First Course on Partial Differential Equations- II

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Lecture - 23
W5L3 Eigen Value Problem 1

Hello everyone, welcome back. In today's lecture, we discuss the eigenvalue problem or the Laplacian.

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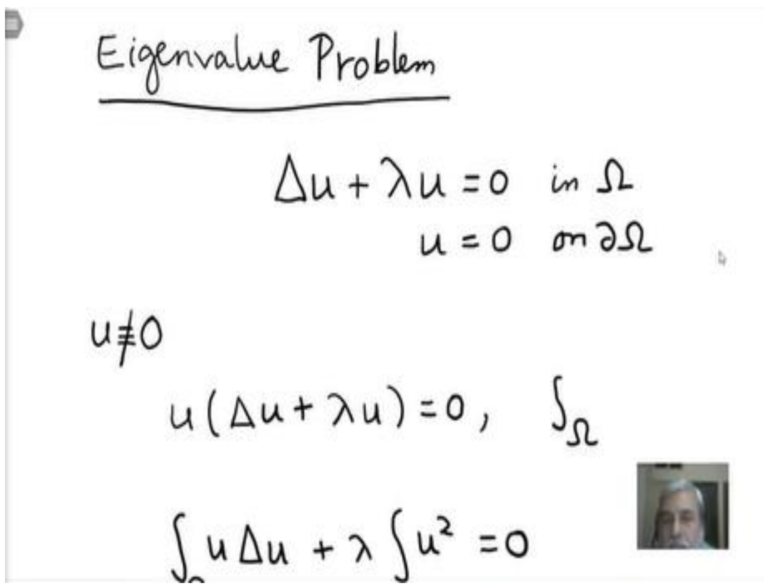

So, it is a small digression from our lecture so far. This is also an important and interesting topic but unfortunately, we cannot provide details of this discussion. So, this is only a brief discussion and there are some interesting problems I just want to indicate. So, consider this second order equation, Laplacian u + lambda u = 0 in omega and u = 0 on del omega. So, again as usual omega is a bounded smooth domain in R n or smooth boundary.

So, one smoothness for the use of divergence theorem and other things. So, this can be viewed as an eigenvalue problem for this operator Laplacian just similar to the eigenvalue problem for a square matrix. So, I would like to find whether this problem has non-trivial solutions. So, u equal to identically 0 is obviously a solution, so we will be looking for non-trivial solutions and whether they differ on lambda or not that we have to jump.

So, before getting further, so let us try to see if there are any restrictions on the values of lambda. Suppose there is a solution which is not identically equal to 0. So, you multiply the given equation by u, that is new Laplacian u + lambda = 0 and you integrate both sides and the domain omega.

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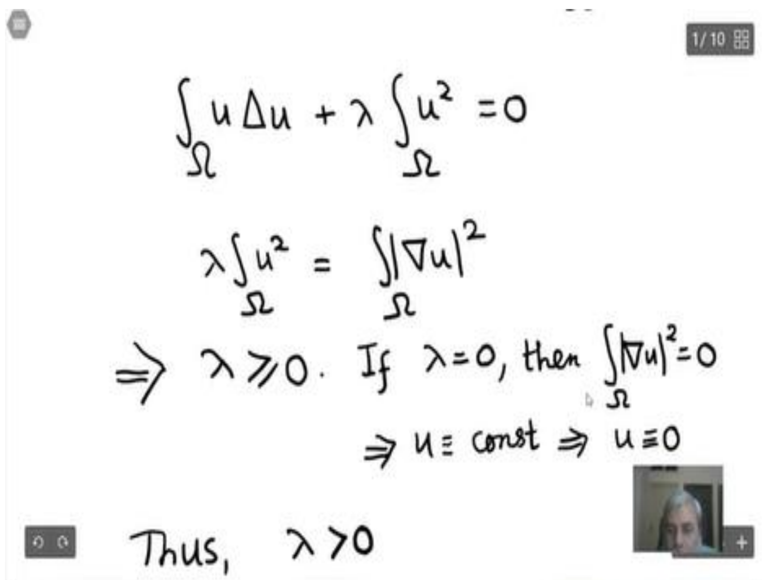

So, then we get integral omega u Laplacian u + lambda, so u into u, u square, so lambda is just a real number and then on the first integral of Laplacian u. So, you apply Green’s formula and that will produce minus integral grad u square and there is no boundary terms as we are assuming u is 0 on the boundary. So, just we get that minus integral grad u square, u take the other side so, we get lambda integral u square or omega equal to integral omega grad u square.

