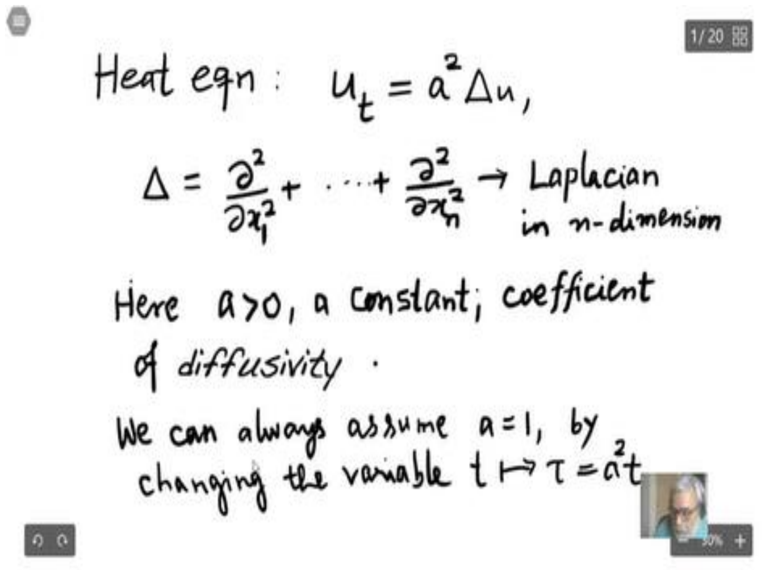
First Course on Partial Differential Equations- II
Prof. A. K. Nandakumaran
Department of Mathematics
Indian Institute of Science, Bengaluru
and
Prof. P. S. Datti
Former Faculty, TIFR-CAM, Bengaluru

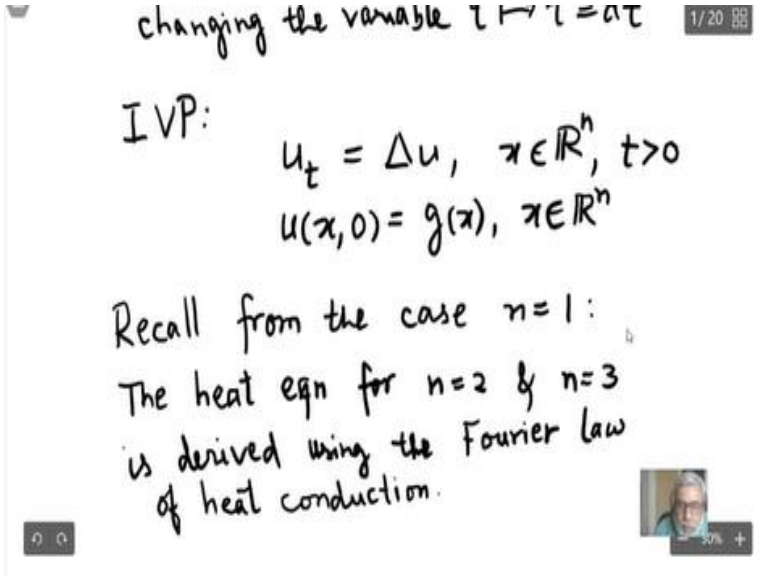
Lecture - 25
W5L5 Heat Equation 1

(Refer Slide Time: 00:47)



Hello everyone, in this lecture we analyse the heat equation in more than one space variable. This is given by u sub t is equal to a square Laplacian u, where Laplacian is given by so, this is Laplacian so here a is a positive constant, a constant and this called coefficient of diffusivity. So, this depends on the material where this flow of heat or temperature we study. As far as mathematics is concerned, so we can always assume a = 1 by changing the variable t to tau which is a square x t.

