First Course on Partial Differential Equations- II
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Lecture - 24
W5L4 Eigenvalue Problem 2

Hello everyone, welcome back. So, in the previous class we are discussing eigen value problem of the Laplace operator.

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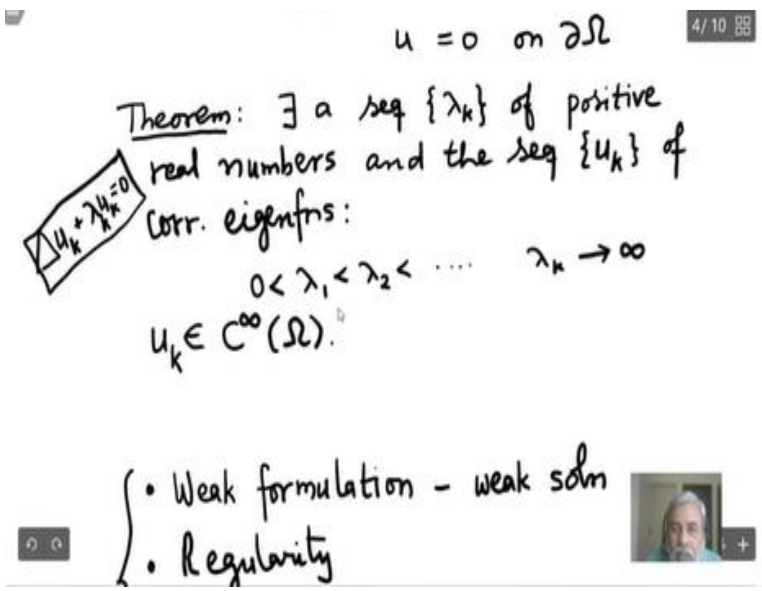

And in a bonded domain we have seen at least in some examples that there are only discrete set of eigenvalues. There is so though we are not prove that but that is the result. So, there exists a sequence of positive real numbers and corresponding eigen functions. And in the end, I remarked that the story is different in unbounded domains. So, now we will consider the case of unbounded domains and I want to show you a uniqueness rejecter and if possible, try to prove it.

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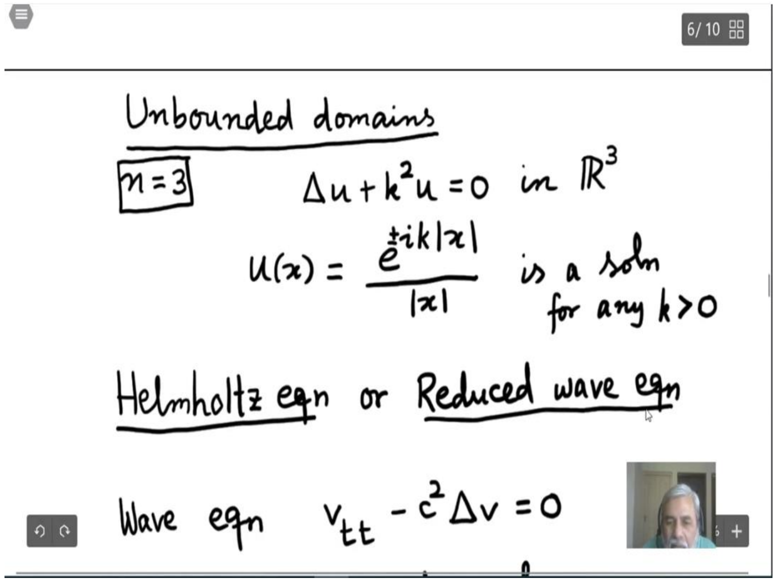

So, now we consider unbounded domains so, more talk or discussion is in R 3 same equal to 3. So, first consider this equation Laplacian u + k squared = 0 in R 3 and k is any real number so, we can take it. So, since you use k squared here, so, can you take k positive. So, you can just again look for the radial solutions and we will find that this u x given by e to the + or - ik mod x by mod x is your solution.

Again, this is true for any k possible. So, the already we see a big difference that it is no longer only a discrete set of positive real numbers, but now the entire positive real axis. So, here so, we can take real and imaginary parts. So, the just for convenience, and return this e to b + or - ik mod x, and this equation is called Helmholtz equation. And if you go for n different from three. So, instead of this sine function, we end up in Bessel functions and that is a little messy.

So, our discussion only concentrates on this n = 3 and this equation is called Helmholtz equation. It is also called a reduced wave equation for the following reason.

