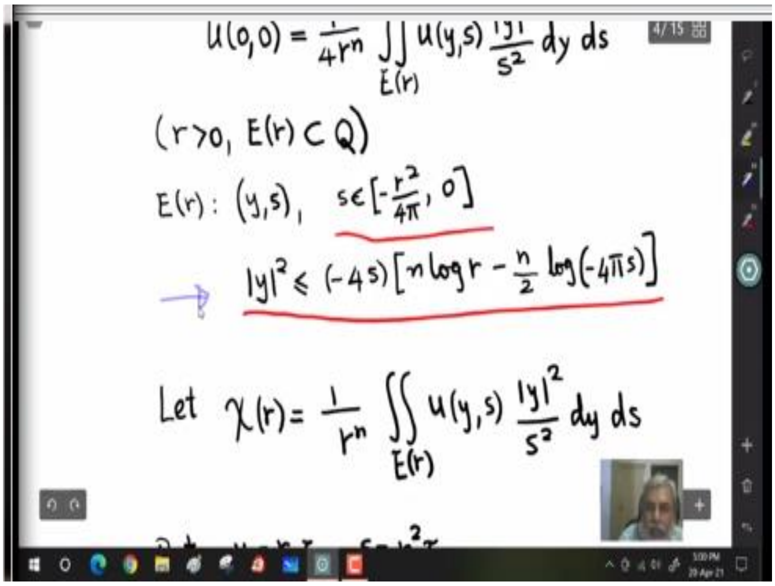

So, just again so y, s, so now t = 0, so s belong to this closed interval and we have this sphere in r n. So, now centered at the origin and with this radius square.

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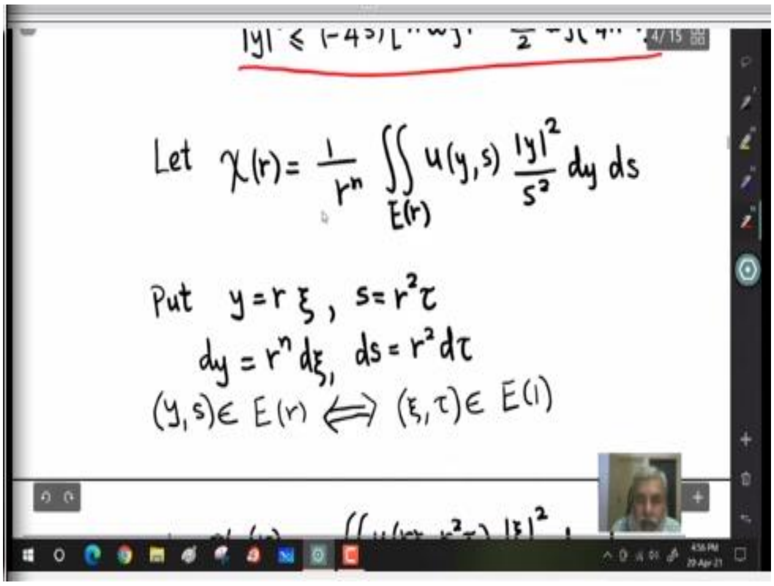

So, the idea of proof is you consider this function chi of r, so it is same as the right hand side except for the factor 4 that is a constant. So, just only did for the time being, so the idea is to show that this integral this function chi of r as a function of r is a constant. And then that will reduce that will immediately deduce the result. And to show that this function chi of r is a constant, so would like to take a derivative with respect to r but that r occurs in the integral, so that is in the domain. So, then that is very complicated, so we will try to push this r in the integrand, so that we can easily integrate.