And since u is not identically 0 so, this coefficient of lambda is a positive number and the right-hand side is there a non negative number. So, immediately, get lambda has to be non negative. So, if this given problem has non zero solution then lambda has to be greater than or equal to 0. If I gave lambda equal to 0, so just to plug in this previous line, so we get that integral or omega grad u square is 0 and that implies u is a constant but again since u is on the boundary.

So, we conclude u identically 0 and again that is not interesting. So, if we want a non trivial solution, then the lambda must be positive. So, here we are assuming, not only there is a solution also these integrals are finite and other things. So, just forget about those things for the moment, just assume all these computations are valid. For example, they are valid if u is continuous up to the boundary and this grad u is also continuous up to the boundary.

So, you can assume that there is a solution which is C 2 in omega closer. So, next question is, if lambda is positive is there a solution and is it true for all lambda positive or only certain values of lambda.

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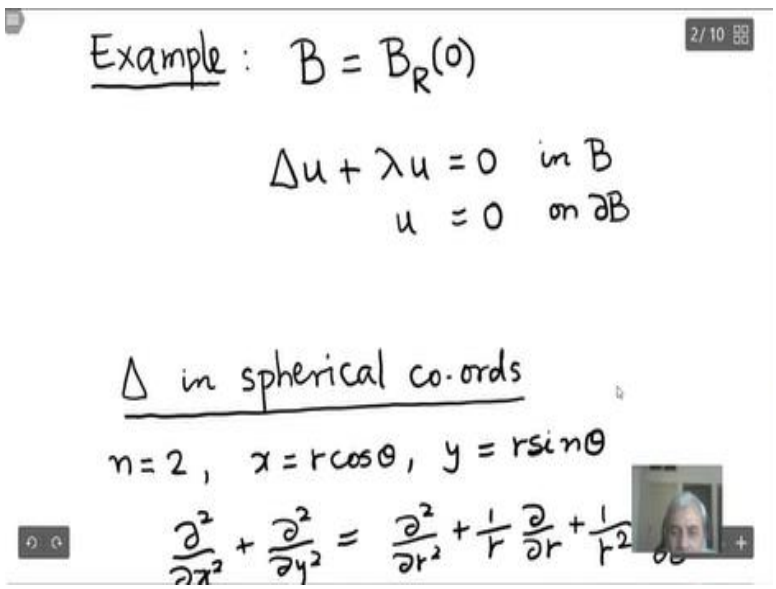

So, at this lecture will not be able to answer this question, so we look for some examples, we will come back to this problem later. So, let us see what happens in certain situations. So, we just take a simple case of the domains namely we take omega to B a bound of radius R centred at the origin and then consider this eigenvalue problem, Laplacian u + lambda u = 0 in B and u = 0 on del B. So, it is a small digression, let me just explain what the Laplacian in spherical coordinates?

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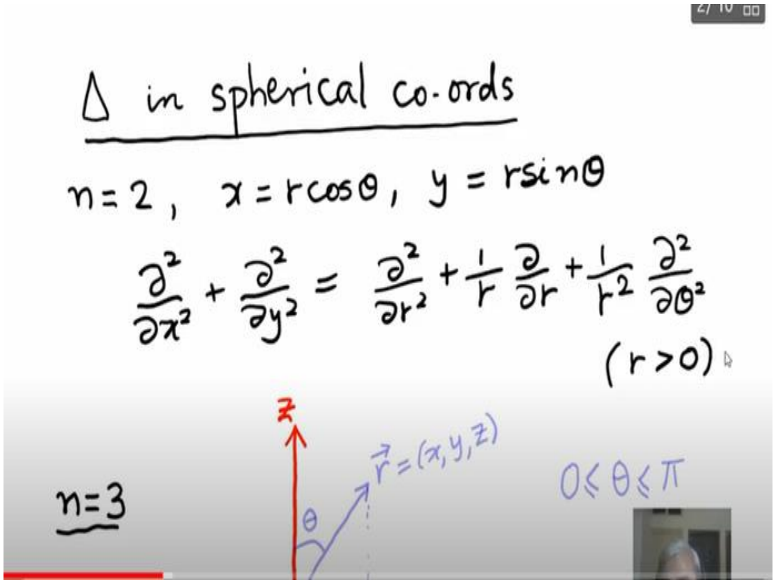

These are usefully transformations some of you already might know this. So, for example, n=2 to the method, will use the variables x and y, so these are x = r cos theta and y = r sine theta and the Laplacian in two dimension transforms into a second order operator in r n theta, this given by Del square by del r square + 1 by r del by de r + 1 by r squared del square by del theta square, so r is positive.