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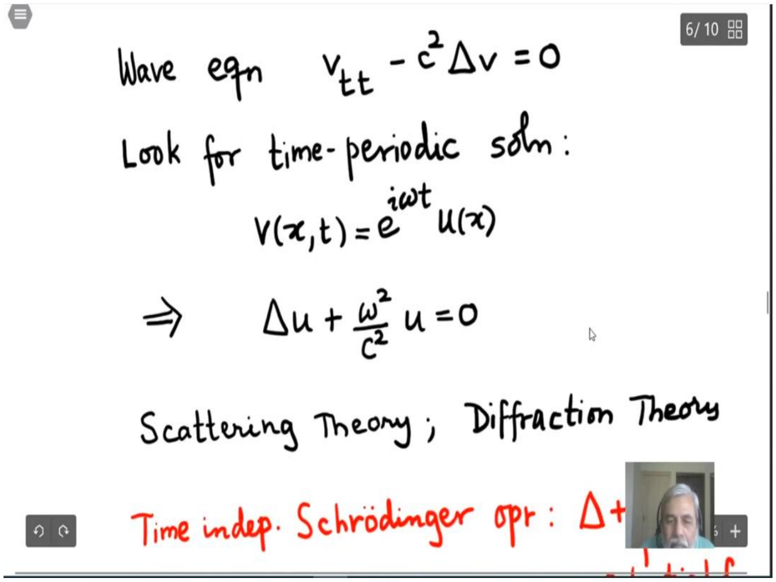

So, now, you can see that the wave equation so, v tt - c squared Laplacian v and look for a solution which are time periodic functions. That is the look for solution of the form v x, t = e to the i omega t u x working, and then simple computation show that this u satisfies the Helmholtz equation with get squared replaced by this omega squared by c squared is omega d period of that period are occupied by omega d the time period of the solution.

And this Helmholtz equation plays an important role in scattering theory and diffraction theory can so if you look at some good books on equations of mathematical physics or even theoretical physics where the scattering theory and diffraction theory are done. So, you will see this mention of Helmholtz equation and disconnection to wave equations and other results.

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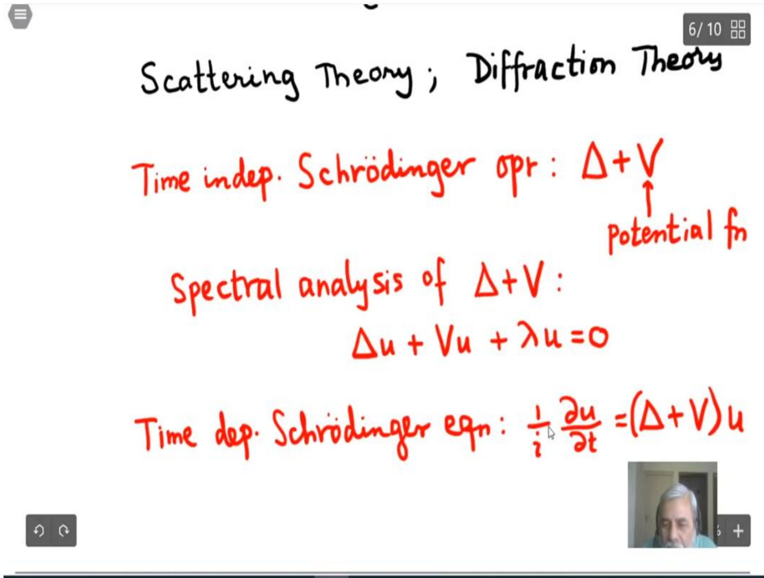

So, as a remark also mentioned that so, this operator Laplacian + V. So, V is a given function called potential function and that is referred to as time independent Schrodinger operator. And spectral analysis namely if you look for the Eigen values are non trivial solutions, look at a spectral lens of this operator and that plays an important role in time dependent Schrodinger equation. So, namely one by i del u by delta t + Laplacian V times u.

So, when we 0 is referred to as Schrodinger equation in free space so, that is also come across the spectral analysis of Laplacian in that case. So, this equation has applications in different areas so our concern here is, so it is very similar to the Laplace operator except this term k squared. So, it says many properties of the harmonic functions, sign without so since it is a constant coefficient operator, it has fundamental solution.

