First Course on Partial Differential Equations- II
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Lecture - 26
W5L6 Heat Equation 2

Last time we were discussing initial value problem for the heat equation. And in this lecture, we continue to do that.

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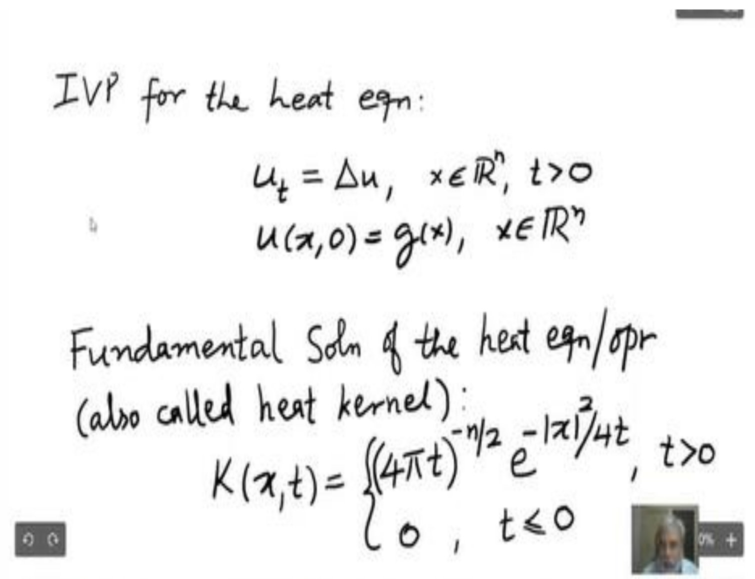

So, recall, we are discussing this initial value problem. Last time we were discussing the initial value problem for the heat equation in the whole of R n, so this was the equation we consider. And the fundamental solution of the heat equation or operator also called heat kernel.

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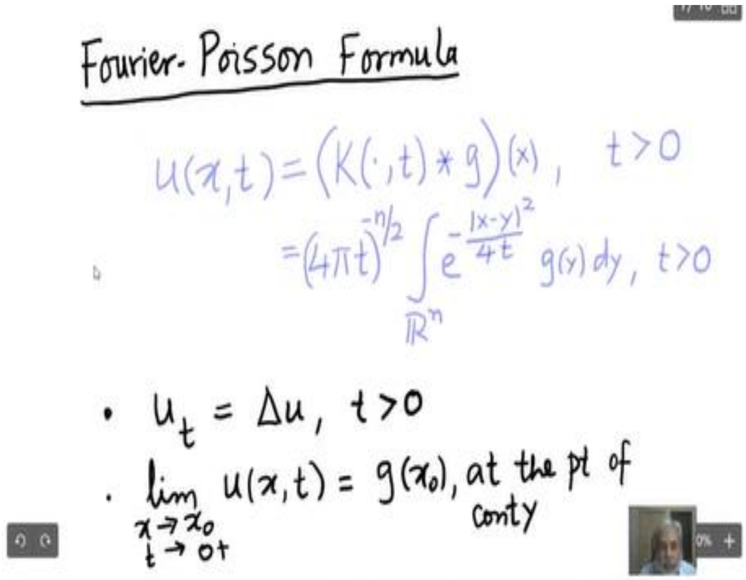

This was defined by this expression and using this fundamental solution this Fourier Poisson formula for the solution of heat equation was derived. And again, recognise that this in the Fourier Poisson formula this solution u is defined as the convolution of the fundamental solution with initial data. And then we verify that this u given by the Poisson formula is indeed solution of the heat equation for t positive t and at t equal to 0 it satisfy the initial condition in this limiting sense.

So, that should be borne in mind it is not get you directly put t equal to 0 because this k has a singularity at t equal to 0. So, we have to intricate this initial condition in this limiting sense, so that is important. And in fact, it is more local in the sense that wherever g is continuous this limit force 2. So, in particular if g is continuous everywhere then this u given by the Poisson formula satisfies the initial condition in this limiting sense.

In proving these two statements we made use of some important properties of the heat kernel the fundamental solution.

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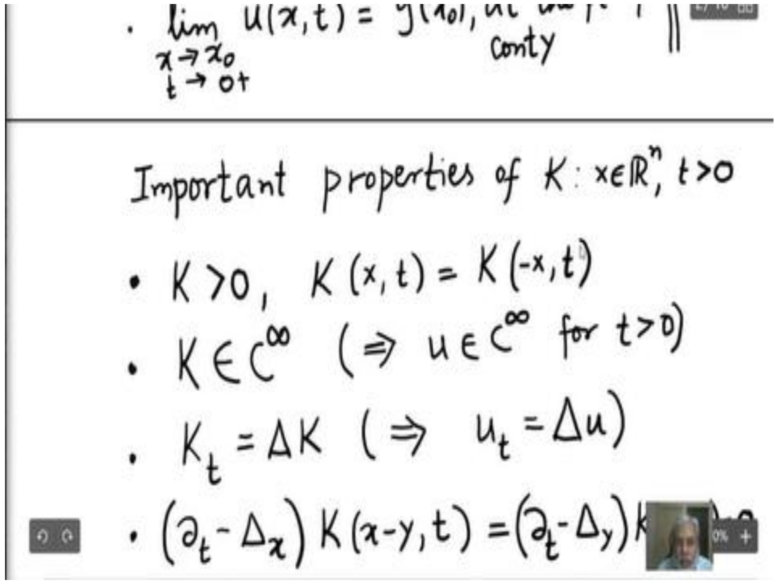

