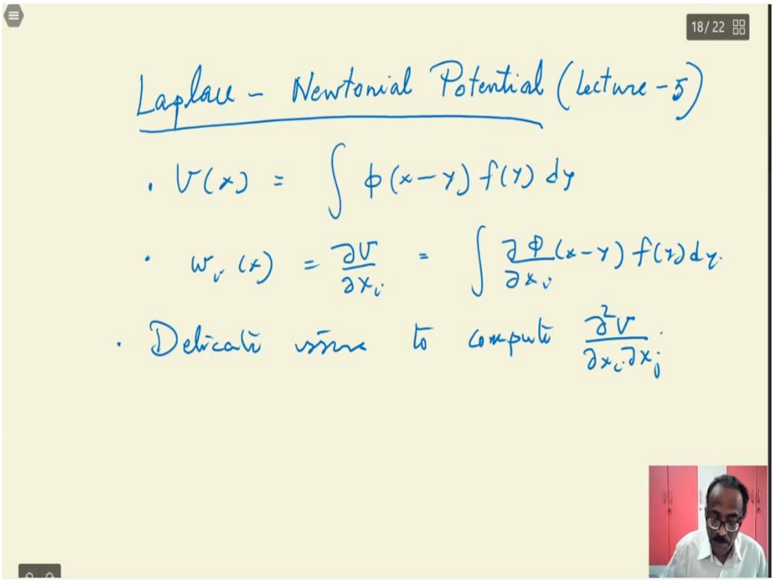
First Course on Partial Differential Equations- II
Prof. A. K. Nandakumaran
Department of Mathematics
Indian Institute of Science, Beangaluru
and
Prof. P. S. Datti
Former Faculty, TIFR-CAM, Bengaluru

Lecture - 22
W5L2 Newtonian Potential 5

(Refer Slide Time: 00:20)

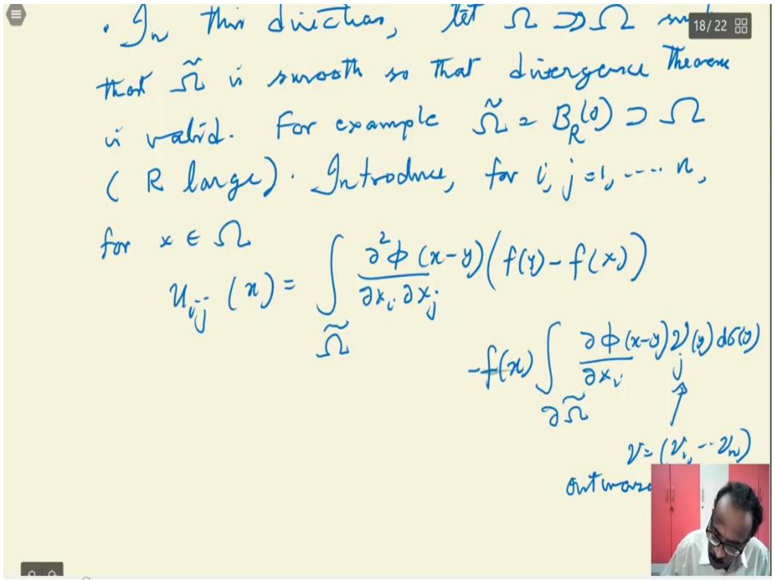


Morning, so we are come to the last lecture on Newtonian potential, in which we eventually complete the proof of the solubility of our Poisson equation. So, we will complete the using the not only Newtonian potential introduced. Let me again because important, so we recall the Newtonian potential, V x equal to this so we complete it not that the everything is over but then we will not do it further.

And in the last class we actually computed w i of x = dv by dx i, and there is no need of any additional assumption you will see the difficulty here this is nothing but d phi by dx i of x - y f of y dy. Now the delicate issue is this today's lecture delicate issue to compute, because you want to solve the problem you have to compute d square v by dx i dx j rather you want to compute your Laplacian. So, at this point we want to tell something.

That the there is some mistake in the proof given in our book. So, we will be giving the correct proof here so there are some points which we have not exactly noticed it. So, that also indicates how delicate this proof if you are not worried about that one. So, the proof is delicate so we will compute this using the elder continuity.