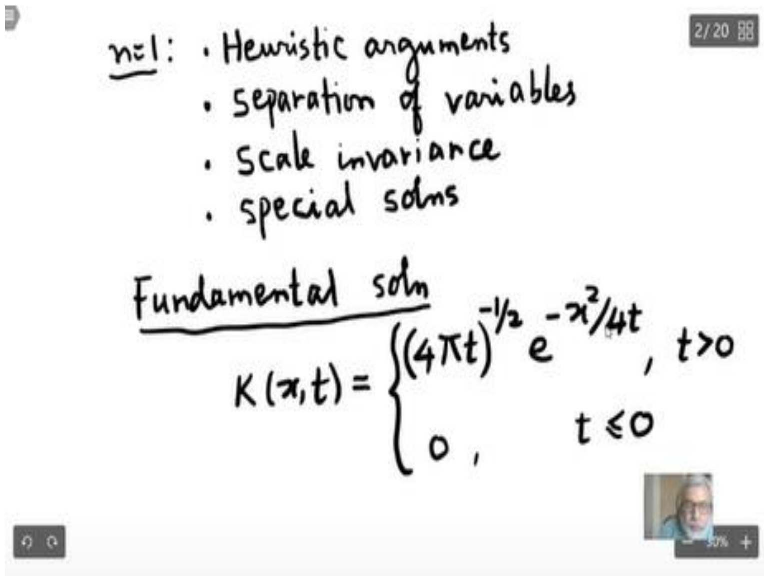
(Refer Slide Time: 03:57)



So, we first concentrate on the initial value problem, IVP u t is equal to Laplacian u, so x E in R n at t positive with an initial condition given by the function g of x, x E R n. So, to a large extent there is no difference between the analysis of this initial value problem for a general n with that of n = 1. So, recall from the case n = 1, so this equation was derived. So, you please go through the portion from the first part of this PDE course where we studied this case n = 1 in detail.

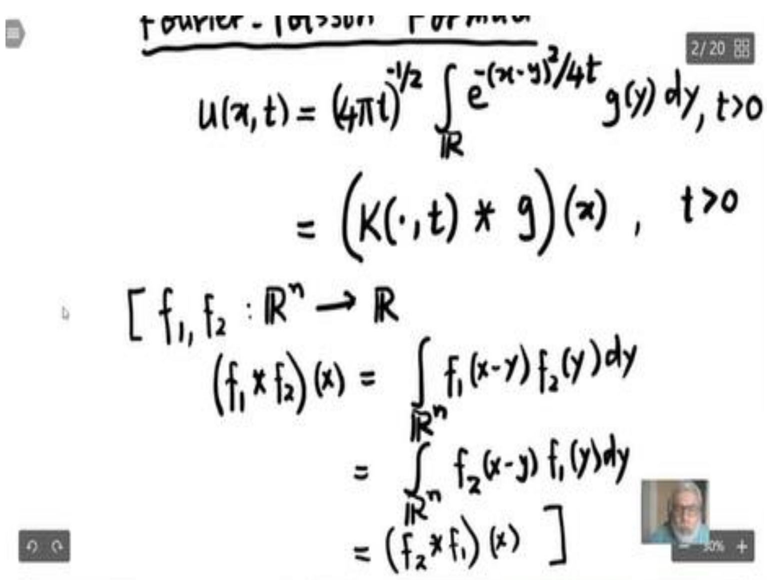
And the same procedure will be applied even for any general (()) (06:08). First of all, the physical situations the equation, the heat equation for the physical dimensions. So, namely n = 2 and n = 3 is derived using the Fourier law heat conduction. So, this was derived in the first part of this course for n = 1 and same procedure if I apply for n = 2 and n = 3 they are physically relevant cases of heat conduction.

(Refer Slide Time: 07:31)



And then for n =1 using some Heuristic arguments and separation of variables and some scale in variance, and then we look for some special solutions. So, by doing all these things for n =1, we derive first of all we obtain the fundamental solution. So, we denote it by K, so it is a function of x and t, so this is given by 4 pi t to the minus half exponential minus x squared by 4 t. So, this is for t positive and 0 for t less than equal to 0.