So, in general this change of coordinates into polar coordinates or spherical coordinates in general. This transformation is singular at r = 0. So, whenever we will find solutions in these new variables r in theta, we should always make sure that the solution is bounded at r less to 0. So, we will see as we go along.

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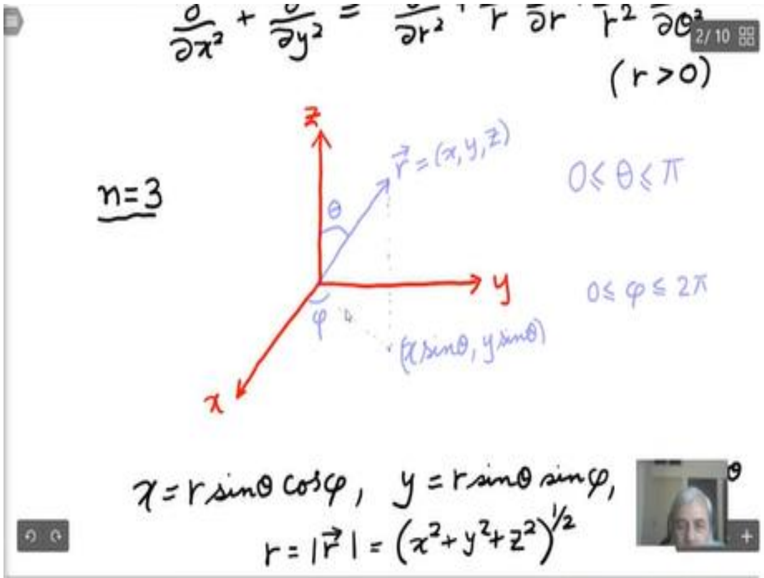

And in n = 3, just let me mention that also. So, here this spec in three dimensions x, y, z using these as coordinate are x, y, z and this vector r which is x, y, z makes the angle theta with the z axis and then you project this vector onto the x, y plane. So, and that is x sine theta y sine theta and, in this x, y plane you use the polar coordinates. So, this vector makes angle phi with the x axis and then obviously with y axis it makes related angle.

So, the transformation in n = 3 is given by x = r sine theta cos phi, y is = r sine theta sine phi and z = r cos theta. So, this r is the length of this vector, that is x square + y square + z square, root and the Laplacian in this case.

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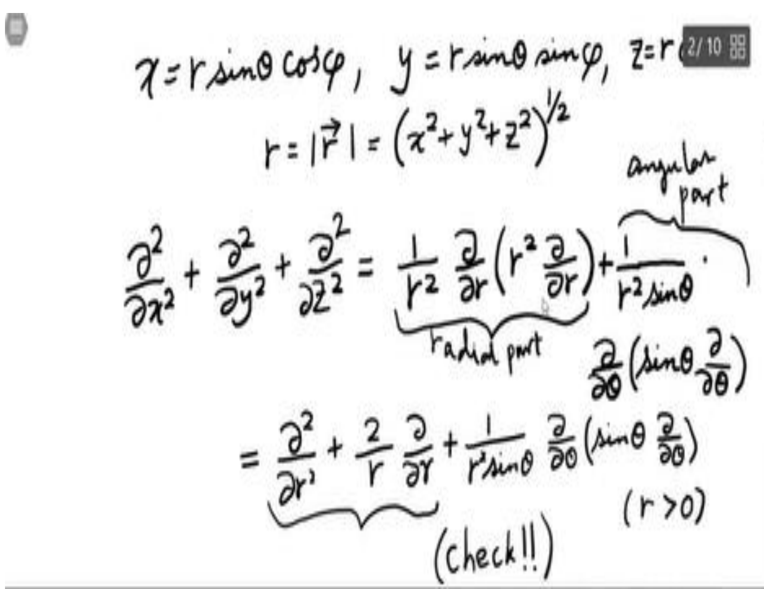

So, this is in rectangular coordinates del square by del x square + del square by del y square + del squared by del z squared = 1 by r squared del by del r, r square by del r plus again there is a radial part, so radial part and the rest is the angular part this whole thing. So, radial part I just write expanded, this is to use one thing. So, in general for n dimensional case instead of 2, we get n-1 and similarly you can write in any time x axis.