(Refer Slide Time: 02:17)



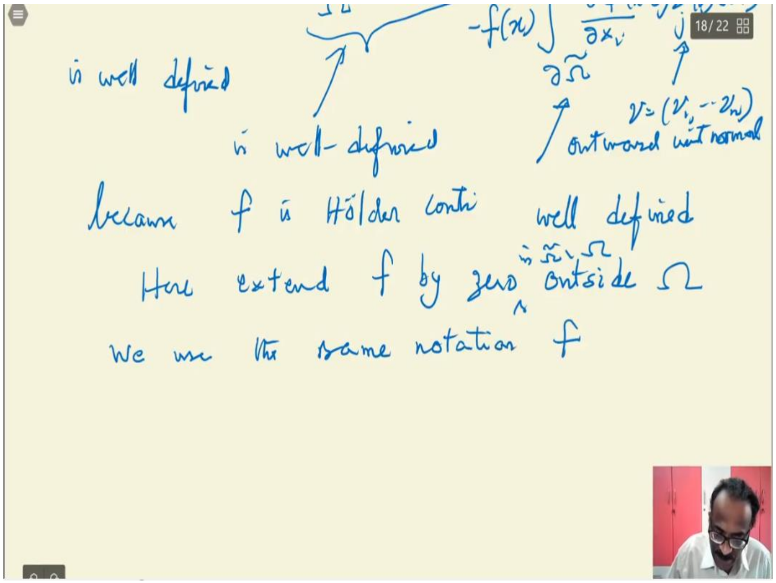
So, in this direction we will state that there are now soon, introduce. Now, before introducing in this direction let omega tilde be a bigger domain containing omega, may be purely fully contained in omega. You can take it if you want it such that, omega tilde is smooth so that omega tilde is smooth enough I am not specifying is what smoothness it is so that divergence theorem is valid. You know that C1 or whatever smooth divergence theorem any domain theorem is valid.

That is the material theorem is valid. For example, you can take for example so that is why it is a kind of a little delicate the boundary thing you can take omega tilde is equal to a ball of radius R with R containing omega that is R is large. You can choose because omega is the bounder domain R large you can do that. So, now introduce a class of functions introduce for i, j = 1 to n so we are going to compute what is d square v by (()) (04:04) up to n.

And define for introduce for x in omega. So, you are define x is in omega some u ij of x for equal to, so this is the thing and this is in omega tilde. I am defining d square phi by you will see why you need this x i, x j but then at a value at x - y and f of y - f of x this is the internal thing and we also need to take care of the boundary. So, you will have one more term f of x = f of x what is the term you exactly wanted on the boundary of omega tilde this is in omega tilde.

This is to take care of the boundary by d phi by dx i at x - y nu j of y this is the outer normal this boundary d sigma y this nu is equal to outward unit not nu in outward unique number unit normal is a standard notation.

(Refer Slide Time: 05:56)

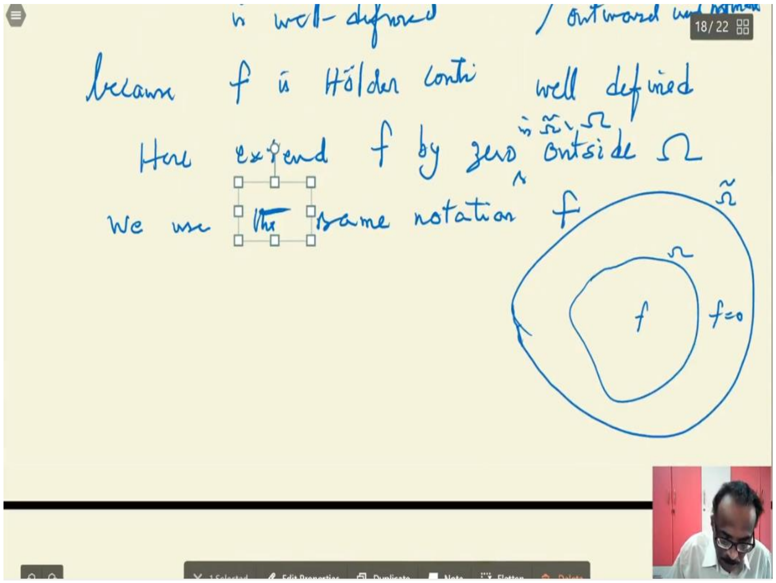


So, now look at the terms here is well defined. Why it is well defined? This there is no problem well defined because it is on the boundary and the singularity is only at x = y which you will end this is locally integrable d phi by dx i. Here this is not locally integrable but you have yesterday's remark this is well is well defined. It is well defined that we already made the comments yesterday defined because f is Holder continuous.

So, the singularities so there is a lack of local integrability here. But that is taken care of by this Holder continuous, so yesterday we have seen that so Holder continuous of course f is bounded. So, look at the first term anyway first term. So, what is f outside omega? So, here extend f by 0 outside omega a 0 in omega tilde - omega that is 0 outside omega. And we use the same notation, so we do not use any other notation. We use the same notation f.