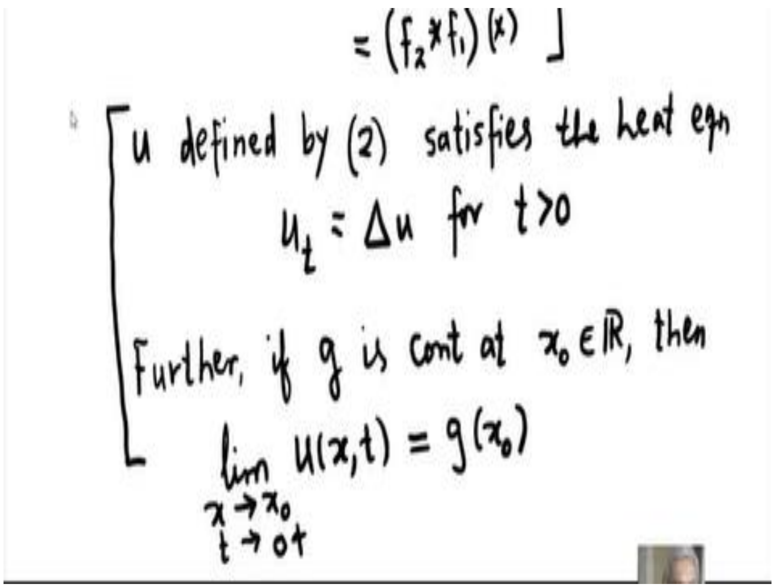
(Refer Slide Time: 09:53)



And then we derived this Fourier Poisson formula, so u of x, t = 4 pi t - half integral R e to the - x - y square by 4t and the initial condition g y dy. So, this can be written as, so K so as a function of x, I put a dot here and t. So, this is for t positive star of x. So, convolution of this fundamental solution of heat equation or heat operator and we also call it heat kernel because that appears as kernel in this integral operator.

So, just recall so if f and g are let me use f 1 and f 2 are functions from in general R n to either the real value or complex value does not matter then we define its convolution as this integral you know x - y which is same as by change of variable. So, this is your 2 of x - y of 1 y dy provided the integral is finite. And that is again f 2 convolution f 1. So, now, so this was the case n = 1 and so what we really showed was so this let me call it my sum equation, so this is just 1 this is 2.

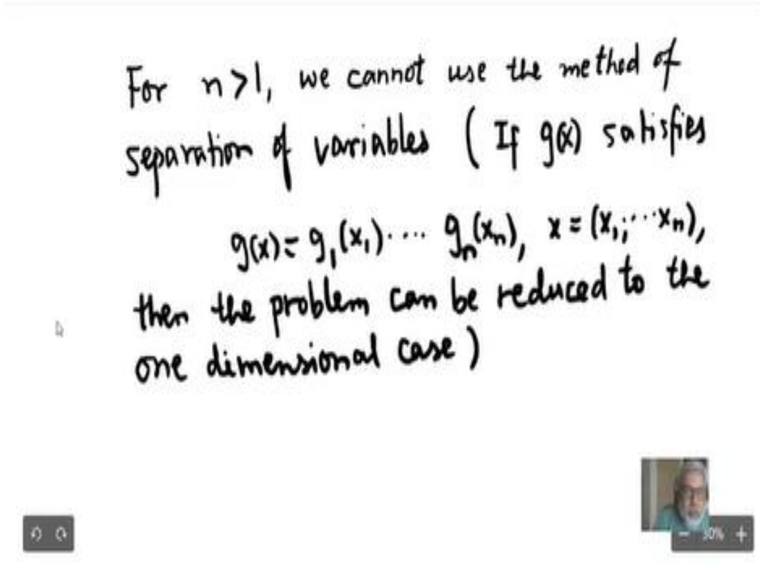
(Refer Slide Time: 13:55)



So, u defined by 2 satisfies the heat equation u t equal Laplacian u for t positive and satisfy the initial condition in the limiting sense in the sense of limit. Further, if g is continuous at x 0, so, this is very local result, at x 0 belonging to R then limit u of x, t x tends to x 0 and t tends to 0 plus is equal to g of x. These verifications depend on some crucial properties of the heat kernel the fundamental solution that we now list it was explained in detail in the first point.