So, there will be n-1 angular variable, this is just so in case there is a possibility that the theta and phi variables might have got interesting. So, you just check that, so it is just good exercise in differentiation. So, express just left-hand side operator in terms of this r theta. So, in this brief discussion of Laplacian in spherical coordinates. So, again let us go back to the eigenvalue problem.

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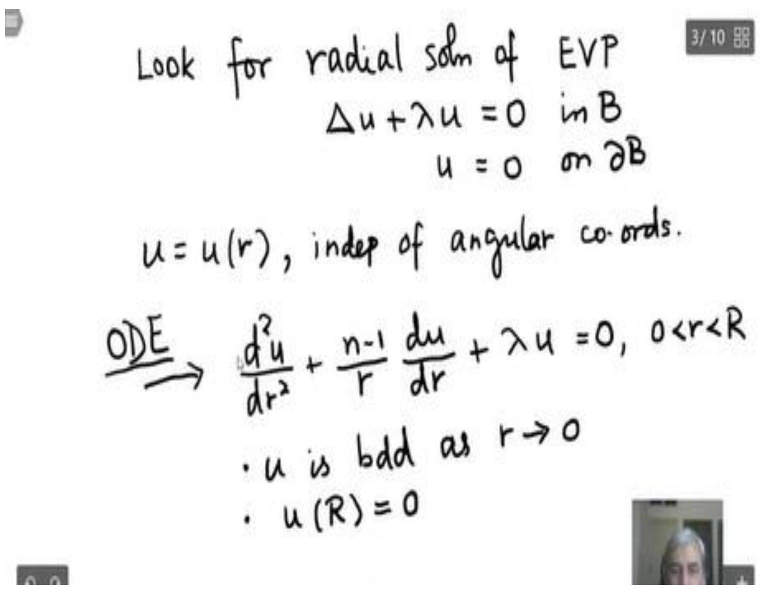

Again, let me state that so Laplacian u + lambda u = 0 in the ball and u = 0 on the boundary of the ball. And since this ball is symmetric about the origin and this invariant under rotations and same thing is true with the Laplace operator, it is reasonable to look for a radial solution of the eigenvalue problem. So, we look for a solution, u as a function of r only, so independent of the angular coordinates. So, in the n dimension, then u satisfy this ODE.

So, this is an ordinary differential equation of second order. So, d square u by dr square + n- 1 by r du by dr + lambda u = 0. So, in the study of harmonic functions this term are not there, so could integrate this easily and then we obtain the fundamental solution of the Laplace operator, but in this case there is a lambda and lambda as we know is paused. And as I say we should look for a solution which is bounded as r tends to 0.

And in order to satisfy this boundary condition directly the boundary condition on the boundary, so, u of R has to be 0.

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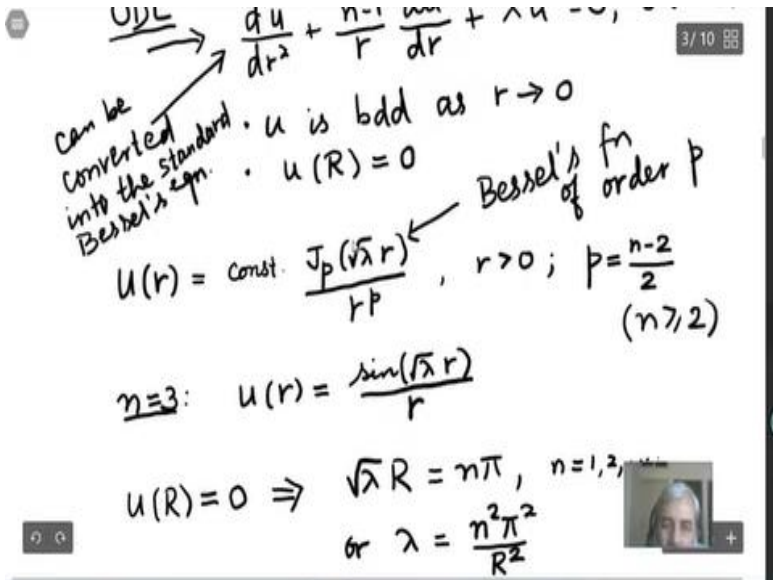