So, let me again state them for the reference. So, for x belongs to R n and t positive. So, that is the only relevant region for t less than or equal to 0 K is anyhow 0. So, let us concentrate only for x in R n and t positive. And for this in this region K is strictly positive because it is given by an exponential function and if you look at this exponential factor, say norm x square so this is the standard norm in R n. So, K is symmetric with respect to x.

And K is just infinity function because we know that the exponential is a C infinity function and this t since we are restricting to t positive. So, all these quantities are C infinity functions so is their product. So, C is in g infinity from K is the infinity function in these reasons and that implies this function given by the Fourier Poisson formula that is namely the solution of the heat equation is also a C infinity function for t positive. So, bear this in mind.

Because we; are assuming the initial condition only continuous and bounded, so that this integral is well defined. So, for t positive this u becomes a C infinity function. And that we call it smoothing effect coming from this heat curve. And in this reason again the fundamental solution satisfy the heat equation more generally we have this one. So, we; can take K the function of x – y here and if you take this heat operator.

And differentiate with respect to x variables here and differentiate with respect to y variable here, even that e g.

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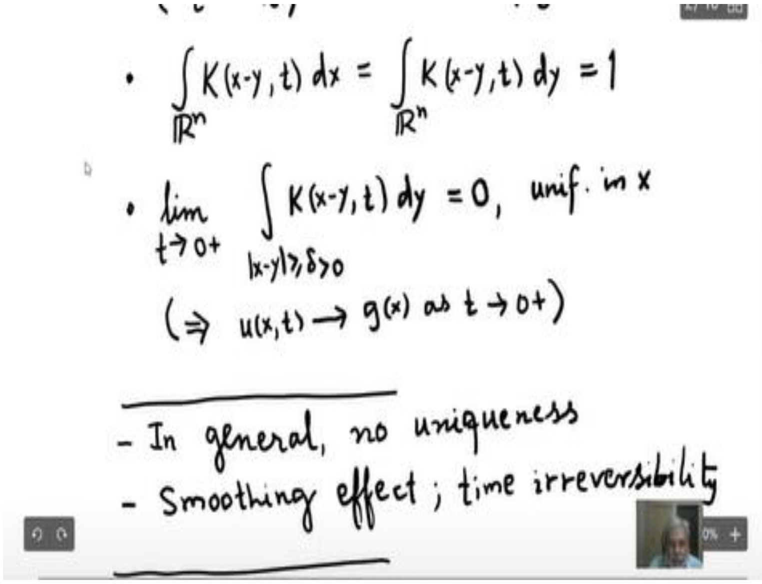

And this is another important property so it is the total integral for R n is 1 and this is another important property. So, it is like kind of averaging any function we integrate with this as kernel so this is an important property. So, if you stay away from a positive distance from any point x, so then this integral tends to 0 as t stands to 0 and this is used in this verification of initial condition in the limit x. So, this is again important.

And as given by the taken of example in general there is no uniqueness and for uniqueness, we need to impose certain conditions on the solutions and in one dimensional case we have discussed this in detail and same thing applies to even intimation. So, if you restrict for example the solution in the class of those functions with exponential growth then there is uniqueness. And as I said here so this K is a C infinity function then u is a C infinity function.

So, that is referred to as smoothing effect and this physically this equation represents time irreversibility in the sense. For example, suppose we are given temperature in a rod at time t equal to t 0 we cannot say what was its temperature sometime back? So, that is referred to as time irreversibility. So, only if this heat equation we can only predict what happens for future times not backward time.

This is the story of the heat equation in the entire domain. So, we can ask now the question what about bounded domains?

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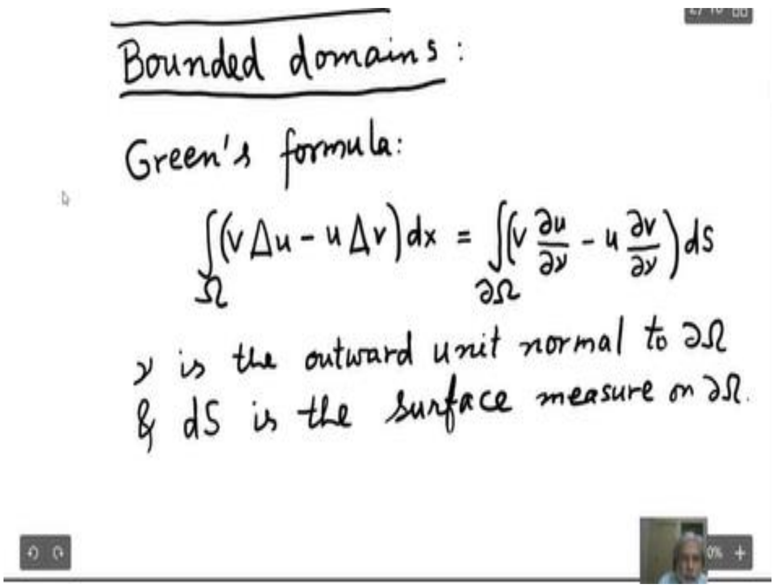