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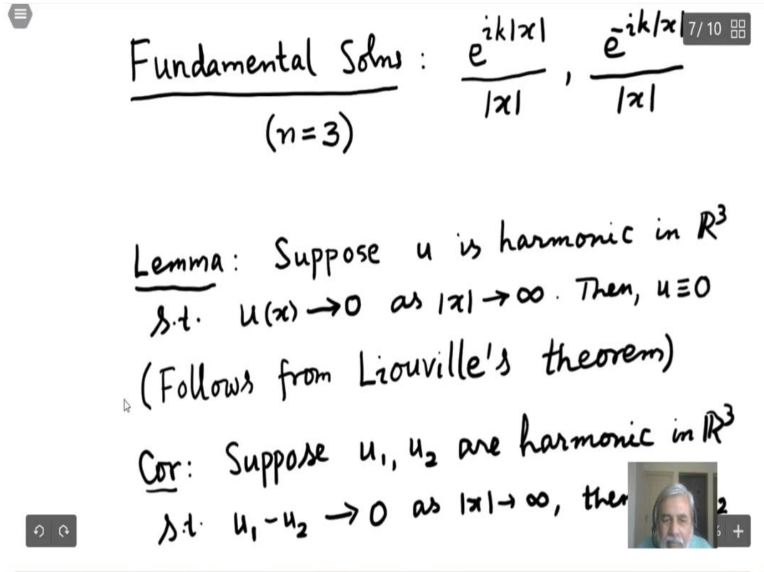

In fact, there are two fundamental solutions. So, you can check that again remember n = 3. Otherwise, you will end up in Bessel functions. So, these are the fundamental sources of the Helmholtz operator. So, they namely Laplacian + k squared. And recall one unique result for harmonic functions, this way already done. Suppose you is harmonic in R 3 the whole space and u x tends to 0, as more tends to infinity, then u is identically 0.

And this follows from Liouville's theorem. So, once we know that u x tends to 0 at mod x to infinity, so, u becomes bounded. And by Liouville's theorem, it is a constant. And since you go to 0, that contract must be zero that is just one line proof using Liouville's theorem. And that has again interesting corollary some kind of uniqueness of harmonic functions. So, u 1 and u 2 are harmonic in R 3 and their difference goes to 0 as mod x goes to infinity, then again u 1 and u 2 are same.

So, that kind of uniqueness results, we have for the harmonic functions. And such a uniqueness result first for the Helmholtz equation. So, if you just take this Laplacian + k square u = 0.

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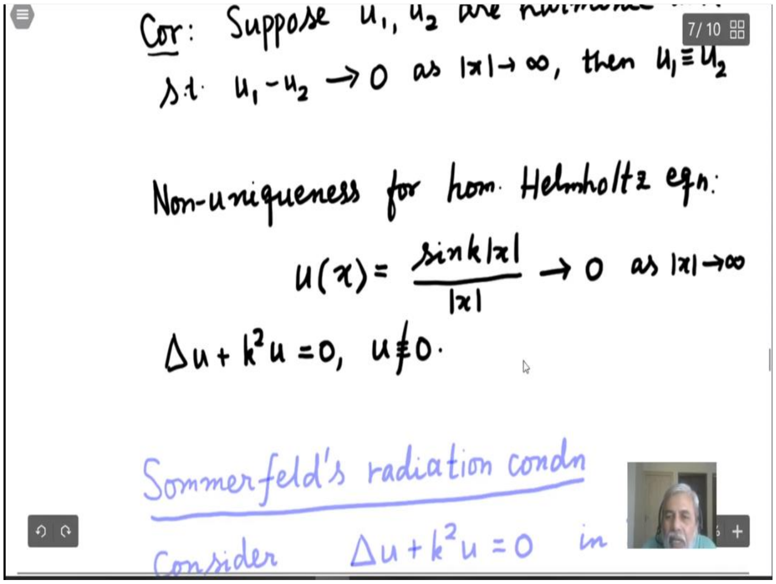

So, here we have a solution, again, remember n = 3. So, consider this function u of x = sin k mod x. So, I am taking the imaginary part here that is sine a mod x by mod x and that goes to 0. But this u is not identically 0. So, we have a 0 solution and a non trivial solution and both goes to 0. So, since uniqueness is an important result to one, so, what additional conditions one should impose on the solution of the Helmholtz equation in order to get uniqueness.