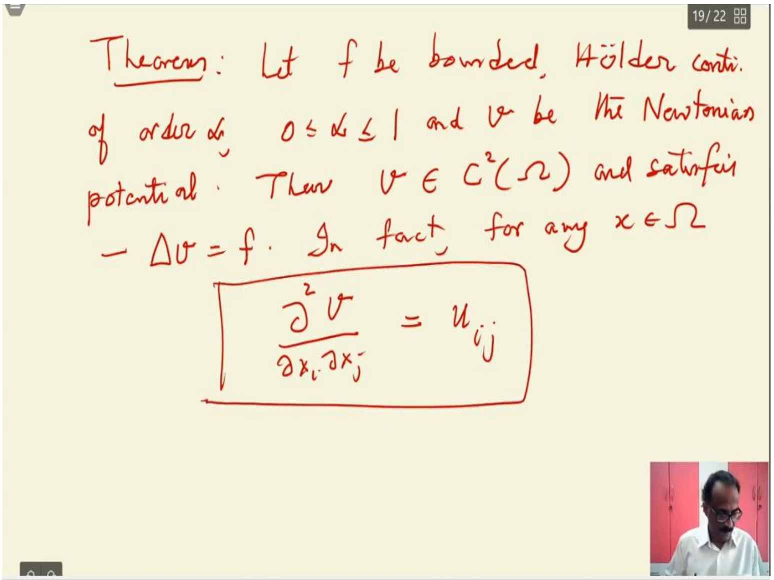
So, if you look at the first integral in the first integral this first term outside omega, it is 0 of course x is fixed which is in omega. So, the first term is actually in omega, it is not in omega tilde so the important term is actually in omega. Only the remaining term is in omega tilde and then that is something like omega tilde, and this is also a term in d omega tilde. So, these are all will the kind of final analysis which you need an extended domain to do this final analysis.

(Refer Slide Time: 08:35)



So, that is it, so f is, so you have an omega here and then you can make. It can be about a big ball so you define omega f = 0 here, f is given here so you do that what. So, I call this so you look here so this is a definition of u ij. So, you keep that as I said you have to be careful in defining everything, they say call it star if you want it. So, that definition of u ij, so now I will state my theorem main theorem.

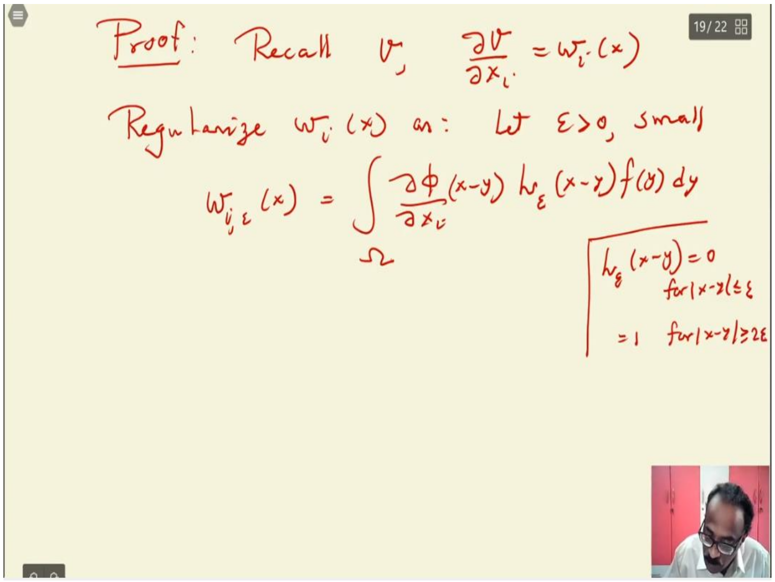
(Refer Slide Time: 09:28)



Theorem so definition let me do it theorem so definition let f be bounded whatever we have assembly and putting it in form of a theorem Holder continuous of order alpha filter continuous of order alpha 0 less than alpha in fact you can up to 1 and V be the Newtonian potential. So, this is essentially the final theorem though we state the final theorem which is Newtonian potential defined earlier.

Then what we are telling is that this is what we are going to prove it, then V is in C 2 of omega and satisfies - Laplacian V = f. So, of course we do not derive any boundary condition but we use this to prove the problem revenue problem soon you will see that what. In fact, we have more you can compute every derivative in fact for any x in omega you can actually get your d square v by then after that it is the summing d square v by dx i dx j = u ij where u ij defined earlier

(Refer Slide Time: 11:25)



