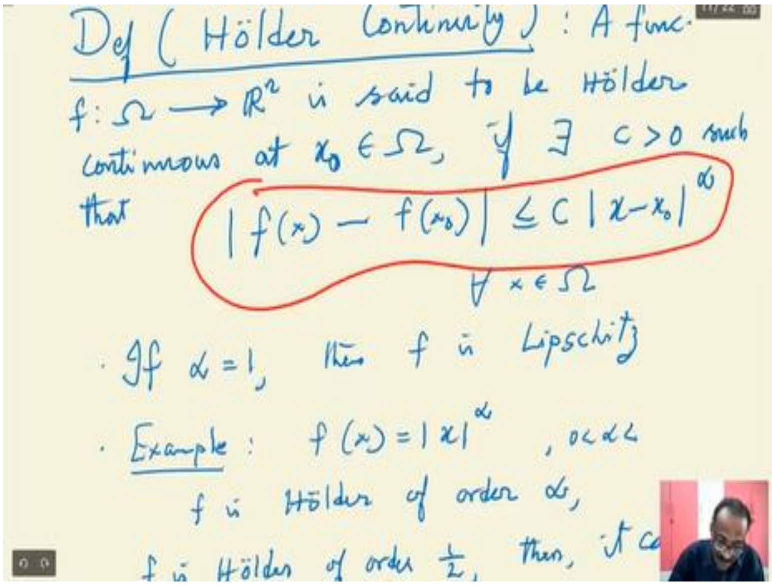
First Course on Partial Differential Equations- II
Prof. A. K. Nandakumaran
Department of Mathematics
Indian Institute of Science, Bengaluru
and
Prof. P. S. Datti
Former Faculty, TIFR-CAM, Bengaluru

Lecture - 21
W5L1 Newtonian Potential 4

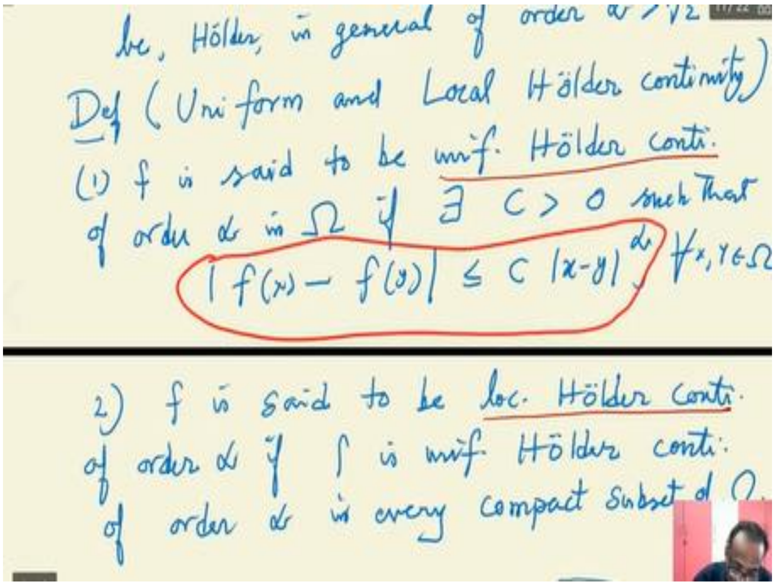
Morning, so last week we have proved the existence of the solution when the source term it is C 1 and then we have introduced the concept of Holder continuous function.

(Refer Slide Time: 00:37)



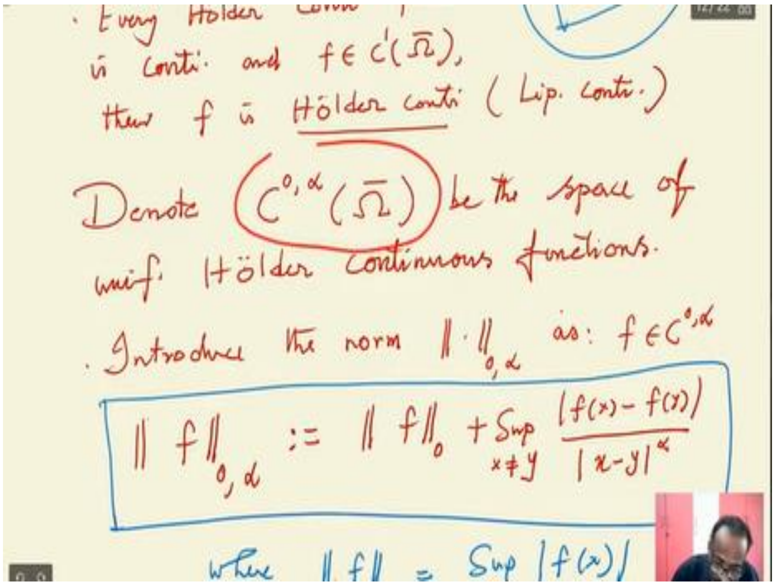
So, let me recall once again the definition of the Holder contiguity. So, we have a f is set to be Holder continuous quickly let me go through a continuous set x 0 if there is a constant which satisfies this in equality. So, this in equality is satisfied at x 0.