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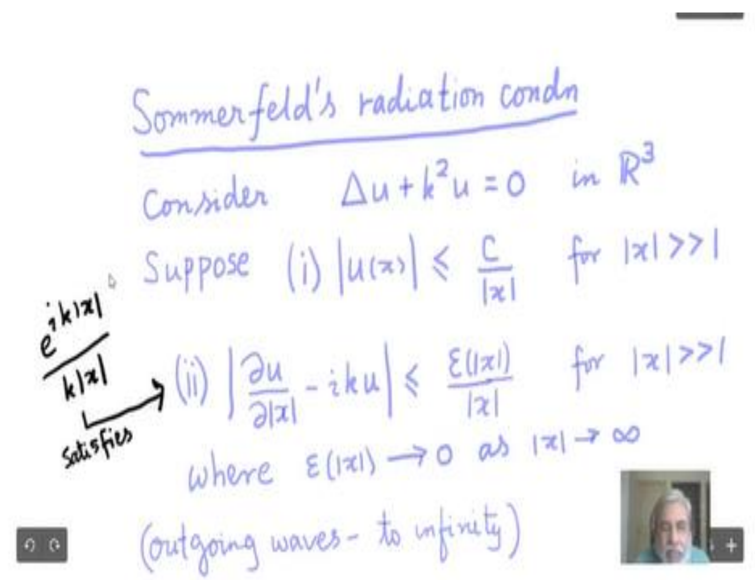

One such condition is provided by the, so called Sommerfeld's radiation condition, also simply called radiation condition. So, let u be a solution of the Helmholtz equation and in addition assume that u is bounded by this C mod x for large x or the key symbol for large mod x. And this additional condition so, this is actually the radiation condition. So, then del u by del mod x, so, you take the derivative with respect to mod x here - ik u that k is coming from the equation.

And this in modulus is less than or equal some function of mod x divided by mod x again for large mod x where this numerator this function epsilon mod x tends to 0, as mod x tends to infinity. and this is referred to as outgoing condition, out going to infinity. So, this terminology comes from the scattering theory and we may as well replace the second condition in the second condition we take instead of minus ik u, it can also take plus ik u.

And that is referred to as incoming from infinity. So, if u is any solution of the Helmholtz equation satisfying these two conditions.

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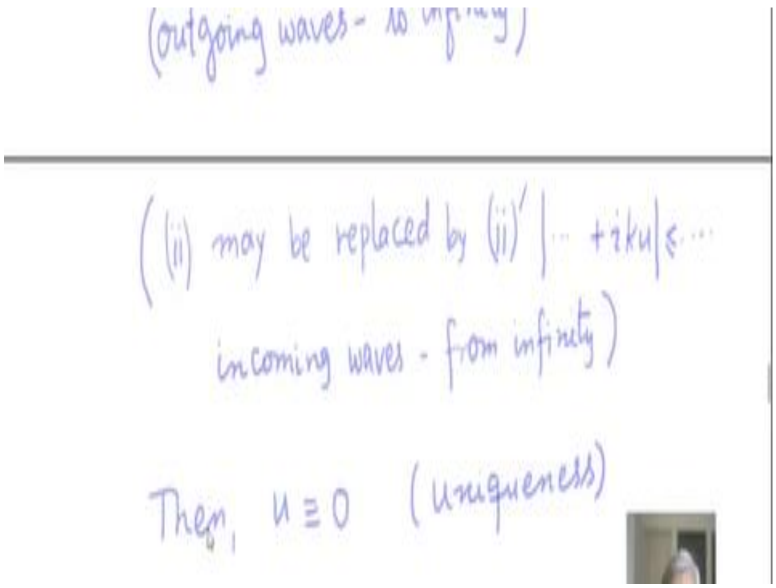

Then the conclusion is u is identically going to 0 and that is uniqueness. So, you anyhow you identically 0 is already a solution with these additional conditions that you is identically 0 and uniqueness comes. One can show that from this condition comes, that is again easy, it is not very difficult to see. So, if you take this e to the ik mod x by mod x that we know already a solution so can multiply by j mod x.

And if you for this function, if you do this computation, you see that this satisfies this condition whereas if we take e to the - ik x. So, it does not allow this condition does not allow that kind of source. So, there is a distinct difference between the + sign and - sign. And this one we can use it for if we will replace by + ik you can make it e to the - ik mod x.

Actually, one can show that if k is positive, this first condition is redundant. So, the second condition automatically implies the first condition. So, I am not going to do that. So, we are going to assume both the conditions and try to prove this uniqueness reason namely u is identically go to 0. So, we are here we have taken the entire space of R 3.