And let us go back little bit for the Laplace equation, so recall this Green's formula. So, suppose omega is a bounded domain with smooth boundary require that smooth boundary so that the divergence theorem can be applied. So, recall this greens formula so integral of any two functions u and v we have this integral v Laplacian u minus u Laplacian v equal to dx equal to ds. So, this is integral over the domain so this is volume integral.

And the right side is integral over del omega. So, that is a boundary of omega so that is a surface integral. And u is the outward unit normal to del omega and ds denotes the surface measure on del omega. Now we apply this green’s formula to the following situation we take u a harmonic function in omega and take v at the fundamental solution call it fine. So, this we discussed in the first part of this course and also will be discussed little bit even in the second part.

So, fundamental solution phi of the Laplacian and then we derive this formula. So, you apply here so this is harmonic so Laplacian u is 0 and this Laplacian phi is also 0.

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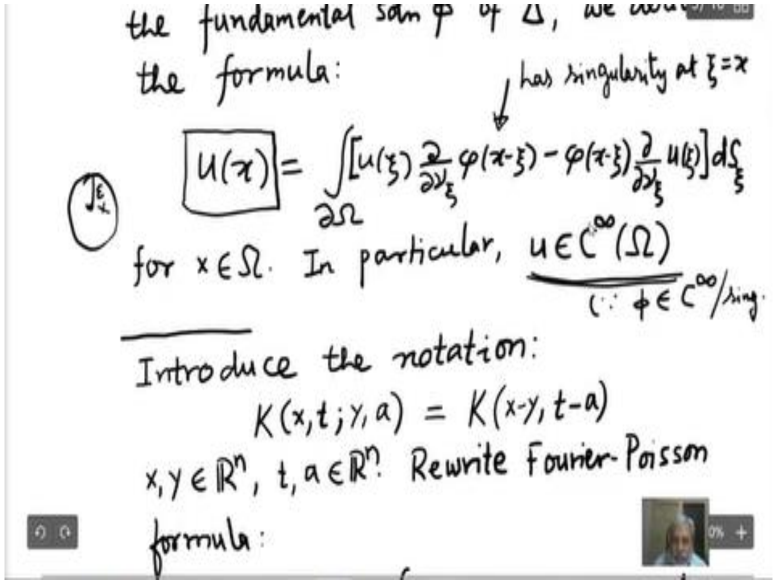

But there is some as you know this phi of x minus xi this has singularity. So, if I consider that as a function of xi for fixed x at xi equal to x. So, what we should do is we should cut off a small ball around. So, first we should integrate over omega minus this bond and then you let epsilon to 0 and that produces this u of x. So, this we have done in detail in the first part of the course and you can recall that.

So, we have a formula for the harmonic function in omega u x equal to u xi. So, this is purely a boundary integral these are called single and double-layer potentials and this should not be taken as formula for the solution because this formula requires both u and its normal derivative on the boundary and both cannot be given simultaneously. So, this is not a formula for this solution but nevertheless this important formula we can use it for some other purposes.

For example, here so Laplace if you use harmonic means just Laplacian u equal to 0 and that requires only a C 2 function but this formula gives us that u is C infinity function. And this is because phi is infinity function of course there is some singularity. But here if you look at this formula xi is on the boundary and x is in the interior, so this the x does not cause any problem here so this phi as a function of xi is a C infinity function and that is how we get u a C infinity function.

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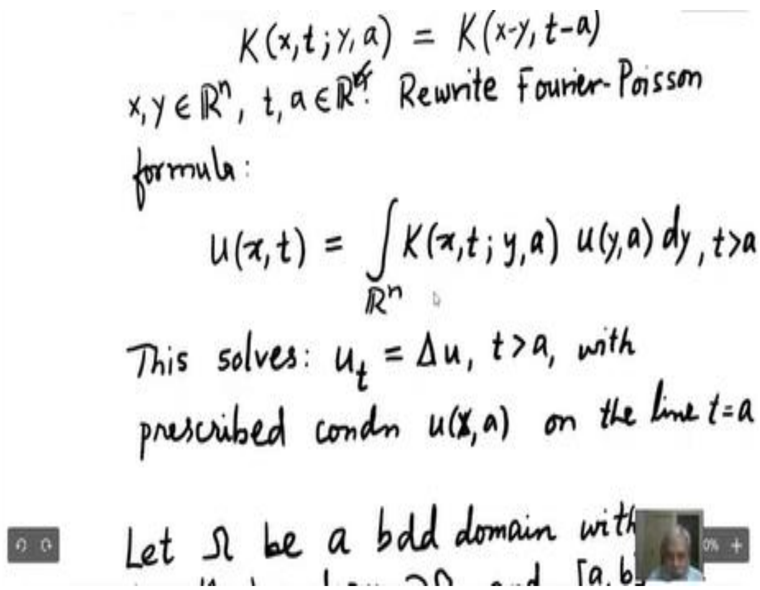