(Refer Slide Time: 00:56)



After that we have introduced the uniform continuity, you need this estimate to satisfy for all x and y and uniformly, so that there is C which exists for that one. And then we have introduced local Holder continuity and the local Holder continuity it means it is uniformly Holder continuous for a very compact subset of R.

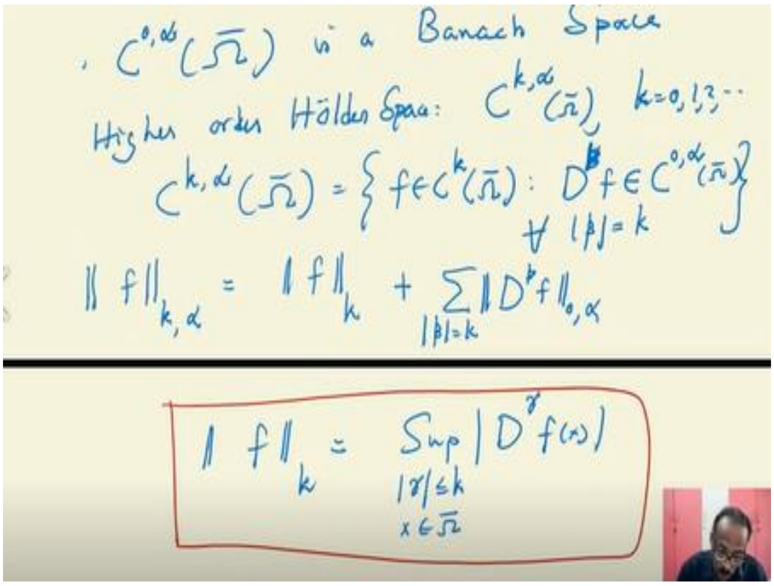
(Refer Slide Time: 01:19)



And then we have using that one; we have introduced the space, we have introduced this space C 0 of Omega, and then for that we have introduced the norm. So, this is the usual norm which you will see and then you see this is the additional term added for this of course. So, this norm is super norm. So, in addition to that you have an extra property you want to whenever you are defining a norm and if you want to recover that property.

That property should be bettered in your norm definition. You do not include that extra property which you are defined in the norm this will not recover that property when you do a limiting analysis.

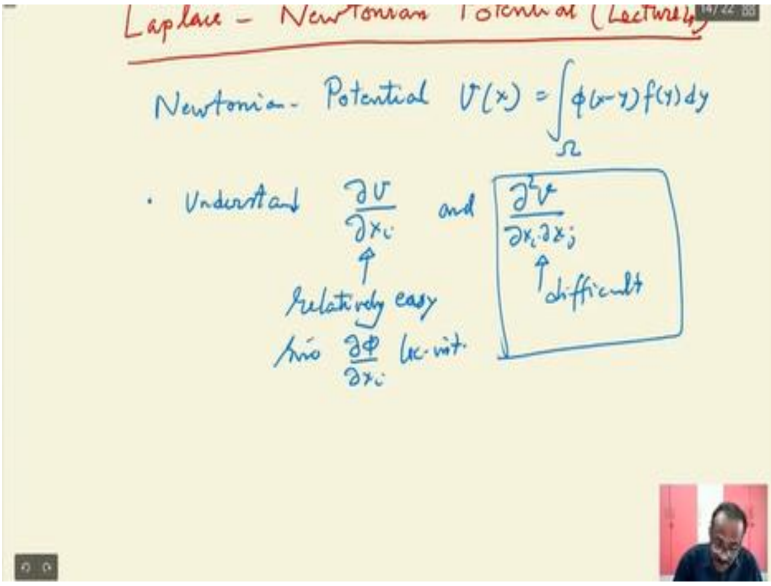
(Refer Slide Time: 02:10)



And this with this it is a Banach space you also define the higher order holder continuous function C k of alpha. So, you are looking for all k times this continuously differentiable functions and kth derivative is holder continuous, of course below that lower derivative it is already derivative exist. So, you do not have to worry about the Holder continuity. At the highest derivative it maybe it will be continuous but it may not be Holder continuous.