And again, this is a very standard equation this one can be converted into standard Bessel's equation and the solution is given by get them here so, one solution of that linear second order equation, so u of r is some constant multiple of this function. So, this is J P root lambda r divided by r to the P and P = n -2 by 2 and remembers, n is greater than or equal to 2, so this is Bessel function of order P.

So, this knowledge of the Bessel function, we can show that this whole thing J P root lambda r divided by r to the P as r tends to 0 is some finite number, so it remains bounded. So, that is what there will be another solution because this is second order equation. There will be two independent solutions, but the other one would be unbounded as r tends to 0 and rejected, so, this is the only solution.

And here I will write down some specific cases for n = 3. So, we get P is equal to half and J half can be expressed in terms of sine function. So, when you do that, for n = 3 we will obtain u of r equal to sine root lambda r by r, so, I omitted that constant, you can put any constant. And you can directly verify, so again this is bounded as r tends to 0, as r tends to 0 it just tends to root lambda which is finite.

So, we can directly verify that this u r satisfied this ODE when n = 3 so, that was here it is 2 by r, so by direct differentiation we can do that. So, it remains bounded as r tends to 0 and what about this boundary condition, so we want to have this boundary condition u of r = 0. And that implies this root lambda big R may be a multiple pi the roots of the sine function or this lambda is equal to n square pi square by R square.

So, only for these specific values of lambda, so this at least in the class of radial functions, so we are able to get that and of course, you can still ask what about the non radial solutions and other things and I will not go into that.

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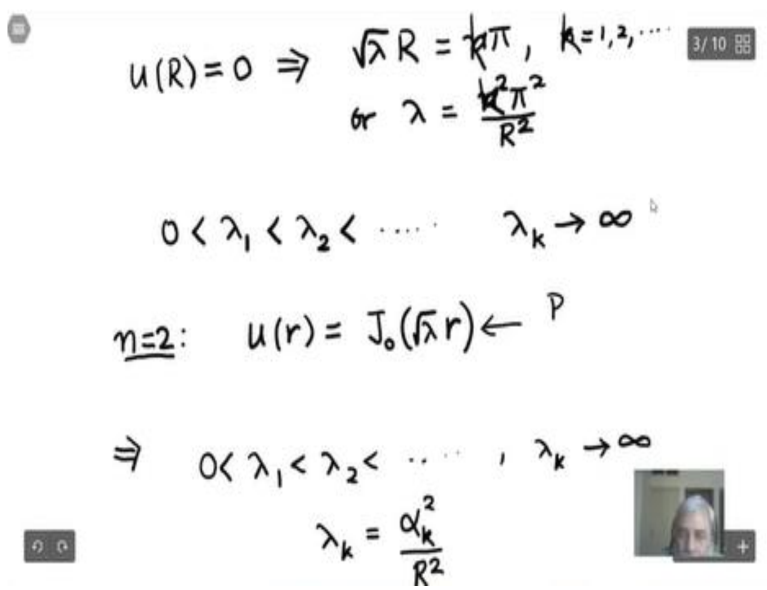

But at least what we obtained here is, so there is a sequence approach to real numbers. So, lambda 1 lambda 2 lambda K and lambda K tending to infinity just to get that maybe I should use a different notation here instead of n. Because n we are using for the dimensions, so let me just check. So, these sequence of numbers tending to infinity, so it is unbounded and for each value of that we obtain an Eigen function.

And for n =2, so this P will be 0 and we obtain this Bessel function of order 0. And Bessel function of order 0 has an infinite number of positive 0s tending to infinity. So, again the same conclusion we opted for n = 3 and = 2. And that is a case for any general n, the only difference being that is given in terms of some Bessel order of some order and all those Bessel functions they will have infinite number of positive to 0s.