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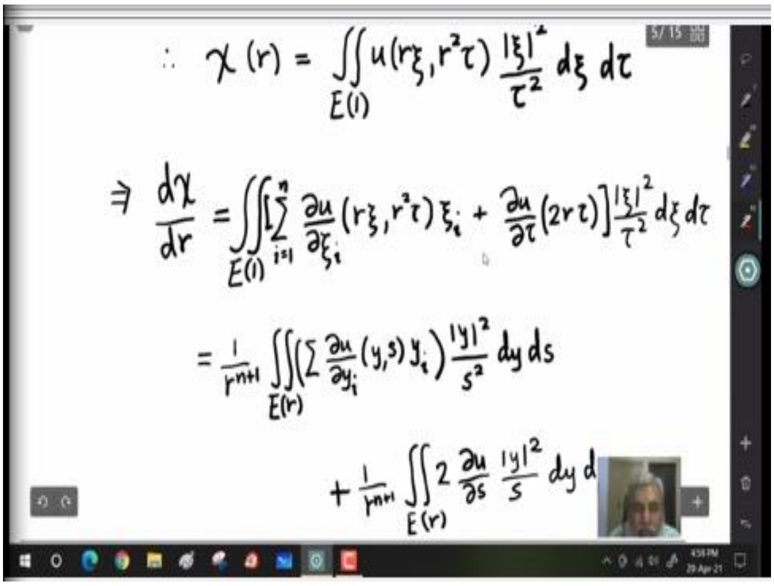

So, for that you change the variables, so y = r Xi and s = r square tau, with this change of variables and this y, s belongs to E r if and only if Xi, tau belongs E of 1. So, that reduces to E of 1 and a simple computation show that the chi r which is given by this is equal to E of 1, u of r Xi r square tau mod Xi square by mod tau square d Xi d tau. So, will be going back and forth with this change of variables couple of times.

And now we can easily integrate this function chi with respect to r because r appears only in the integrand, so we can just apply differentiation and integral sign and so we get this d chi by dr. So, you carefully do it, there are many steps here, so but carefully do it, they are all simple, there is absolutely no problem.

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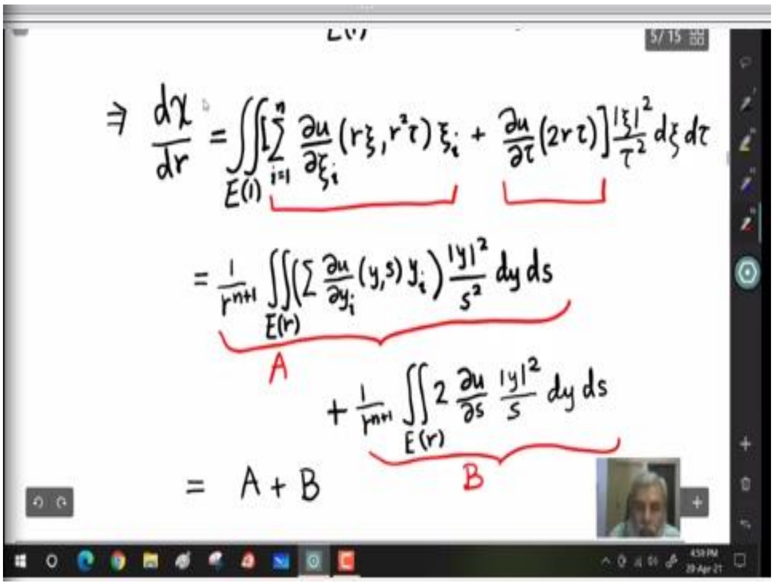

Once you do this thing again you go back to the y, s variables, so that is what I am saying. So, from y, s variable we came to Xi tau and again from Xi tau you go back to y, s variables. And there are 2 terms here as I am writing them separately, so this we are calling as A and this as B.

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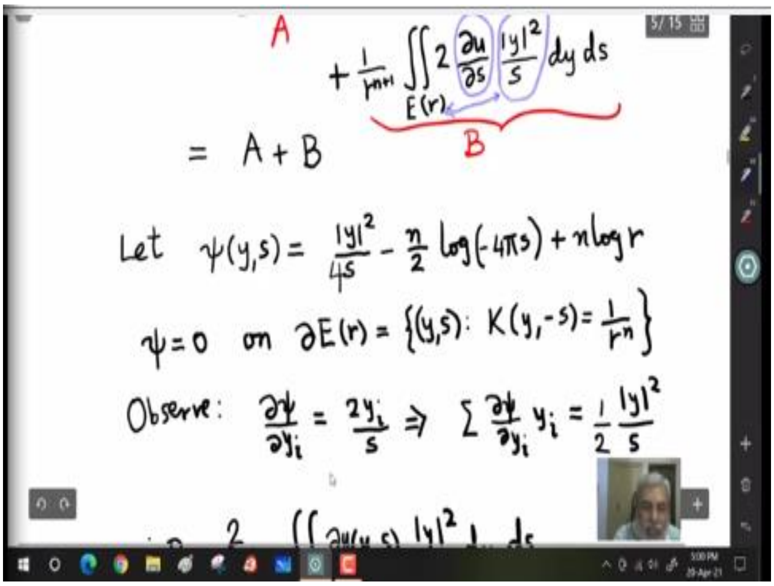