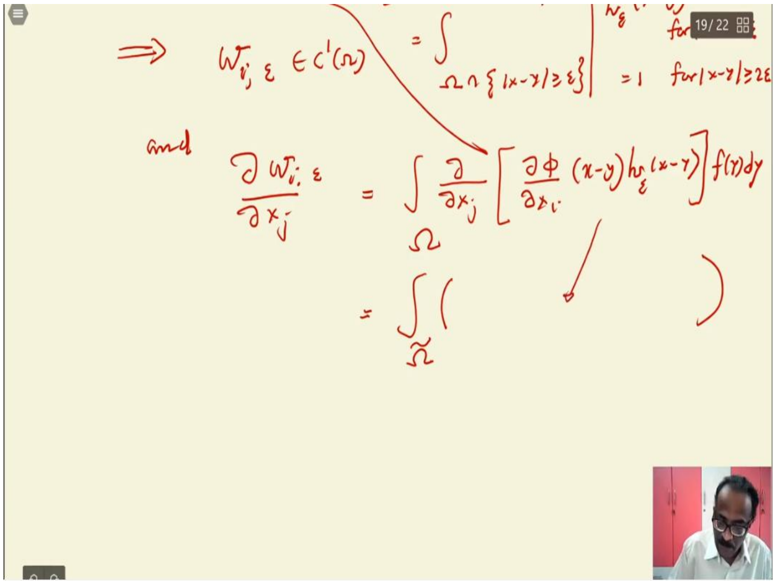
So, let me give you the proof as I told you there is some mistakes in the proof of this book. Proof of this theorem given in the (()) (11:40). Let me so let me recall V so I will not just write again recall V and then you recall dv by dx i which we already computed and given a notation; this is a standard notation. So, we regularize so you see what we want to do is that we want to for compute d square v by dx i dx j.

So, in other words we want to compute dw i by dx j which needs a further regularization and then so this is only d v by dx i you did not need any the Holder continuity but here you need regularization and the trick we have employed that one. In the case you regularize w i epsilon i x as that net epsilon positive small we choose small. So, we regularize it this w i epsilon of x = integral over omega, so same thing d phi by dx i and then you have multiplied of x – y.

So, you and you recall h epsilon h epsilon of x - y and f of y table. So, you see this h epsilon = 0 for x - y less than so recall this property h epsilon of x - y = 0 for mod x -phi, so less than or equal to epsilon that is where you have a troubled value. So, you want to take a differentiation inside then you will get d square phi by dx i dx j and then it has a not locally integrable but then this is inverse 0.

So, you regularize that one and equal to 1 for x - y greater than or equal to 2 epsilon and it is smooth. So, it is one eventually the integral will be only less than or equal to 2 epsilon, epsilon less than or equal to mod x - y less than equal to 2 epsilon so this is what.

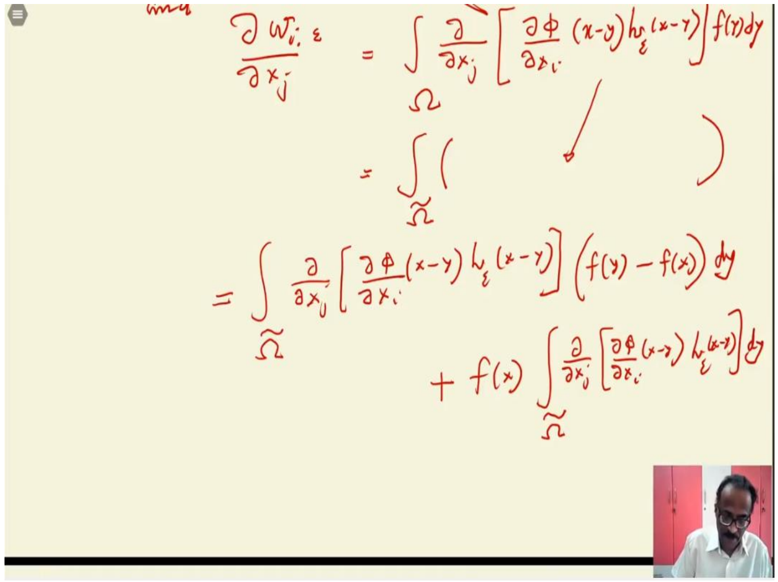
(Refer Slide Time: 14:08)



So, now this is a w i epsilon is because there is no problem here you see. So, this is a so you can take differentiation because of h epsilon vanishes there is no problem of singularity inside. Therefore, that implies your w i epsilon is actually by regularizing this one the singularity is removed where the singularity is there it is completely removed so basically this is an integration. This is basically an integration over omega intersection mod x - y greater than or equal to epsilon.

You see so there is no singularity this implies w i epsilon is C 1 of omega and you can see that d w i epsilon by dx j, I can differentiate now this is integral over omega and d by dx i, I can take this dx j inside of d by d phi by dx i of x - y h is also depends on the h epsilon of x - y you see into f of y dy. So, you can do that. Now look at this one f is 0 outside omega, so this is same as integral over omega tilde, this integral. There is no problem, because f is 0.