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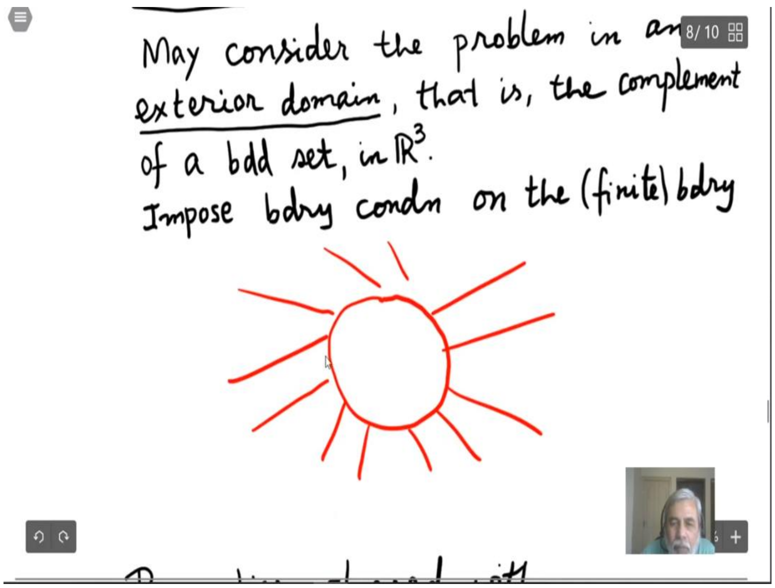

So, that can be replaced by any exterior domain. And the exterior domain by definition is the complement of your bounded set in R 3. For example, the compliment of a ball that is one and one can impose both the condition on the finite quantum for example, if we take this complement of the a bond in R 3. We can put the condition and this radiation conditions then uniqueness one force for such domains.

The only thing is that domain should contain infinity, so it should extend to infinity. That is one observation so we can handle some boundary value problems but in exterior domains.

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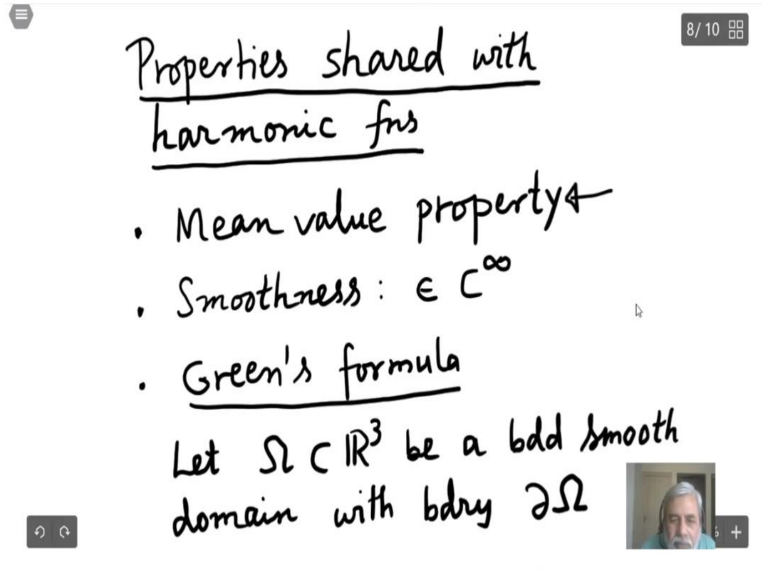

That is one of the reason, and let me list some properties which are shared by the solution of Helmholtz equations with the harmonic functions. So, this solution of the Helmholtz equation also satisfied a mean value property. So, this will discuss in assignments, so you should do as many problems in the assignment as possible, so, we will discuss that thing assignments. What does that mean? In fact, there are a couple of set properties.

We will list them in the assignments and the smoothness is another thing that any solution even if you start with a weak solution that up the Helmholtz equation is automatically his infinity function tends to the regularity of the Laplace operator. So, it is more generally true for constant coefficient elliptic operators. So, this because of that Laplace operator this mode as follows and now, we derive a kind of greens formula this we already derived for harmony functions.

So, for that purpose so, let omega be a boundary smooth domain with boundary again in R 3. So, using these fundamental solutions, e to the ik mod x + or – ik mod x, we can easily derive this Green's formula.