And now we try to do a similar thing for the heat equation, and for that we introduce a new notation we still call it K there should not be any confusion. So, now I write K is a function of four variables x, t, y, a x and y are in R n and t and a are r so just so this is not. And we rewrite the Fourier Poisson formula with this notation and this as u x t is equal to k x, t, y, a u y, a dy. And this formula now gives under appropriate conditions on u y, a etcetera.

So, this sums the heat equation for t bigger than a with prescribed condition u x, a on the line t equal to a. So, earlier we have we have taken the initial condition on line t equal to 0, but we can take that initial condition on any line, t equal to a. So, this t equal to a and you solve the heat equation. So, t bigger than and this is the Poisson formula in that case. So, now back again to a so with this new notation.

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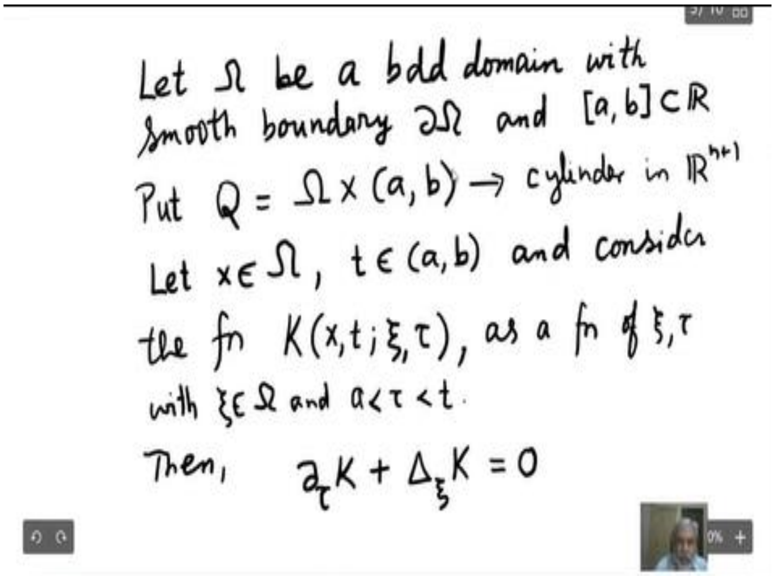

So, let omega be a bounded domain with smooth boundary again that smoothness for the purpose of using divergence theorem and you take any interval any finite interval on the real line. And you put so q equal to omega cross a b, so this is a cylinder in R n+1. So, similar to the harmonic function so we derived here a formula for u of x in a domain.

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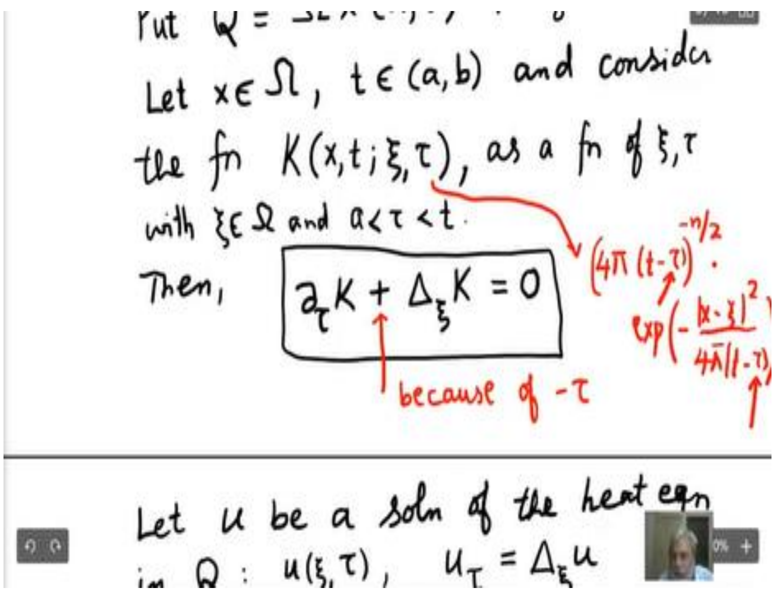

And here also would like to derive a formula for the solution of the heat equation at point x in omega and t in this interval a, b given interval a. And so those x and t we hold them fixed and now I consider this function as a function of xi and tau, and tau is restricted to this tau is less than t. And then it is an easy verification that K satisfies this; what we call backward heat equation, and this plus, so there is a plus here this is because of minus 2.

So, if you write this K take that one here so this is 4 pi t – tau - n by 2 into exponential - x – xi square and I am differentiating with respect to tau. So, there is minus tau here, so this is the one and that makes this plus. So, that is an easy verification you can do that.