So, now we consider the integral, this B term and try to analyze that term in detail. So, this factor is ok, this factor du by ds and that to can replace it by Laplacian u later, but what about this factor? And that factor somehow we want to connect it to this E r and try to express in terms of E r and that is where this function is introduced this psi y, s.

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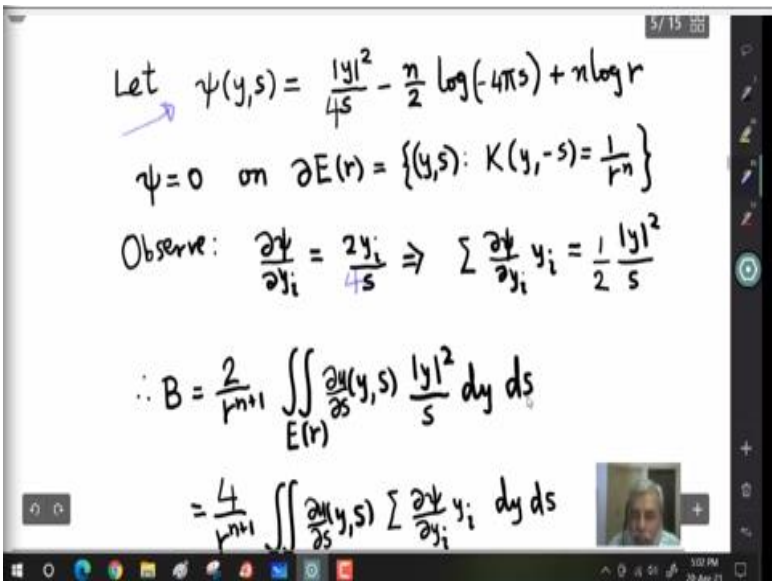

If you go back again, it is not a mystery it is here, so it is comes from this relation. So, it is coming from the boundary of precisely, so define the psi of y, s mod y square by 4s - n by 2 log -4Pi s + n log r, it is precisely that one, just remember that. So, I am just dividing by that and this replacing it by equivalent. And then you see that psi is 0 on this boundary of E r which is given by equality psi.

And now so that is we replaced this mod y square by s in terms of this psi and psi is related to the boundary of E r. So, now you observe that del psi by del y i is 2y i by 4s, let us keep that 4 here. Now again you multiply by y i and sum over all i, so this gives me of one half and that u would be mod y square by s.

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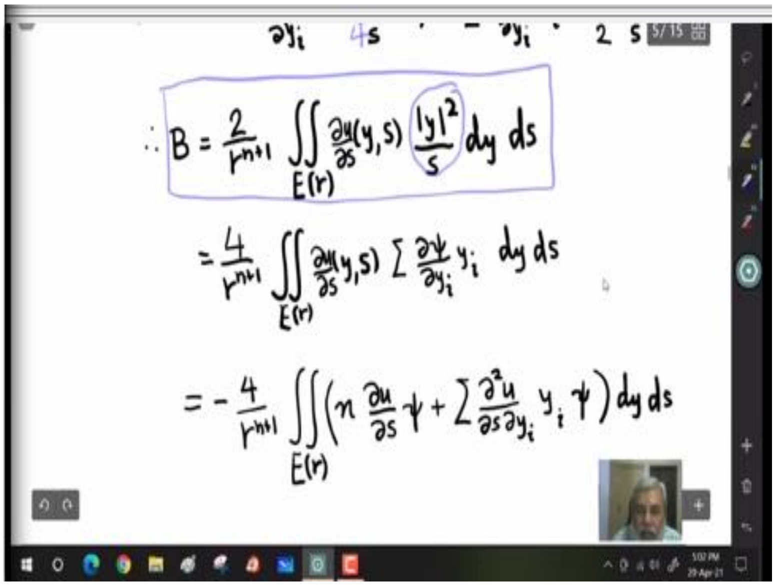