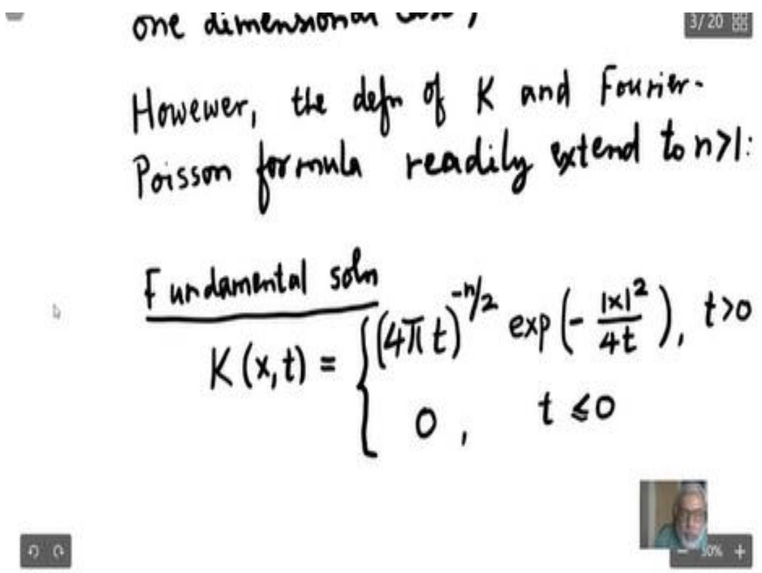
But I will just recall what are those important properties. So, now would like to do a similar analysis for general n. So, obviously we cannot use this separation of variables kind of thing, but however if you look at the formula for the heat kernel and Fourier Poisson formula these two formulas extend readily to higher damages.

(Refer Slide Time: 17:21)



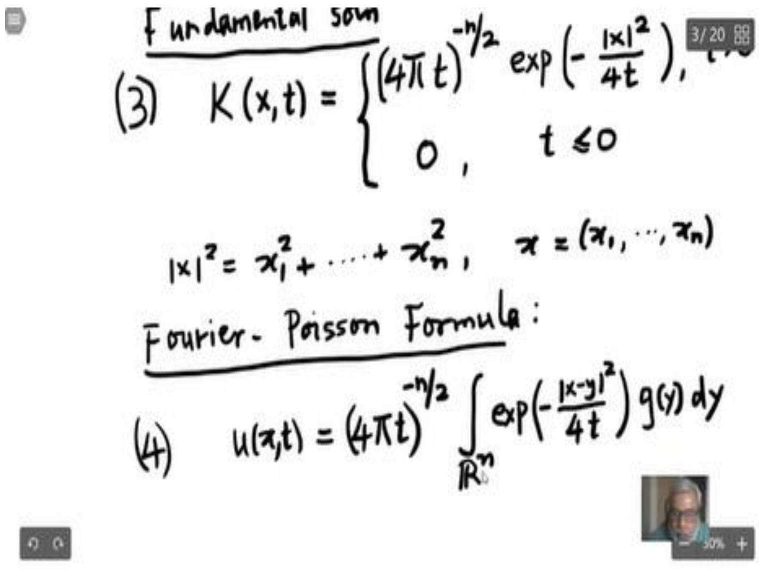
So, for n bigger than 1 I cannot use the method of separation variables. There is one case I will just remark that, so if the initial condition initial function satisfies that g of x is equal to product of n functions of single variable. So, here x is x n, then the problem can be reduced to problem is initial value problem one. But this is very, very specific. So, we cannot use this for a general initial function.

(Refer Slide Time: 19:48)



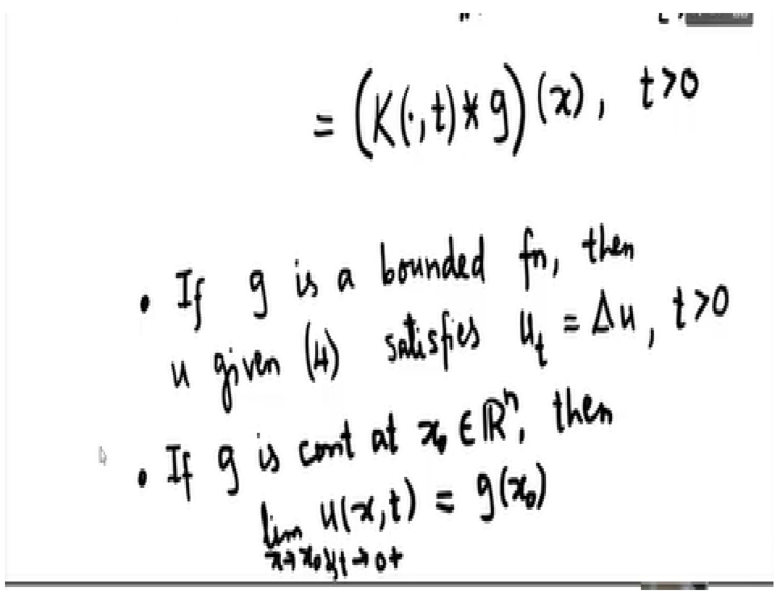
So, however the definition of K and Fourier Poisson formula, readily extend to n lesser 1. So, now we define it so fundamental solution. So, now K is a function of x and t and x varies in R n. So, this is 4 pi t, so this factor remains same but now the exponent changes that is n by 2, the exponential minus normal square by 4. So, again this is for t positive and 0 and t less than or equal to 0.