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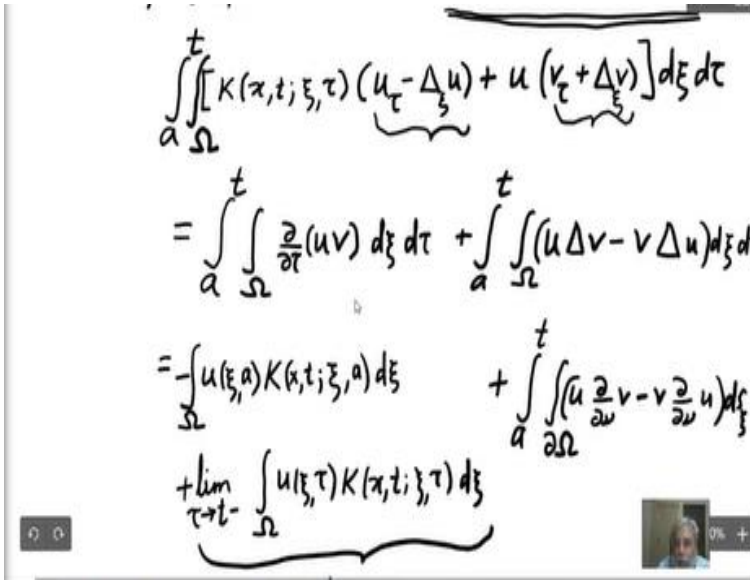

And now let u be a solution of heat equation. So, I take that again as a function of xi and tau, u tau equal to Laplacian with respect to xi variable so that is the stressing of the variables and again very strict to tau variable. And now you integrate similar to the Green’s formula now instead of only Laplacian will bring that heat operator. So, I integrate from a to t integral over omega K into u 2 minus so, this is the instead of Laplace.

And we have this heat operator for and here, similar heat operator for v. So, far heat operating I am just temporarily introducing this notation v of xi tau is K of x, t xi tau because x and t are held fixed. And that so if you take the first part, so this is vu tau + uv tau and that is same as d by d tau of uv and the other part. So, we have u Laplacian v - v Laplacian and u. So, I have now omitted that xi variable. So, but you should keep that in mind.

So, it is only with respect to xi, and this one now we said differentiation with respect to tau and also integration with respect to tau. So, if just evaluation of this for product uv at the end points at this end point there is no trouble here, so this is just minus u xi, a K x, t; xi, a dxi but tau equal to t this v, v is nothing but the fundamental solution it has some singularity, tau equal to t. So, that should be interpreted as you take the limit as tau tending to t from below of this function u xi, tau and K x, t; xi, tau dxi.

So, this evaluation at t should be interpreted as taking limit as tau tends to t because tau, equal to t is a singularity for K, look at here there is a singularity.

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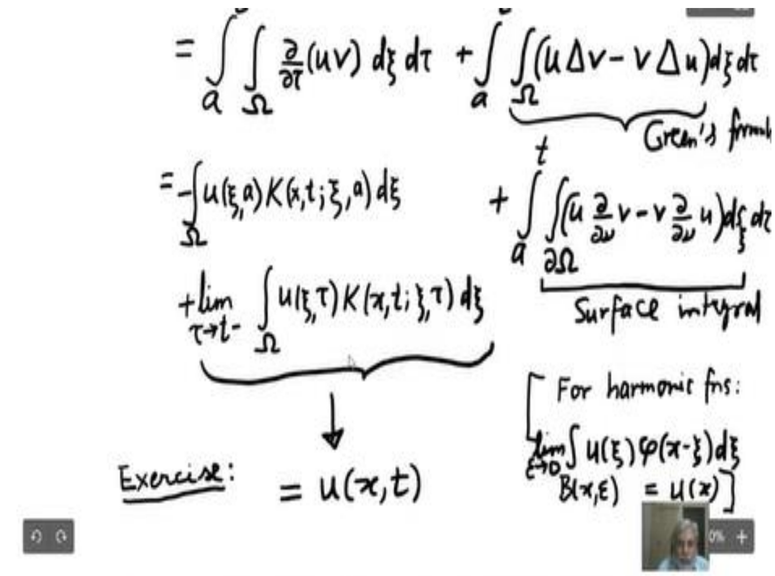

And now that is a good exercise in the analysis. Just show that this limit is precisely u x, t. So, this is for harmonic functions if you recall for harmonic so integral you take a small interval (()) (22:40) u xi phi x minus xi limit epsilon tends 0 for the fundamental solution of the Laplacian is u x. So, depending on what sign you choose it could be minus but it always gives you that. So, in this similar fashion here.

So, this fundamental solution acts in a similar way at the fundamental solution of the Laplacian and produces precisely u of x, t. It is very similar to that and it is a good exercise in analysis and you should try. And for this one there is absolutely no difficulty so we have just apply Green's formula and convert this volume integral into this surface integral. And what is the left-hand side? Left hand side so you are assuming u is satisfy the heat equation that is 0.