So, looking for only those for with the highest derivative kth derivative. All possible kth derivative that is what it is returned by beta here. So, it is returned by beta by for all beta with equal to k. So, the whole kth derivative all possible kth derivative for which you can define norm there.

(Refer Slide Time: 03:05)

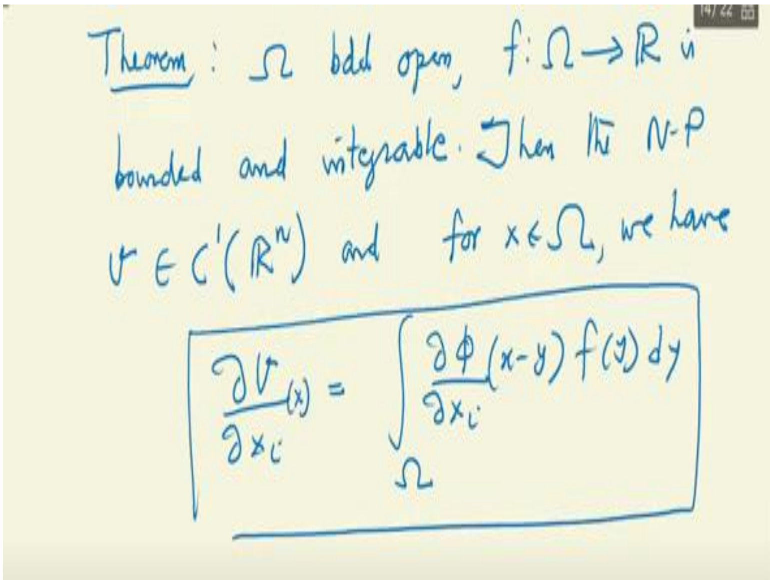


So, now what we are going to do is that so you have two things to be done. So, you have your Newtonian potential, let me again and again called Newtonian potential because that is the main thing, potential V x is equal to integral lower phi of x - y f of y dy. So, you want to understand the derivative. So, we want to understand, understand dv by dx i and d square v by dx i square dx i.

This is bit easy relatively easy because relatively easy is more difficult, the reason is local integrability, relatively easy since d phi by dx i locally integrable. Now you will see there is no differentiability. So, you cannot immediately take that away due to which we have done last time. And if it is differentiable, you could take the differentiation to f not to phi because it is in the convolution form.

So, it is enough to take the differentiation to one of them and f is differentiable you can but here now you have to deal with the phi only. But you have a little extra f I will tell you why it will get motivated later, but that is for this part is enough to this part we will use that Holder continuity part here it is not that important.

(Refer Slide Time: 04:57)

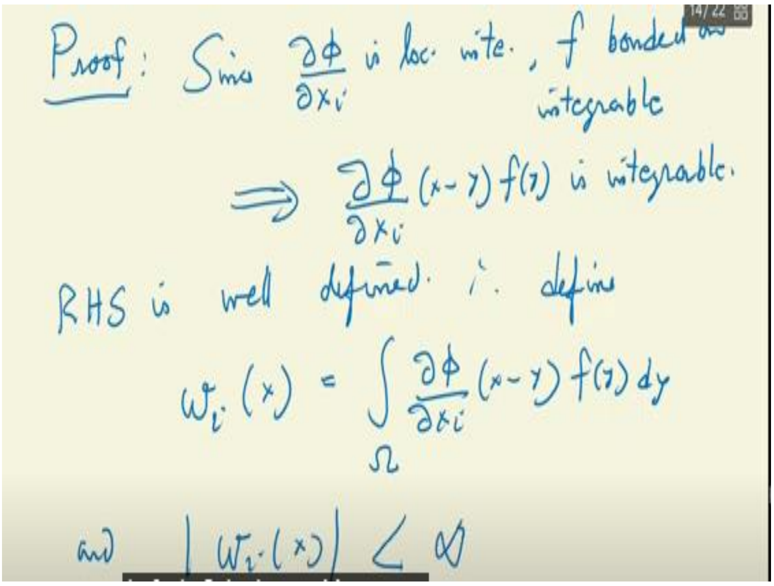


But still, it is not easy because you are to work with the phi. So, you need what is called here is the place, I need regularization. I will explain to you how we do the regularization? So, you want to out this is what in math you will do it. Whenever there is a singularity, you look for a neighbourhood and work try to work outside the neighbourhood and then pass to the limit. That is the technique, we will use it.