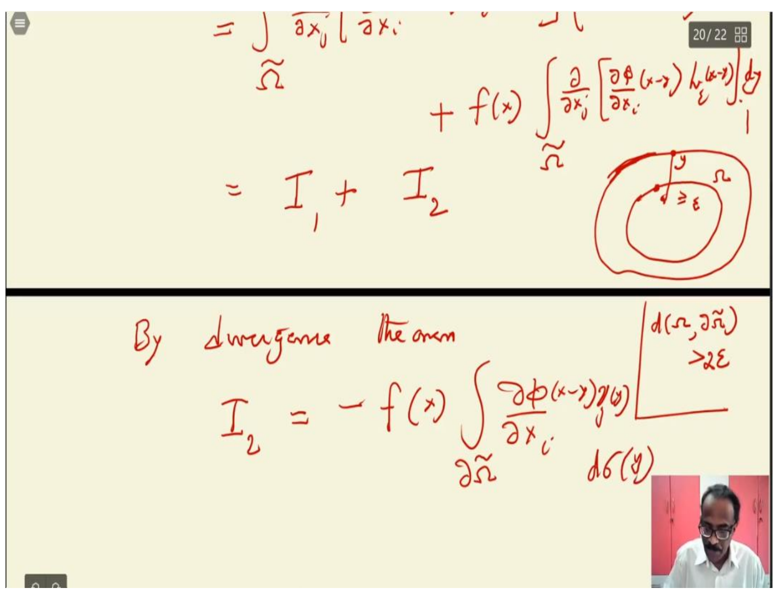
(Refer Slide Time: 16:01)



So, now I will add and subtract a term that is a very important thing. So, this is equal to because I want to take care of my you will see that why I do that one. So, I will do the same thing d by dx j of d phi by dx i of x - y because I want to use my Holder continuity h epsilon of x - y. And I will add so this dx j is not for f and here I will add f of y - f of x, I add a term here the differentiation is with respect to y. So, I add the minus that one subtractor so I have to add plus f of x here.

Integral over omega tilde this is everything in omega tilde of d by dx j of d phi by dx i x - y h epsilon of x - y dy. So, you see this integration is with respect to that one.

(Refer Slide Time: 17:22)



So, look at this one so this integral is let me call it I 1+ I 2. Now look at I 2 do an integration by parts there is no issue at all. When you do an integration by parts this is multiplied by one you see so that will go so only boundary term will come. So, by divergence theorem because it is a divergence theorem I 2 = - f of x the only boundary term will come, the in because there is a full is an exact differential here.

So, the boundary term is equal to d phi by dx i of x - y h epsilon of x - y, this is on the boundary of omega tilde and h epsilon will be one because it is a far away. So, if you look at it here so if epsilon is more things x is inside. So, you have an omega here, and I am choosing omega till I in such a way that and your h epsilon the points will be here. So, you are having one so, it will become one there so the boundary term it will not be coming.

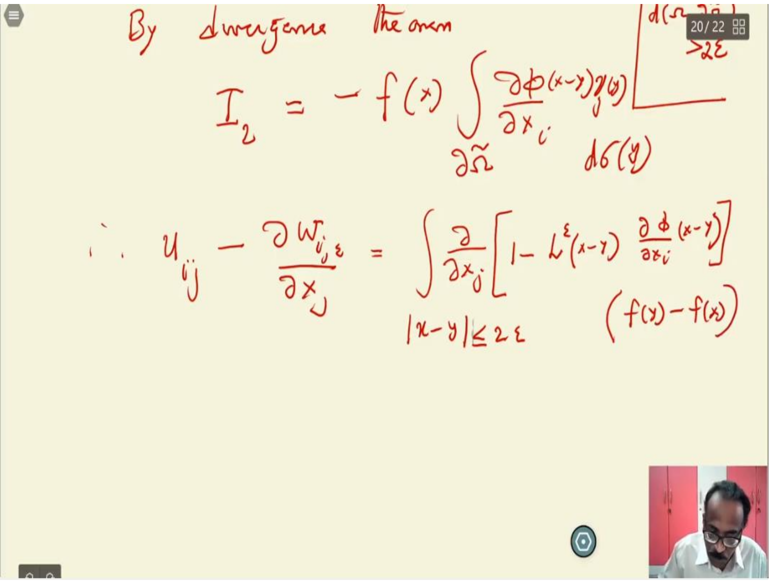
That will take one there so you can have your boundaries of the boundaries far away from this so, you can this one so you can say that for example. You can say the distance from omega to the boundary of omega tilde is probably again strictly greater than epsilon. So, the points here so if you take x here so the y here x here will be distance will be greater than epsilon and in fact you can take 2 epsilon if you want.

So, in that case this will become 1 h epsilon become 1 so h epsilon will become one so this will be nu j of y so this is nu j of y and d sigma y. Is that clear? Because you are integrating here on the boundary and the boundary of d omega that is where you want a different omega tilde. You see the boundary term and you can have the points from here basically more than 2 epsilon that is something. You can take the point you like it so bigger than that one.

So, you have one here and you have now look at this term this term is exactly what is defined the boundary term in your u, if you go about it. So, this term is same as this. See that is exact boundary term which we looked at so the both are same. So, if I compute because I eventually want to know that your du j is d square u by d square V b that is what you want to do it. So, you exactly want to because this is the limit you are looking at it. So, this is the exactly so you want to compute.