And now just now we verify that this is also 0. So, the left-hand side is 0 and now this one gives you u x, t and you bring these two integrals on the other side and we get this formula for the heat.

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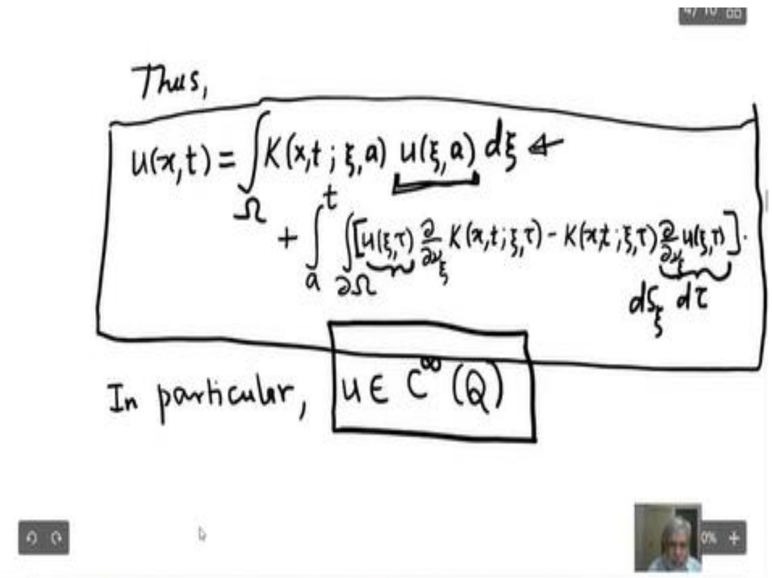

And again, this is only a useful formula but it should not be thought of as a formula for the solution of the heat equation because this part is, this first part is okay, because this we are only giving the condition on the line t equal to a so that is initial condition. But if you look at the boundary it is demanding both the value of the function as well as its normal derivative and we cannot prescribe both of them simultaneously only one of them can prescribe.

So, this is not a solution formula but if you use a solution of the heat equation you satisfy this formula. So, in particular we can use this again to show that. So, just like the free variable K that is the whole the domain is whole of R n. So, we saw that K is infinite implies use infinity and the same qualitative behaviour continues to hold even for bounded domain. So, if you use u is a solution of the heat equation in a bonded domain automatically u is a C infinity function.

So, that is these such results are called regularity results, and both for the harmonic functions and solutions of the heat equation, we see that they are smooth function they are seems to be smooth functions and such is not the case with solution of the wave equation for example that will see later. And with this thing now we will continue to study some more qualitative properties of the solutions of the heat equation.

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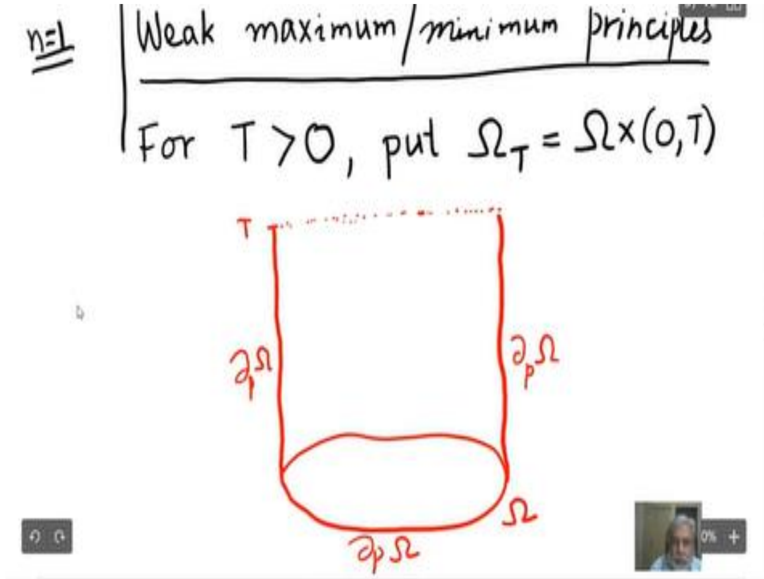

And the first thing we discuss is weak maximum and minimum principles. So, this was again done for case n = 1 in the first part of this course. And more or less the same proof continues to hold there is absolutely no difference. So, again, omega is a bounded domain with smooth boundary and we take some fix T positive and consider this cylinder because this is cylinder. So, its base is omega and height is this T.