So, what we want to claim for our claim is so that they let me take the theorem here. So let me write down the theory omega bounded open as usual open an f from so going to a special case we are considering f from omega to R is so I am assuming not differentiability, but am I assuming bounded and integrable, not even continuity. Then V that then the Newtonian potential Newtonian potential V E C 1 of R N in fact this is defined for all x right.

And for x in omega and we are interested in omega because the x is defined for you look at the definition, the V x is different for all x basically. So, there is no issue but our issue is for all x in omega we have dv by dx i this is what we did it to prove to say this is what we expect because of the local integrity. Let us say x - y f of y dy and I will try to give a proof for this one this is omega. So, this is our optics and this is what you want to prove.

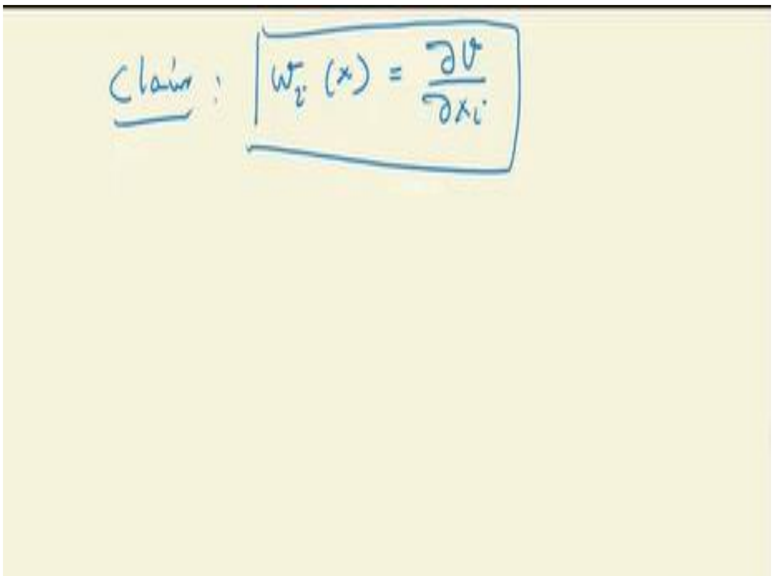
(Refer Slide Time: 07:19)



Let me try to give a proof. First we are to understand that one proof. Since so d phi by dx i is locally integrable is and here we use boundedness f is bounded and integrable. Otherwise to product of two integral function need not be integrable when it is bounded that implies d phi by dx i x minus y f of y each x is integrable that is the first observation. And that means RHS is well defined as a finite quantity.

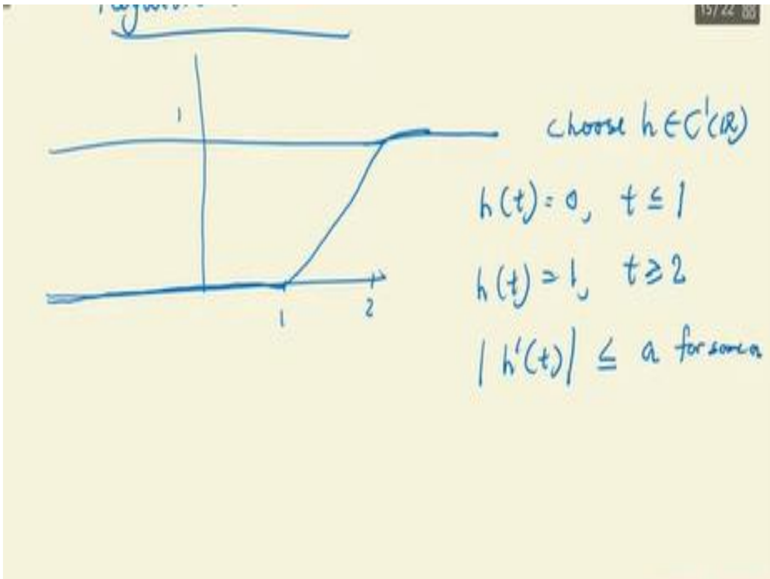
Therefore, define that value we do not know that this dx so we will define the w i of x is equal to integral over omega d phi by dx i x - y f of y dy and one and immediately modulus of w i of x is finite. Integrable means the modulus is finite so it is a well-defined quantity.