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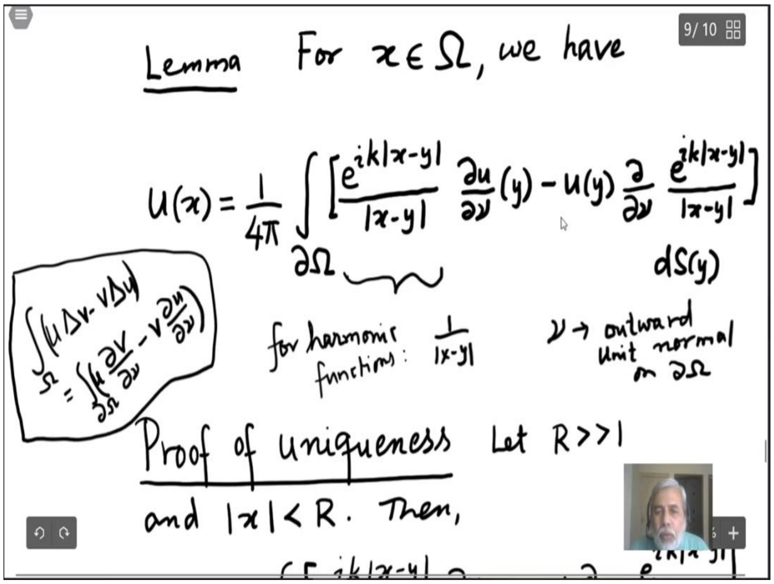

So, here it is. So, u of x is 1 by 4 pi. So, that fundamental solution is coming. So, del u by del nu. So, this integral is only surface integral integrated on the boundary of omega. So, very similar to the one we obtained for harmonic functions. So, for harmonic functions, so, this was only one by mod x. Some constants that is all, there is the only difference. So, this new is as usual an awkward normal to the boundary awkward unit.

dS y the surface media del omega. Exactly the same arguments, so, just now replace this fundamental solution for the Laplace operator by the fundamental solution of the and use this use usual thing this is our Green's identity. So, this omega this is just del u by del nu minus may be there is sign change, but that is one. So, you will use appropriate mean you will get this formula, and that is obviously not a solution formula.

Because this both the value of u and its normal derivative on the surface is required. So that is not a solution formula, but a useful formula. Now using this Greens formula will through this uniqueness. So that is so now you apply the greens formula.

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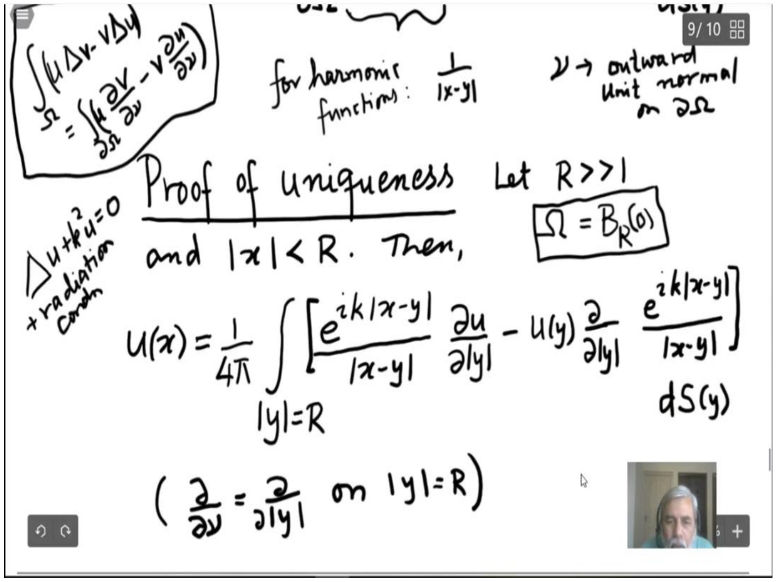

For this omega equal to, you take this, B R 0 take R sufficiently large, a ball centred at origin and radius r. So, using Green's formula so we get now, so proof of uniqueness so you remember that k squared u equal to 0 in R 3 + radiation that is our view. So, by Green’s formula that follow that u of x= 1 by 4 pi. Now integral over this spear mod y = R e to the i k mod x - y divided with mod x-y.

del u by del mod y and the other term, so on since we are on the sphere, the normal derivative is nothing but the derivative in the direction of with respect to mode y. So, that is what we are using here. So, this normal derivative with this integration with respect y. So, this now just concentrate on that. So, this we have to compute.