So, in this B integral, so this is our B, I am going to replace this mod y square by s by in terms of this del psi by del y i and that is what I have done. So, that there is already a 2 there in B and now another 2 is coming from here, so that makes it 4. So, this is sum over i just remember that. And since now it is differentiation of psi with respect to y i variable.

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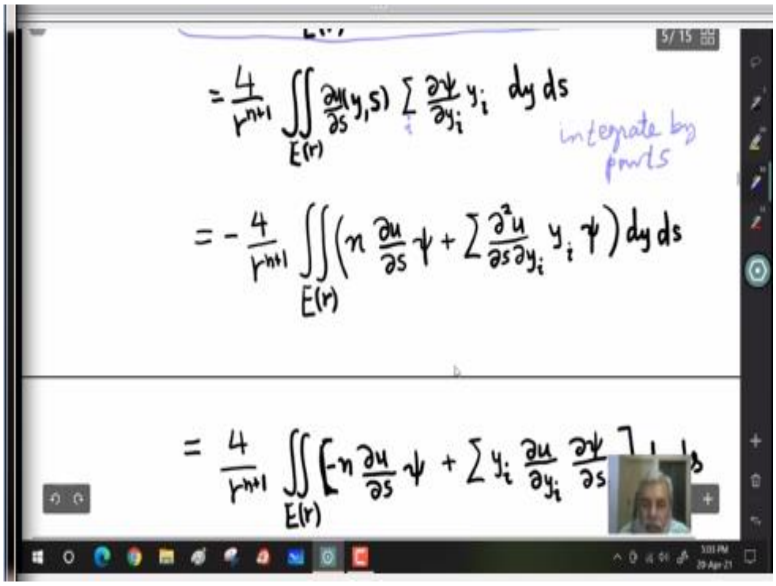

So, integrate by parts that is with respect to only y variables because there is only derivative with respect to y. So, there are 2 terms here**.**

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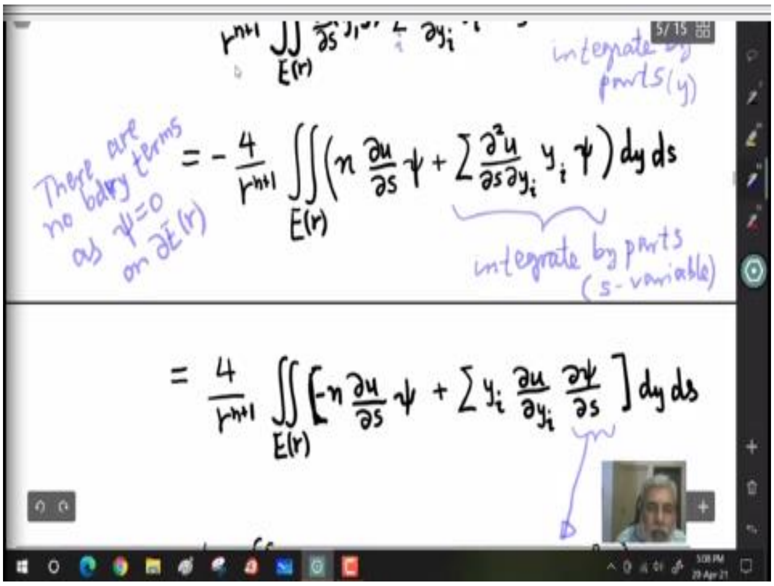

So, when you integrate by parts, so there is a negative sign here and once you take the differentiation with respect to y i and that produces n because there are the n terms. And there is another one, so you are taking differentiation of del u by del s with respect to y i and that will produce this one, so we have 2 terms and now in the second term, so here with respect to y (()) (33:29) and now again integrate by parts, now with respect to s variable. And s variable only here, so this will remain as it is, so I want to take this s derivative to this psi variable, so y does not depend on s, so y remains as it is and now get del psi by del s.