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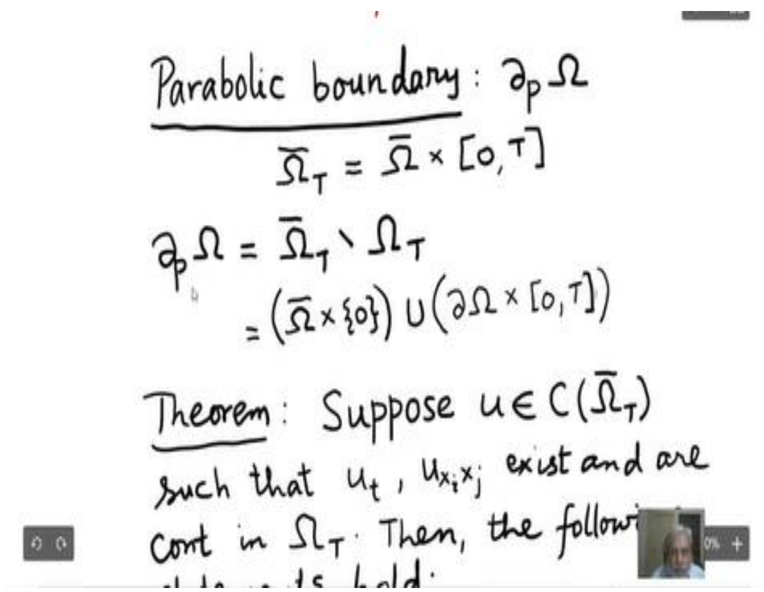

And with respect to the heat equation so there is some part of this boundary so if you consider this closure of omega T this is the one. And some part of this boundary is called parabolic boundary and parabolic boundary is defined by this del P, P for parabolic del P omega is omega

T bar minus omega T and in set theoretic notation. So, you avoid this line T equal to T that part and you consider this part this part and the bottom. You leave the top of the cylinder.

So, that you avoid and you consider the other three sides of the cylinder and that is called parabolic point.

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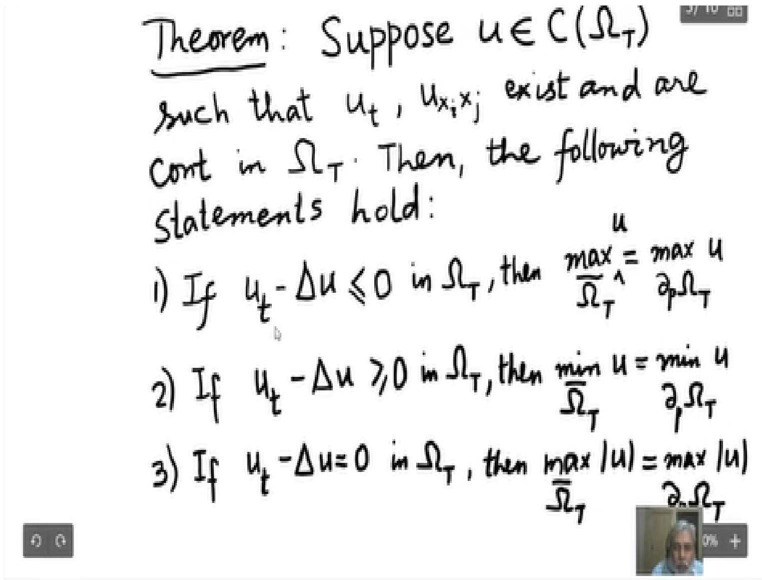

And the result is suppose u is a continuous function in omega T bar and having this existence of u sub t and second derivatives and they are all continuous in omega T, then the following statements. So, if u t minus Laplacian u with less than or equal to 0 in omega T and then the maximum of u in omega T bar is same as the maximum of u in on the parabolic boundary parabolic and similarly, if u t minus Laplacian u is bigger than or equal to 0 in omega T.

Then the minimum of u in the whole omega T bar is same as minimum of u over the parabolic boundary and if you satisfy the heat equation in omega T then the maximum of module. So, both because then this will be equivalent to both less than or equal to 0 and bigger than equal to 0. So, the both the maximum and minimum of u in omega T bar are same as maximum and minimum over parabolic boundary and that is stated as so you can just take more.

So, the small u contains both maximum and so these are called weak maximum and minimum principles. So, the importance of this statement is the maximum of u certainly occurs on the parabolic boundary, but it does not rule out the occurrence of maximum in the interior that is why it is weak. The strong maximum principle concerns what happens if the maximum occurs at an interior point if you compare this for example with harmonic functions.

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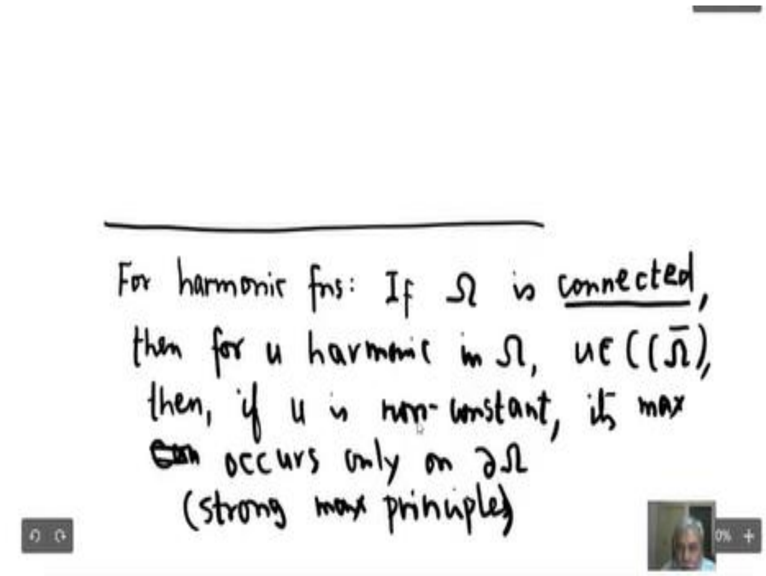