This is what d epsilon by dx j so you want to study the limit of dw i by dx j and that should be your d square v by dx i dx j.

(Refer Slide Time: 21:16)

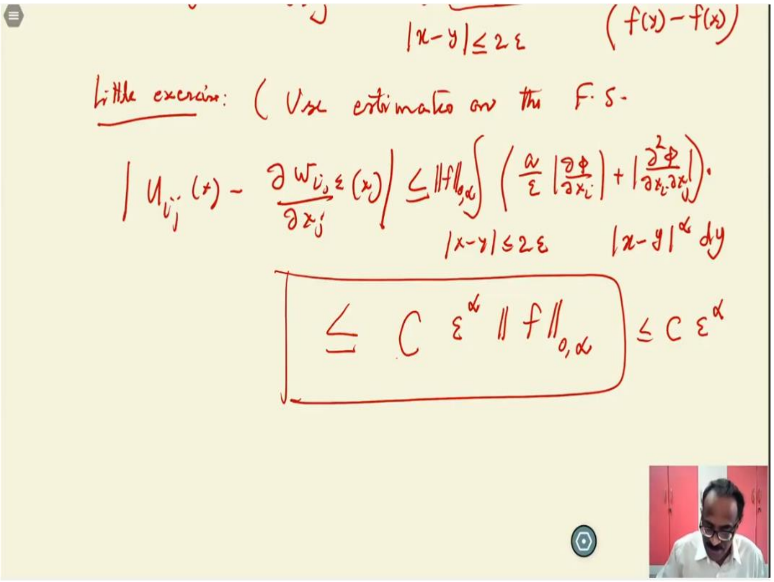


So, if I subtract so therefore, so therefore if I do that one so you exactly you got that second term to be like that so if I subtract my u ij eventually I want to understand this convergence dw i epsilon by dx j these boundary terms will get cancelled. And look at here in this term you just do the subtraction and h epsilon is 1 whenever mode x - y is there. So, if there is no h epsilon this is exactly this term in here.

So, you see the terms are constructed when it this term is the same as now if you look at this term this is same as which h epsilon and h epsilon = 1 with the mod x - y greater than equal to 2 epsilon so you exactly mod x - y greater than equal to 2 epsilon so it will cancel. So, only you will get x integration x - y greater than or equal to 2 epsilon and d by dx j so you are subtracting same term only mod sorry less than or equal to 2 epsilon.

Because for greater than or equal to 2 epsilon h epsilon equal to 1 the term contributed for u ij and w i epsilon by dx j are the same so it get cancelled. So, you only get a epsilon contribution on this interval, 1- h epsilon h is epsilon I used up so you use that of x - y you see d phi by dx i of x - y, so this integration is also here into f of y - f of x. Now we are done almost your proof is here almost the proof is done here.

(Refer Slide Time: 23:27)



If you look at it so do an estimate so a little exercise. This we have done it in the previous theorem but you can do the same thing if I do this exercise you use the what do you say Holder continuity and the estimates on the photo use estimate on the fundamental solution if you do that one, I will get my ui j of x dw i by epsilon by dx j at x will be less than or equal to look at this differentiation. So, you have to first differentiate this bracket should be like.

First you have to differentiate this one when you differentiate this h epsilon you know that it is bounded by a over epsilon which we have used in that case. So, you will get when you differentiate this here so what you will get is maybe some constant may be coming here but that is not important you will have to epsilon so integration is overlap. Do the first term when you differentiate this one that will bring some a over epsilon.

Because it think and that bound on this d phi by dx i and then when you take this differentiation this term is less than or equal to 1, 1 - f epsilon is less than or equal to 1. But then the derivative is d square phi by dx i dx j and there is a problem with dx i dx j but then you have your so maybe you put a constant here and then you will have here you will have mod x - y power alpha. So, this extra singularity here will be taken care of by this one you see.

So, this is for the here you do not need it but here this extra singularity coming for the d square phi by dx square extra singularity here will be taken care of by this thing. You see that is why so this alone will not suffice but if you look at my previous theorem and its proof instead of d phi dx i you got phi here instead of d square phi by dx i dx j you got d phi by dx i. So, the additional term of mod x - y power alpha was not necessary there because you had a better estimate.