(Refer Slide Time: 22:02)



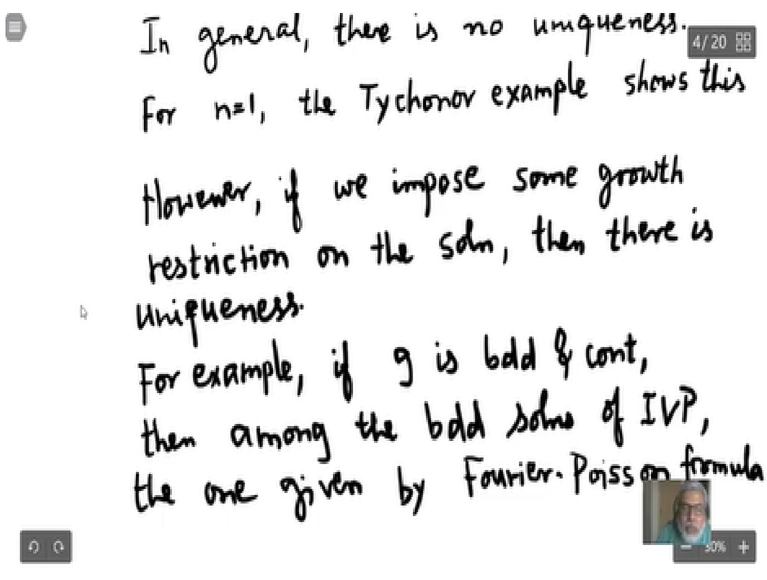
So, here norm x square with the standard norm Euclidean norm. So, write this x and the Fourier Poisson formula also need. So, again this is 4, so u x, t so let me write one scene and now the integral is over R n. So, that is the only difference, so it is a multi-dimensional integral so exponential - x - y squared by 4t, g of y again g initial function dy. So, this is multi-dimensional integral, though I have written only one integral here it is integral over R n.

(Refer Slide Time: 24:04)



And this is again for t positive. So, this again we can write it as convolution of K t star g x. The verification that u is indeed a solution of the heat equation at least for some good function g is exactly similar to the case n = 1. So, let me just mention that as so very so if g is a bounded function an R n, then u given by 4 satisfies the heat equation for t positive. And again, as in the one-dimensional case, if g is continuous at x 0 in R n, then limit of u x, t as x tends to x 0 and t tending to g of x 0.

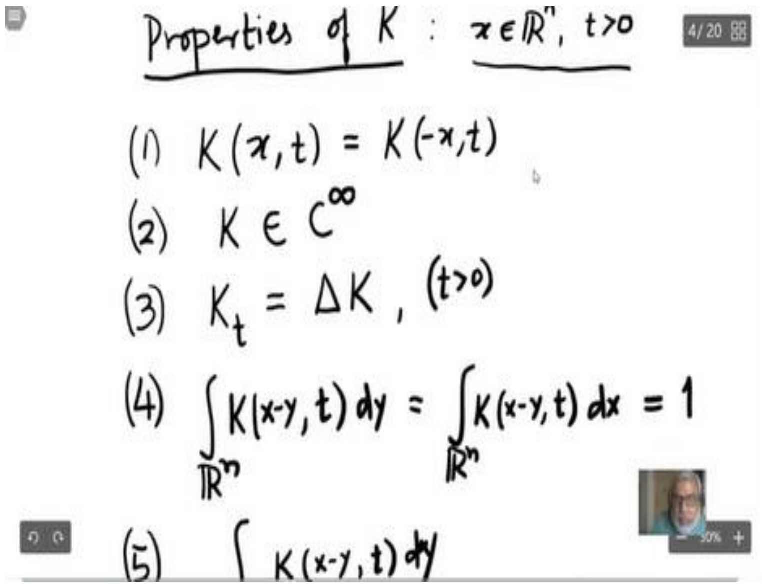
(Refer Slide Time: 26:49)