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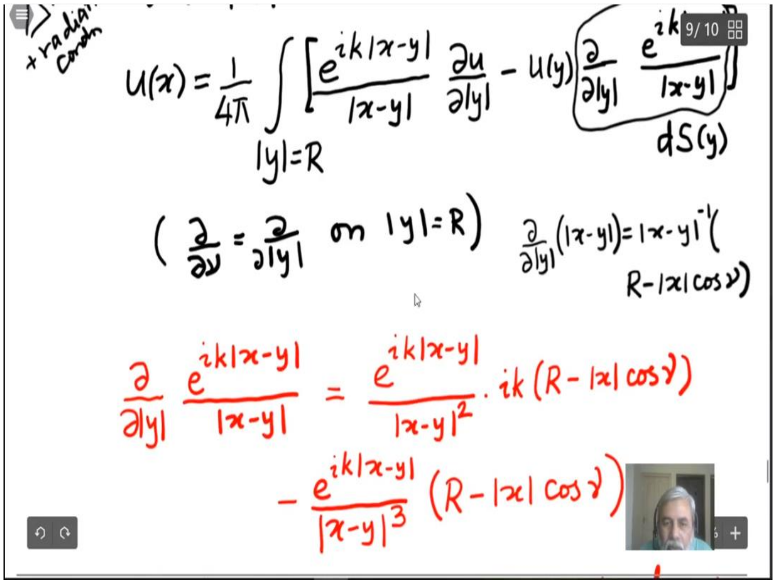

And that requires the computation of, so there is a mod x - y, it is not simply mod y. So, let me just write that, del by mod y some simple calculation is equal mod x -y inverse into R - x cos gamma. Do that thing then the differentiation is straightforward. So, I have in fact written here what are the normal derivatives of this function.

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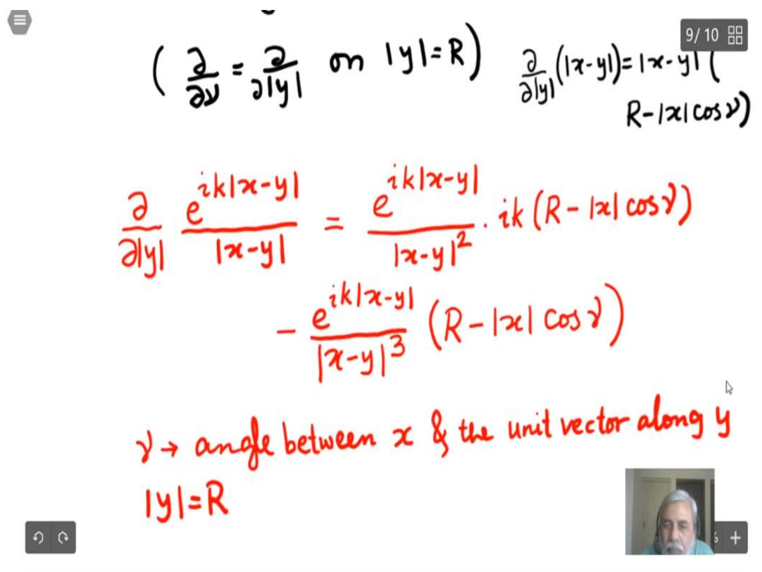

So, you just see that and this gamma is nothing but the angle between this vector x and the unit vector along y. I remember mod y is equal to R so that is already I am using. Otherwise, you get the mod y here. And so, this let me also say that I am already using that mod y is R. So, you do this simple computation, and then you plug in this expression in the integrand.

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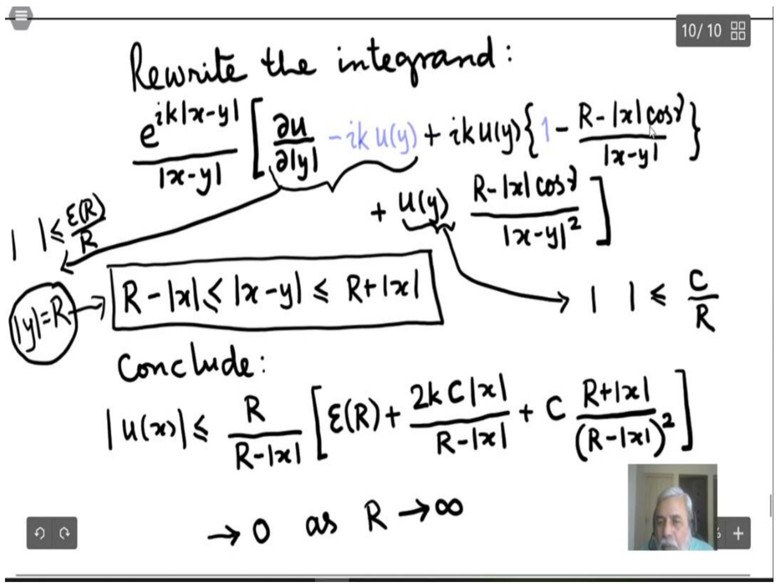