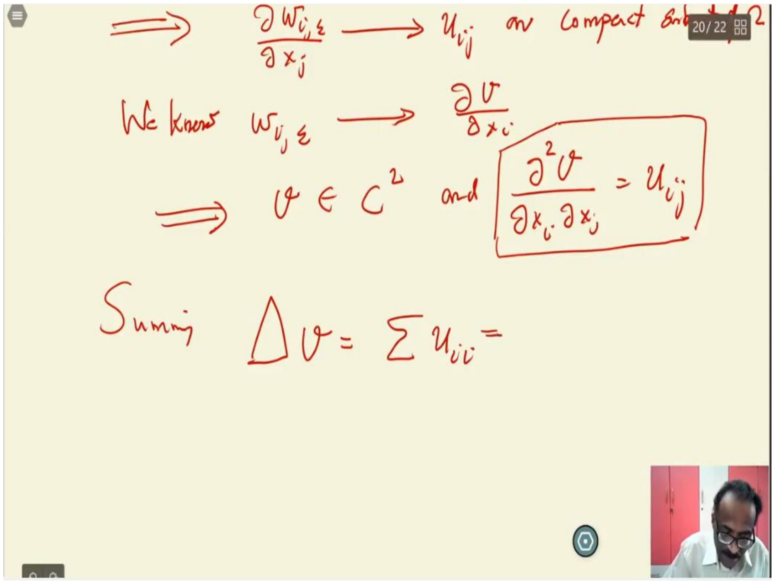
So, now if you look at it, this one this is less than or equal to so if it is phi, you it gives you something like epsilon square for n greater than or equal to 3, but for d phi by dx i it can be estimated by epsilon in this interval. So, that epsilon and this epsilon cancel and this will be of the form epsilon square it will have nothing it will be order because near the origin you have a problem but that will be cancelled.

But away from the origin away from that epsilon here you have a good estimates and it will be bounded. So, you can show that in the earlier case it was phi and d phi by dx i so that it produced some epsilon here and it epsilon square cancel epsilon here d phi by dx i that you got an epsilon there. So, look at the proof earlier and now but you do not need here but this gives you this one this extra things is given here.

So, if you do that analysis if you are not convinced, please do that one so it will be some other constant and then is precisely this norm of f. So, this is constant into some epsilon power alpha so basically you will have a norm f of if you want it. So, you can actually get this constant epsilon power alpha into 0 to alpha so but whatever it is this is basically less than or equal to constant into epsilon power alpha.

So, if you want so you so that is the constant, we already included this constant here so if you multiply and divide it so I actually added here. So, that is how you get this one so you take a norm f it is overall for here so this is it so this is boundaries the constants are keep on changing that what.

(Refer Slide Time: 28:29)

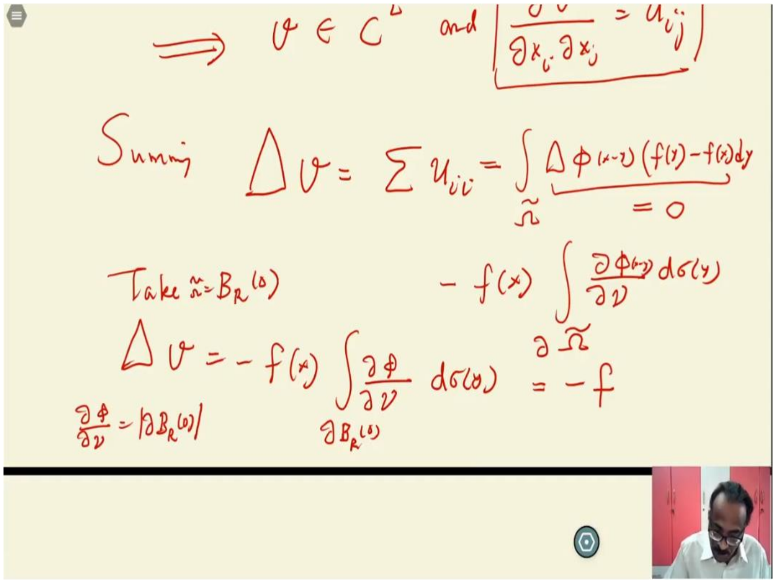


So, this implies immediately this implies now it is the same thing so getting this estimate towards the difficult thing so that implies your dw i epsilon by dx j converges to u ij on compact subsets of omega these neighbourhoods as any compact subsets of omega. So, we already know that w i epsilon converted we know this we already proved last class we already know that w i epsilon converges to dv by dx i.

All these put together now gives your V is in C 2 that is exactly you want it and your d square v by dx i dx j = u ij. So, that is it. So, you can read that one. So, you have computer here this exactly the proof is delicate but end of these conclusions are the similar thing. Now summing so if you how do you solve the problem summing so Laplacian of V = summation of u ii. Now we look at it what is this summation.

This summation if you look at it u ij so you have to summit only when i = j. So, when you summit only when i = i so not this one what is the definition yeah so if you submit u i you see this is only d square phi by dx i dx j that means d square phi by dx i square so that will be Laplacian. In general, you cannot integrate the Laplacian because of the singularity under the integral sign but f y – f x is held are continuous. So, this whole term is locally integrable. So, you see that one and then you have your; this boundary term.