Since the initial value problem, we are studying is a characteristic initial value problem, so we cannot expect so in general there is no uniqueness. So, for the case of n =1, so we exhibited for n =1 the Tychonor example show this. So, however, if we put some growth restriction on the solution, however if we impose some growth restriction on the solution then there is uniqueness. So, for example if the initial condition g is bounded and continuous.

Then among the bounded solutions, so this is the restriction now among the bounded solutions of the initial value problem, the one given by Fourier Poisson formula is the only solution.

(Refer Slide Time: 30:43)

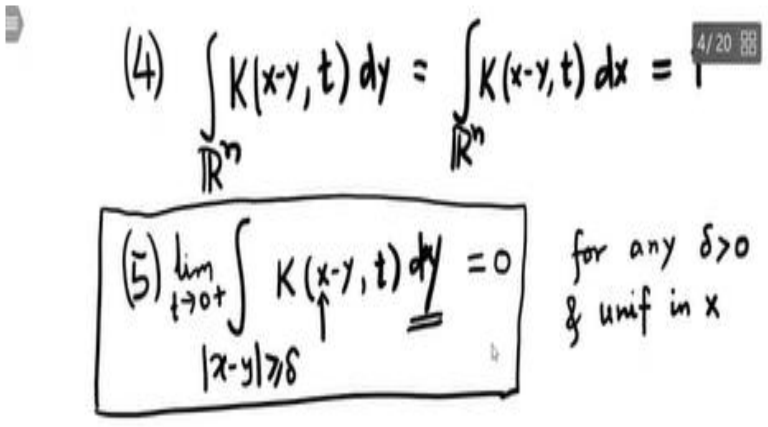


The verification of all these properties they depend on these crucial properties of the fundamental solution. So, I just state them so properties are which are used in deducing all these statements. So, let me just mention them, so K is symmetric. So, this is x, t = K - x. So, let me just restrict myself to x bigger than R n and t positive for t less than or equal to 0 it is any of 0 function. So, in this region, this K is C input function of x.

So it is infinitely differentiable both with respect to x and t variable when you restrict t to only positive values here satisfies the heat. So, let me stress again this. So, I have stated that and this integral of R n, over R n K of x – y, t dy. So, any of from the first property it is symmetric with respect to the x variable. So, I can change that so K of x – y, t dx, so this is 1. So, in this case I am integrating with respect to y variable so this integral is 1 for all x in R n.

And in the second integral I am integrating with respect to x variable so this integral is 1 for all y in R n and then crucial property this one. So, if you integrate not on the whole of R n but you just leave out a ball. So, this K x – y, dy. So, this is usually we neglected, but since K is given by an exponential so K is a positive function.

(Refer Slide Time: 35:14)



So, you just integrate this fundamental solution not on the whole space R n but you leave out a small bond so you integrate over the exterior of your bond and then this one limit t tends to 0 +0 for any delta posed. And you are integrating with respect to y variable and there is a x here so this convergence is also and uniformly in x. So, that limits usually when you use epsilon delta definition to prove this thing.

That given any epsilon that delta and this delta are different of course. This is any given delta that does not depend on x. So, that is the infirmity with respect to x so it does not depend on x. So, this is a very crucial property that is used in, of course all properties are used but this is quite a crucial one. So, if you see 4 and 5 the forces it is the integral is 1 for all t positive. This property 5; so that if I integrate giving a bond then this in the limit this integral is 0.

So, that shows that this integral is concentrated only around x. So, it that can again be seen from the formula for K which is exponential. With that I will stop here and we continue from this in the next class. Thank you.