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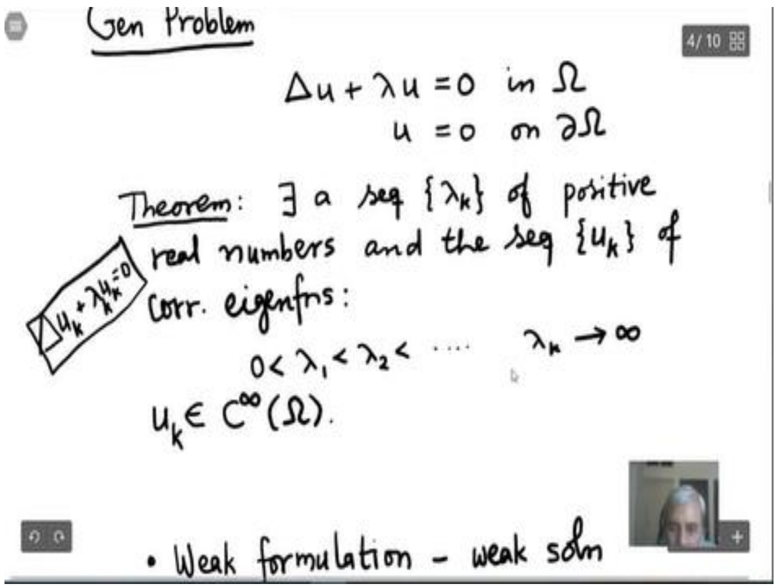

So, now come back to the again general problem. So, after seeing these two examples, let us now consider the general problem. So, again a bounded smooth domain you take a boundary condition. So, let me just take the theorem, the theorem is not very precise, I am just concentrating on the existence of this eigenvalues. So, there exists again a sequence lambda k, a positive real number and the sequence u k or corresponding Eigen functions.

So, that means we have just Laplacian u k plus equal to 0 and u k r 0 on the boundary and further so, there is some regularity, so these u k are all C infinity function in omega. So, proof of this theorem is a very much non trivial, I just would like to impress upon you. So, a usual method of proving this theorem is as follows.

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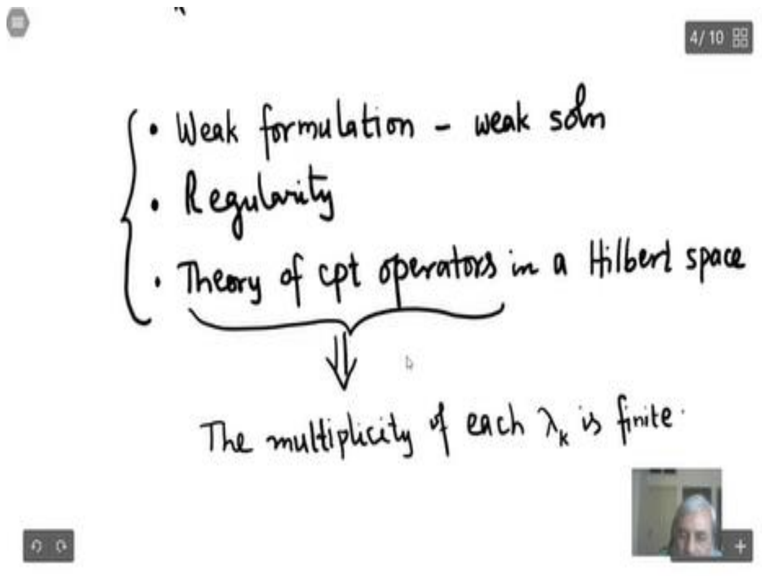

First you go to the weak formulation and obtain a weak solution and then you use regularity results to prove that the solutions are actually infinity functions and then in order to get this existence of infinite number of eigenvalue we need to use theory of compact operators in a Hilbert space. So, you can see how technical did the proof and this theory of compact operators also tells us one more thing, this is interesting.

The multiplicity; there is an underlying Hilbert space here multiplicity of each lambda k is finite. So, this is an important consequence of this theory of compact operators, what does that mean?