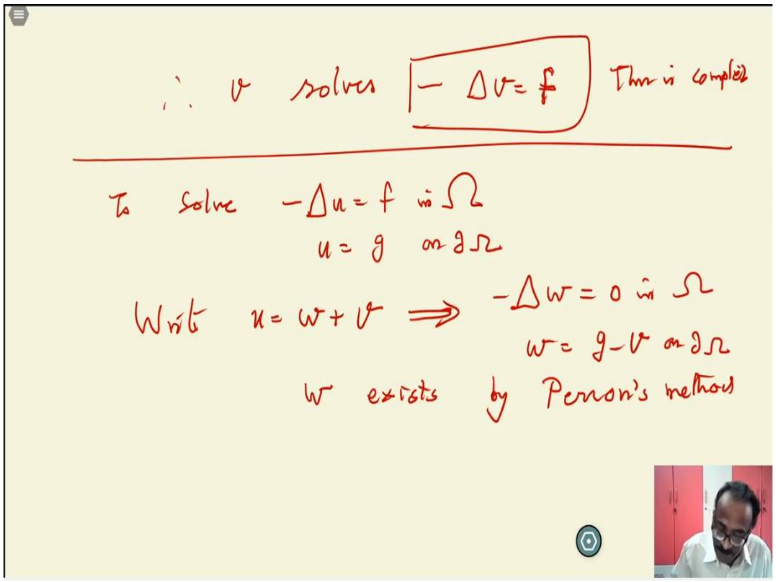
(Refer Slide Time: 30:53)



So, if you submit you exactly get this term, what you get is integral over omega tilde Laplacian of phi and x - y and this is together is locally integrable f y – f x dy and then you will have your - f of x and then there also you are summing you get d phi by dd over omega tilde boundary of omega d sigma you see (()) (31:30) and then this is locally integrable and this is equal to 0. So, that is 0 so your Laplace n of v is = - f of x integral d phi by dv.

And then take omega tilde is equal to B R 0 omega tilde = B R of 0, and then this will be on the boundary of B R of 0 d sigma y and in PDE one several times you have seen this is nothing but this average. So, and this actually one so d sigma y so this is equal to minus f. So, that is it because d phi by d nu you can turn to B the modulus of B R also 0 you have done that computation. So, it solves the problem.

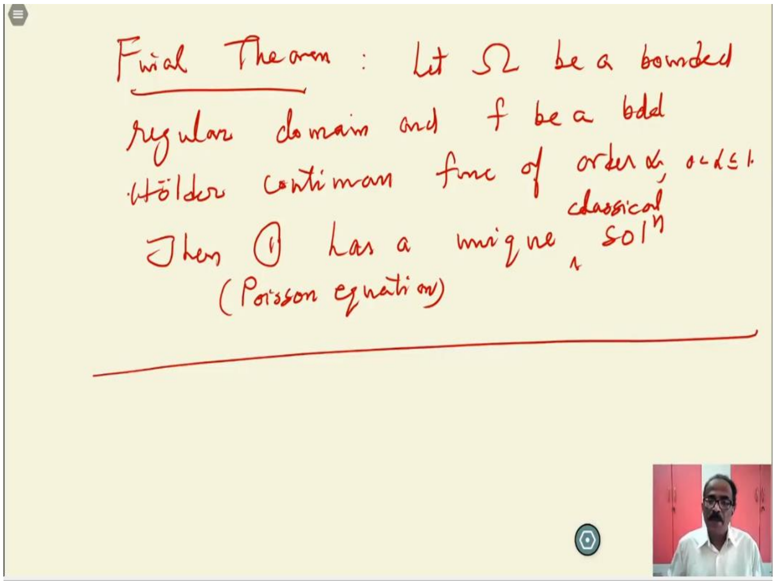
(Refer Slide Time: 32:36)



Therefore, the Newtonian potential V solves - Laplacian of phi = f. So, the theorem is complete and now we complete the proof of your thing now you so you want to solve this is our final theorem to solve - Laplacian of u = f in omega and u = g on the omega. Now it is fine so you have minus Laplace this boundary condition you do not know. So, you write u equal to you are looking for a solution in this form implies your Laplacian of w - Laplacian of w = 0.

Because Laplacian of V = f is already taken then your w should be g - V whatever value it gives down the boundary on d omega this is and this w exists by perron, that is it so you see that is the way you complete the proof perrons method. So, you first solve this equation use that V and use this g - V as the boundary value solve for w then you can see that Laplacian of V + Laplacian or w = f and u = g because V get cancelled.

(Refer Slide Time: 34:18)



So, this is the final theorem and which we will complete here final theorem. Let omega be a bounded regular domain you need that boundary regular domain with all these last PDE 1 and PDE 2. So, this is the culmination of the whole theorem, regular domain and f be a boundary. Regular boundary Holder continuous function of some order of order alpha 0 less than alpha less than or equal to1then this problem so let me call this problem one as a unique solution.

Uniqueness for all seen last PDE 1 course has a unique solution then the Poisson equation this one is the Poisson equation. It is a unique classical solution so that is it. So, we will stop here and we will continue with something else and, thank you.