So, let us rewrite this integrand, after plugging in this computation and you also subtract this term, this term is not there in the computation and you we add it, because this is what is the term that appears in the radiation condition. So, you do this computation and you observe this and now it is just use the hypotheses. So, this term for example, in the absolute value, we are writing that epsilon R by R.

That is the radiation condition, and also this u of y, wherever u of y. That is also in absolute value less than = C by R. And those are the conditions given in the uniqueness result. Mode u of x is less than equal to C by mode x and this derivative is less than or equal this term and that is what I am just using. And now you estimate the index 3 is after rewriting this integrand in this fashion, now, you take the absolute value, and this simple inequality.

So, wherever mod x- y appears. So, we can bond it both below and above. And again remember, so, I am using here that mod of y is for R, so, just remember, so, wherever mod y appears, you just replace it by R. And now you plug in this expression for the integrand and you the hypothesis, this estimate and again estimate on the solution u and these are easily it is just cos gamma is bounded no problem. So, this r - x and again this one mod x - y you can use this estimate.

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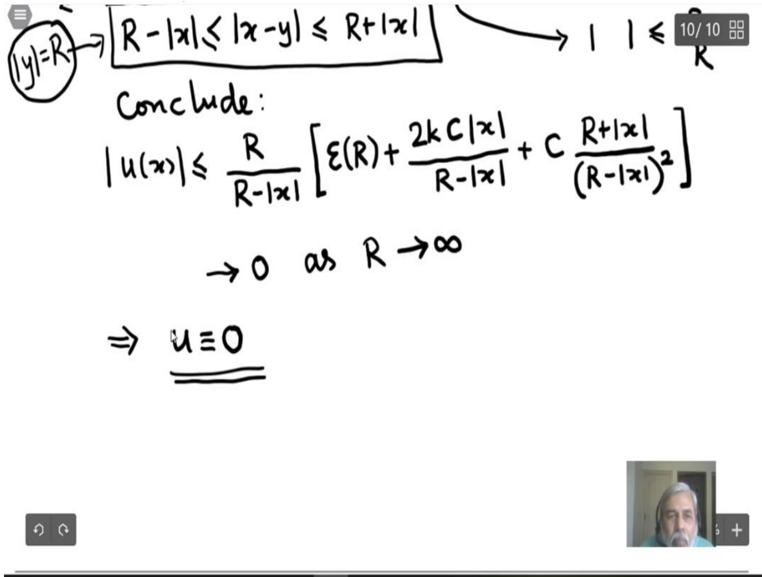

So, in conclusion, what we get is mod of u x is less than or = R by R - mod x in the bracket epsilon R and that is from here and 2k c mod x by R- mod x and x come from there and then the last ones who integrate we also get an R term that is coming from here, that the surface area. So, do these things carefully. And you get this estimate. And now by hypothesis, our epsilon R goes to 0 as opposed to infinity and rest can see the denominator there is any additional power of R.

So, they all go to 0. So, here for example, the second term so, you get an R square, but there is one r in the numerator and that goes to 0 and similarly, the third term, so the right hand side goes to 0. And since x is in R 3 arbitrary point, we conclude that u is identically 0. So, that is an interesting uniqueness result concerning the solution of the Helmholtz Equation. Of course, there are many interesting things one can do with Helmholtz Equation.

But there is not much time to go into those directions. But those of you who are interested to dig more, they can refer some good books on theoretical physics, especially in connection with scattering theory diffraction theory. And so that is, all I wanted to say regarding this Helmholtz equation. So, as you notice, just in these two classes, I have said very little, but I suppose some of you might have got interested in looking at more of this spectral analysis of the Laplace operator.

And in general, at least constant coefficient elliptic operators and they are useful in mainly, situations, as I said, if we want to apply Lazarus theorem of semigroup theory, we do need a complete knowledge of this spectral analysis. Thank you.