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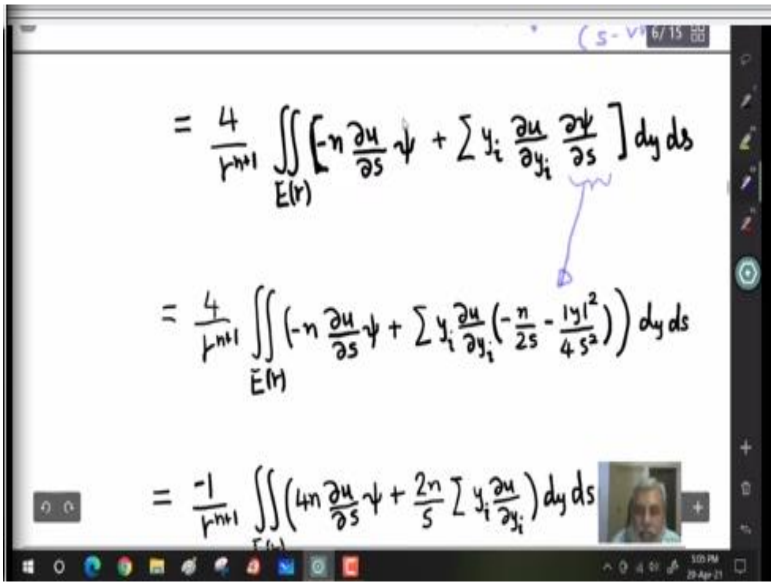

Just keep observing the signs. And now again you simplify little bit and now you substitute this del psi by del s, so psi is here again, this is the function. Now you take differentiation with respect to s variable and that is what you get here.

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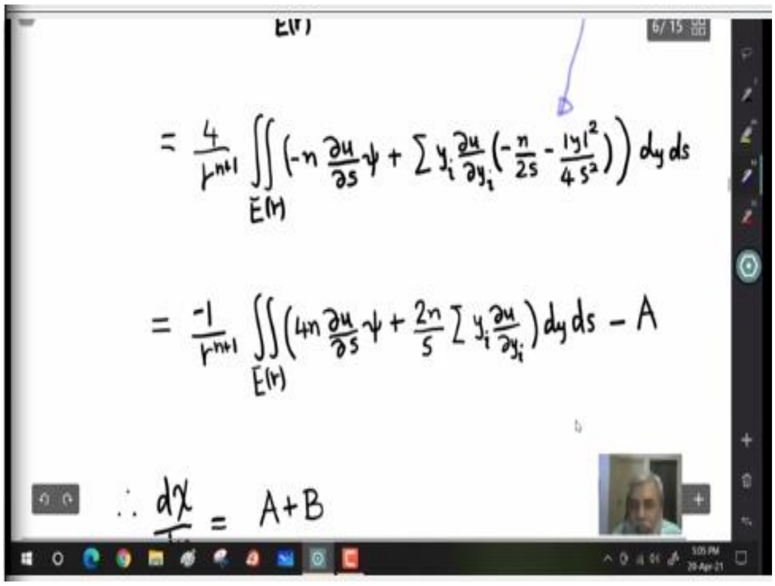

And now to simplify, so you get a nice expression, so this the first term gives me this external term and second term is precisely A. Again you go back and check here, so this is A, so del u by del y i, y i mod y square by s square and that gives me A, that is -A.