(Refer Slide Time: 09:00)



So, you have to show so the claim but you want to prove is claim w i of x is equal to dv by dx i you want to show this. So, here this way now we may require some regularization procedure. So, I will do so let me, so let me do regularization first because there is a singularity at x equal to y. So, I want to remove that singularity, regularization to do the analysis so I choose a function.

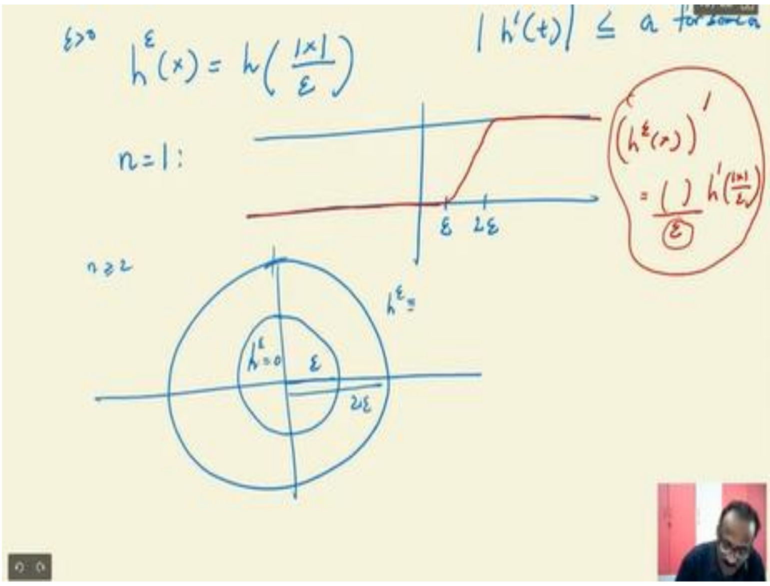
(Refer Slide Time: 09:45)



So, let me geometrically show you so I want to choose a function. This is 1 so this is 1 here 2 here. So, I choose a function which is 0 here smoothly varying and 1 here. So, I am choosing choose h E C 1 of R, h at t it is equal 0 for t less than or equal to 1 and h at t is equal to 1 for t greater than or equal to 2 and h E C 1 and you so you have a bound there. So, you assume your h prime of t so it is bounded, I think.

Because it is a C 1 here 0 here the derivative will be 0 so naturally it will be bounded. So, I assume this is for some a so this can be done.

(Refer Slide Time: 10:55)



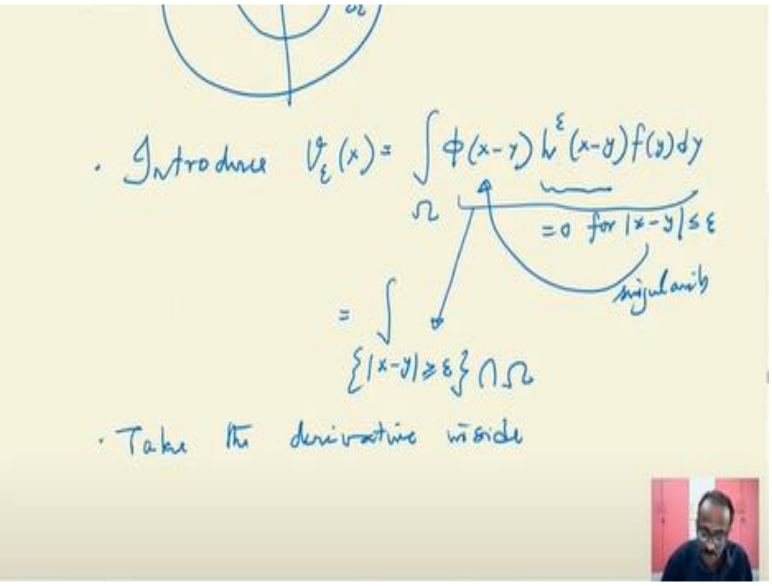
So, with this I get define my h epsilon, so I can define for h epsilon not only in R, I can define h epsilon in R n you will see I will define h of mod x by epsilon. So, when n = 1 how do you get this one n = 1 it looks like this. So, you will have epsilon here this is epsilon positive n = 1 to epsilon. So, it will be like this it will be in that here the derivative if I look at this derivative if you compute h x epsilon prime.