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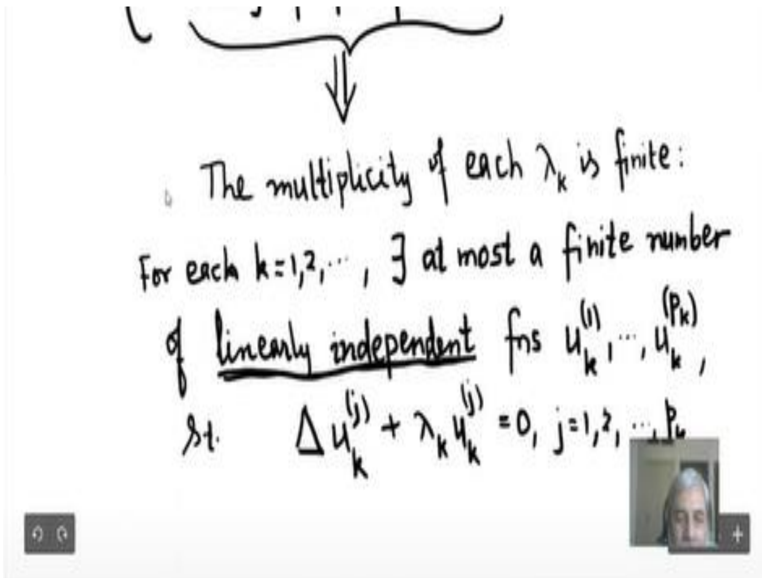

So, there exists for each k there exists at most a finite number of linearly independent, so to talk about linear independence we have to have an underlying vector space, in this case it is a Hilbert space linearly independent functions. So, let me call that u k 1 etcetera u k, let me call it some P k, so that depends on k such that Laplacian u k j + lambda k = 0. So, this linearly independent. So, why do we need this?

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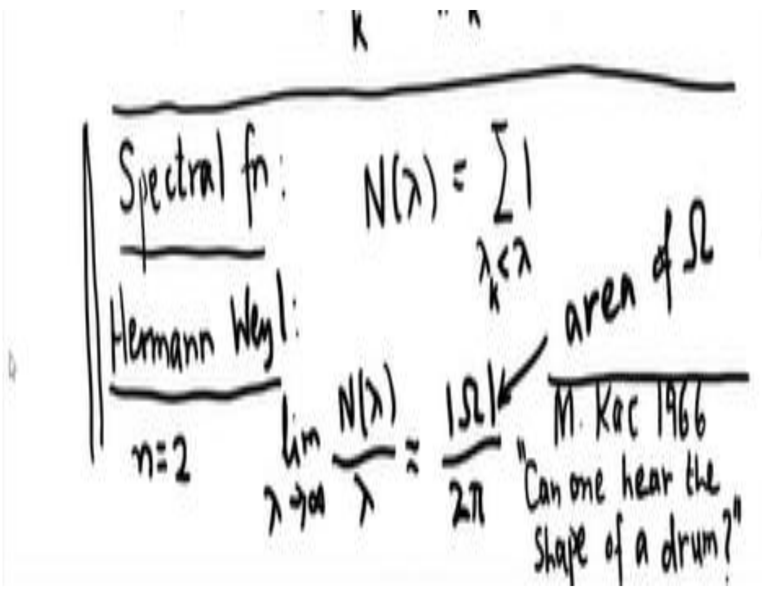

This kind of detailed analysis of the spectral. So, this is by the by called spectral theory not only just for the Laplacian in general for a second order uniform elliptic operator, so, where is it used? So, let me just briefly explain that, before coming to that there is another interesting line one can go. So, here again let me call a spectral function, so you have just defined this N lambda is equals, so you just count given any positive real number lambda, you just count the number of eigenvalue less than lambda.

So, it is just like counting the prime numbers up to some positive number. So, there is an interesting result regarding this proved by Hermann Weyl, so it was constructed by the physicist Lorenz, so this is two-dimensional case what Hermann Weyl did, but then it was extended to higher dimensions. So, this limit N lambda by lambda as lambda tends to infinity is, so this one is area of omega, very interesting result.

And there are further extensions of this result, not only for the Laplacian, but also for the second order elliptic operators. So, there is a old paper by Mark Kac which appeared in 1966 and the title of that is, Can one hear the shape of a drum? So, it appeared in American Mathematical monthly, so you can easily find in Google search. So, you will see some history of this spectral function and this Hermann Weyl results and extensions and what is this problem etcetera etcetera. That is an interesting story about this spectral function. So, where do you use this spectral theory?