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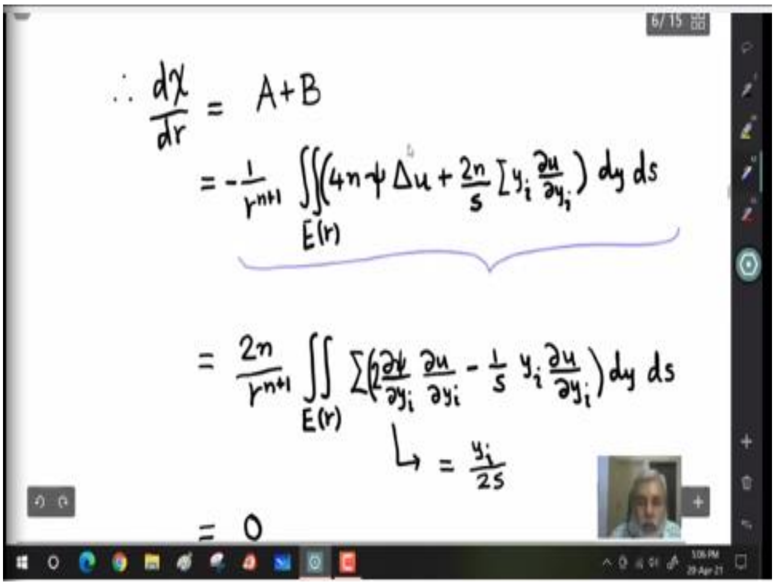

So, therefore, so this all computation is for B and d chi by dr is A + B and now I have B is equal to something -A, so that A + B will be this expression and just what I am writing, this is the expression. And so far we are not use that u satisfy the heat equation, so this is for the first time we are using.

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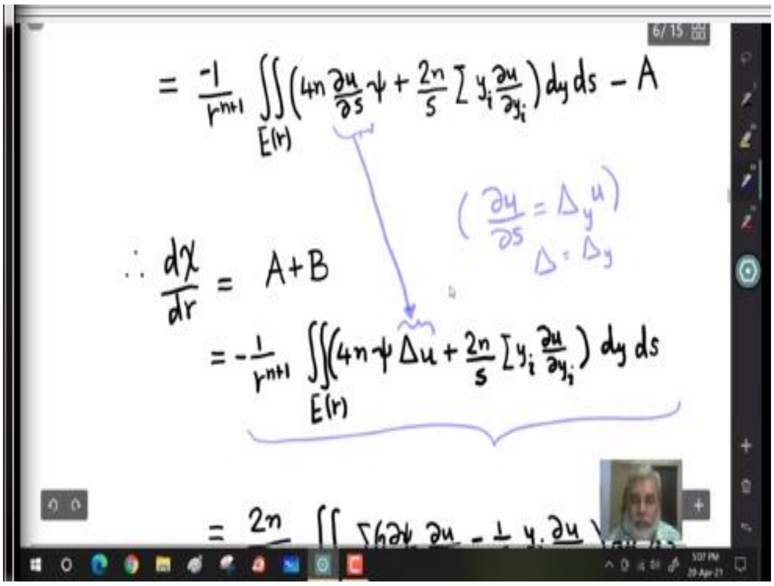

So, this del u by del s is replaced by this, so del u by del s here we are using pi, (()) (36:13) we write that, so all are this y.

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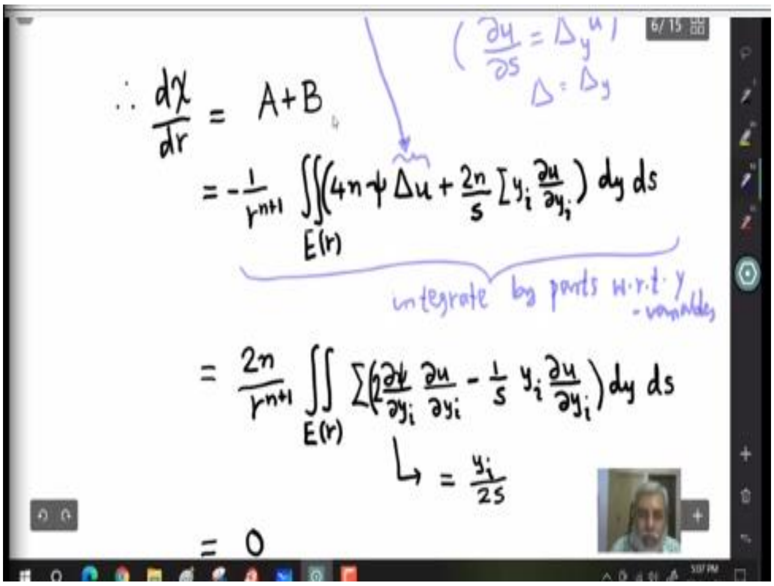

And since again Laplacian is there**.** By the by just notice here we are integrated by parts, but there are no usually there are boundary terms, so let me just comment, there are no boundary terms as psi = 0 on the boundary and again let me stress that.