So, harmonic functions also we saw that let me just write it somewhere here. So, for harmonic functions so if omega is connected so that is then for u harmonic omega and of course what we have continuity of boundary then if u is non-constant its maximum can occurs only on so, this is the strong maximum pressure. So, when omega is connected this, if omega is connected a harmonic function cannot have a maximum in the interior unless it is a constant function.

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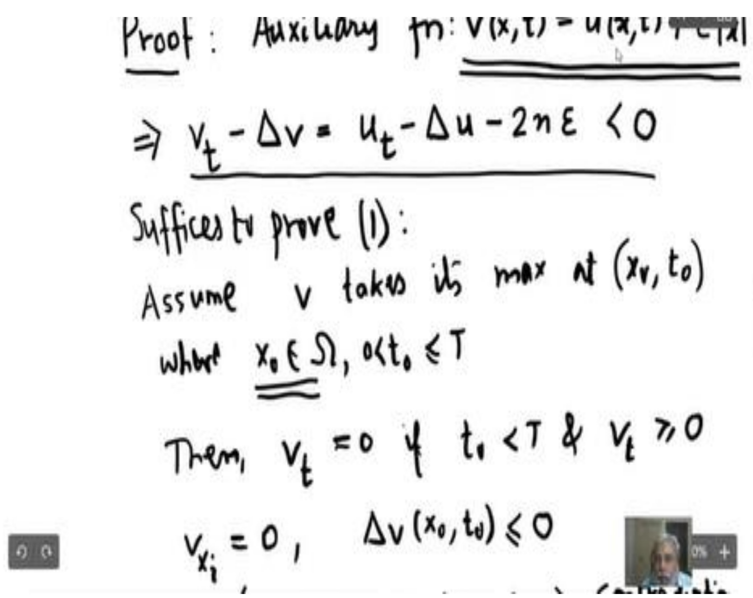

So, similar statements are true even for the heat equation but these are more technically involved and involve construction of special solutions and that will avoid this first part. So, and the proof of this theorem is very, very simple. So, it is just follows from taking this auxiliary function and even let me remark, so it suffices to prove the first part. For the second part follows if you replace u by minus u and third part follows if we replace, I take u n minus u both.

So, suffice is to prove one and for that you consider this auxiliary function and you see that this is. So, now assume v takes its maximum at x 0 t 0 where x 0 is in omega and t 0 we want to rule out you want and t but suppose we take that.

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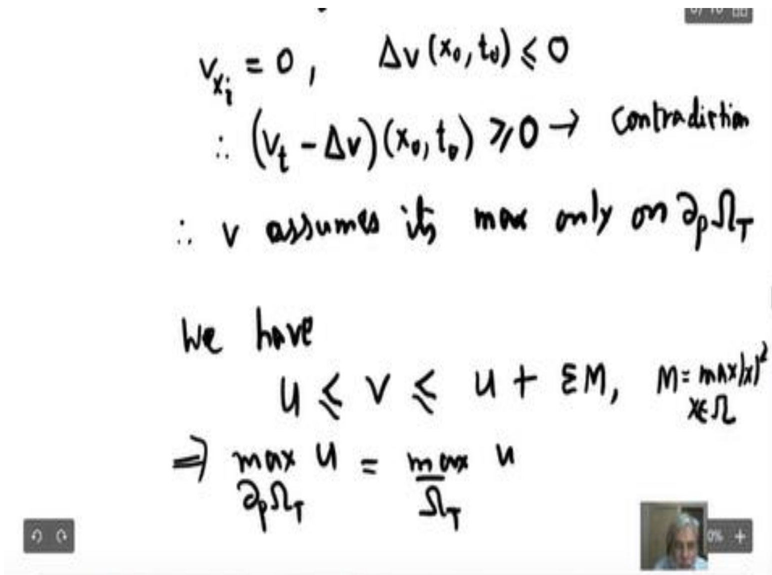

Then this v of t is always either 0 or it can only be positive, 0 if t 0 is typically less than T and v t since this is an interior point, so we get this v i x i are all 0 and the hessian, let me just write that Laplacian v i guess 0 t 0. Since it is a maximum so therefore v t minus Laplacian v x 0 t 0 and this is a contradiction to this. So, we cannot assume maximum in the interior, so therefore we can assume it is maximum.

So, to conclude the proof just you see that so we have put v = u + epsilon norm x square. So, we have v less than or equal to u and this is of course v, this we have v is equal to u plus Laplace squares so u is always less than or equal to and this is less than equal to u plus some epsilon let me call it so M is the maximum of mod x square x in omega and from this we can easily conclude that. I stop here and we will continue from this next time.