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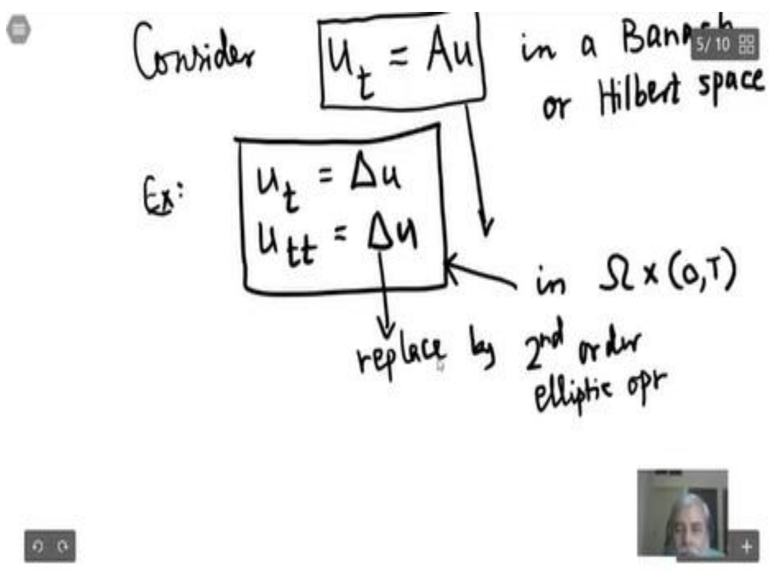

So, consider this evolution equations, again I am being quite vague here in a Banach space or Hilbert space. So, you have studied system of first order equations ODE. When it comes to

PDE, so, we are moving from finite dimension to infinite dimension. So, this A is a linear operator. So, in our example we have this heat equation or more generally a parabolic equation and again a wave equation.

So, these can be brought into this form and you want to study this one, where do we study in some domain bounded domain or unbounded domain in R n and in some time interval. The standard procedure of proving existence, uniqueness and continuous dependence on solution, so this Laplacian, it can replace by second order elliptic operator, maybe with one variable coefficient.

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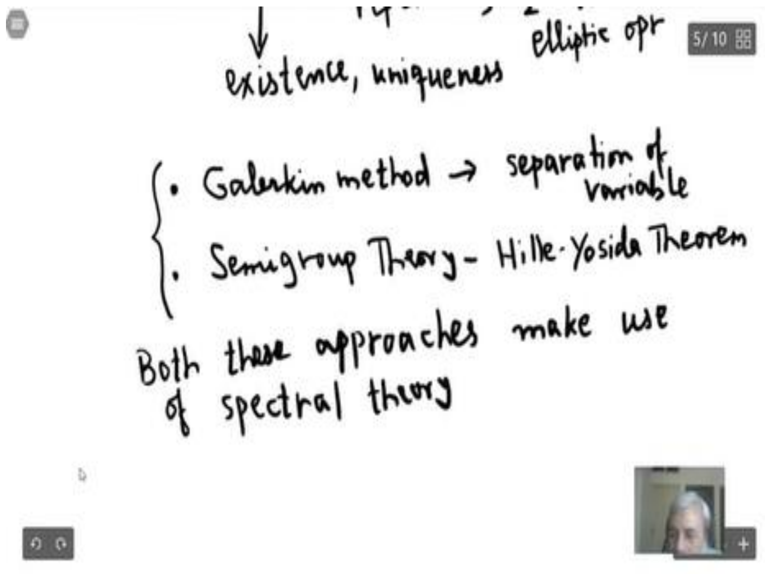

So, in general; will not be able to obtain a solution in explicit form. So, we have to prove the existence and uniqueness results by some techniques. So, there are two ways of obtaining search results; one is so called Galerkin method. So, this is essentially separation of variables method and another one is semi group theory. So, here essential it is Hille Yoshida theory and put these approaches, they use approaches make use of the spectral theory.

So, it is essential to have the spectral knowledge of this operator A in general, so when you consider a general linear operator, so we have to make use of the spectral theory corresponded to that, so this is. And with one final remark, this so far, we have only considered bounded domains. And this story is very much different if you go for the unbounded domains. So, this I will take up on the next class. Thank you.