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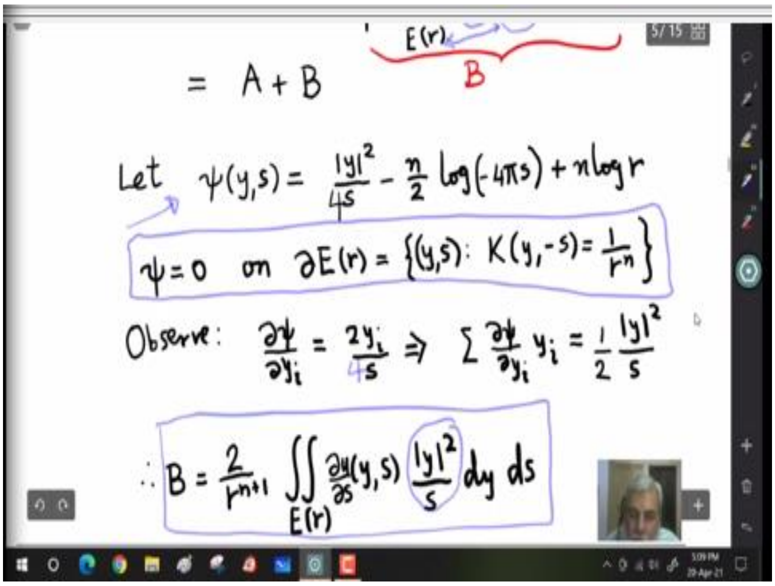

So, this boundary is given by this, so boundary of E r is nothing but 0 set of this nice function, so boundary is smooth. So, this integration will parts is justified and there are no boundary terms as psi 0 r. So, remember this just use everywhere, so whenever you integrate by parts that psi appears and psi vanishes on the board. And same thing is true here again there are no boundary terms and you just get this term.

And finally again you go back to the definition of psi and in differentiate that with respect to y i, we have already done and that is precisely y i by 2s, there is a 2 there. So, we get y i by s y i by this. So, each term in the summation vanishes and therefore we get d chi by dr = 0.

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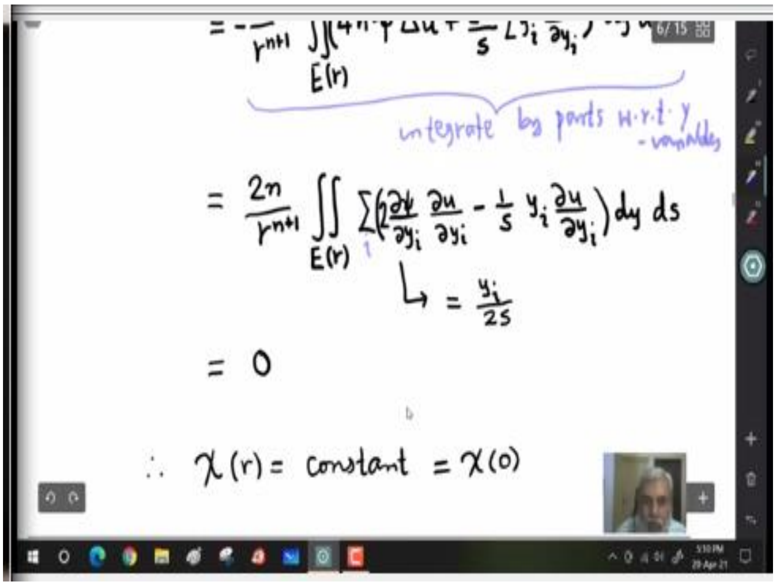

So, we will continue from this point and conclude this prove the mean value property, thank you.