If you compute it will bring some x i is here. So, there will be something here but divided by epsilon. So, you compute this one and h prime so you can compute mod x by that. So, you can do something will be there you will see. So, but then so the derivative as epsilon 2 goes to 0 the derivative even though this is bounded that there will be a factor epsilon here. In higher dimension it will be something different for each derivative you can compute to that one.

So, for higher dimensions how does it look like? So, higher dimensions it will look like n greater than or equal to 2. So, it will be h epsilon will be something like that you will have a ball of radius one and you may have a ball of radius. So, this is one this is two and so h epsilon this is the ball over radius epsilon this is ball over radius 2 epsilon. So, h epsilon is equal to 0 here h epsilon equal to1.

These are all tricks you regularly using PDE and 0 equals 0 less than or equal to h epsilon equal to 1. So, you can do that also because here 0 less than or equal to h they are equal to 1 and smooth. So, you have a smooth compactly supported function. The important thing is that in a neighbourhood of epsilon it will be 0. So, and you know that phi has singularity when x equal to y. So, if you are considering phi of x - y it has a singularity. So, you use h around that point and that singularity remove it.

(Refer Slide Time: 13:42)



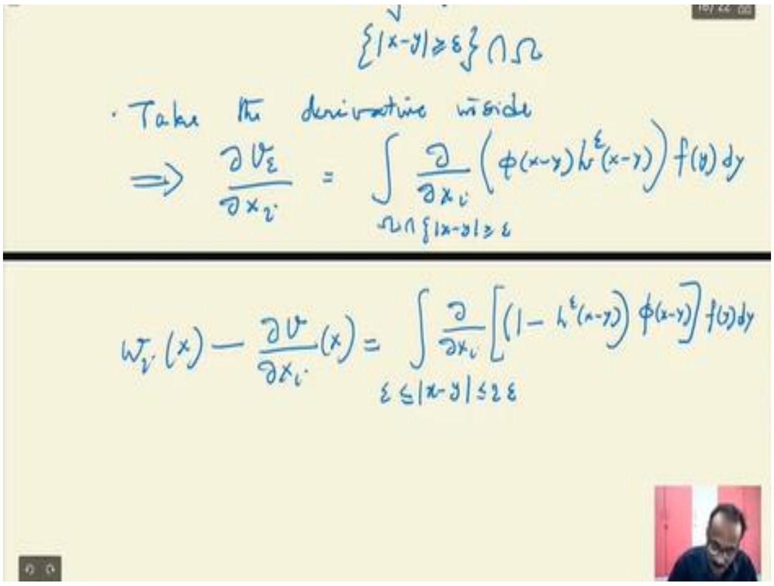
So, with this let me introduce you introduce an approximation. Introduce w i epsilon of x is equal to or I can introduce something else. So, first we will introduce both w not epsilon in order to play epsilon. So, that V epsilon is the problem. So, introduce V epsilon of x is equal to integral over omega phi of x - y is h epsilon of x you see this is where I am using it as h epsilon of x – y, h epsilon I put epsilon up.

So, let me follow that h epsilon x - y f of y dy equal to, now we look at this one this is equal to 0 for mod x - y is less than or equal to epsilon and this is where the place it is a singularity. So, there is no singularity for this function. So, in fact this is nothing but integral of mod x - y greater than or equal to epsilon this set intersection omega and then this is the term here. So, you see so there is no singularity.

So, you can take immediately the derivative inside and inside it is everything is nice, so you can take that derivative inside. Because there is no singularity now smooth derivative inside, and this is a convolution form. So, it is enough to take the derivative here whenever you have a convolution form and if you are taking the derivative or to take the derivative with respect to that and to get me to imply your dv epsilon by dx i.

You see by regularizing I can do that I removed this one, so you want to prove because integral over omega d by dx i of phi of x - y and h epsilon of x - y in case I epsilon I write it below do not worry about it f of y. Now I want to show that this precisely converges to w i which is nothing but my dv by dx i. So, that is the basic game I want to show epsilon.

(Refer Slide Time: 17:35)



So, I want to compute the now w i of x which is defined minus dv by dx i that x is equal to now look at here again as I said there is near the neighbourhood there is no problem it is x - y less than or equal to epsilon greater than or equal to epsilon because it is the same thing. I did not write it here, so it is basically integral mod x - y equal to epsilon. Now when I do subtract this thing h epsilon is equal to 1 for mod x - y greater than or equal to 2 epsilon.

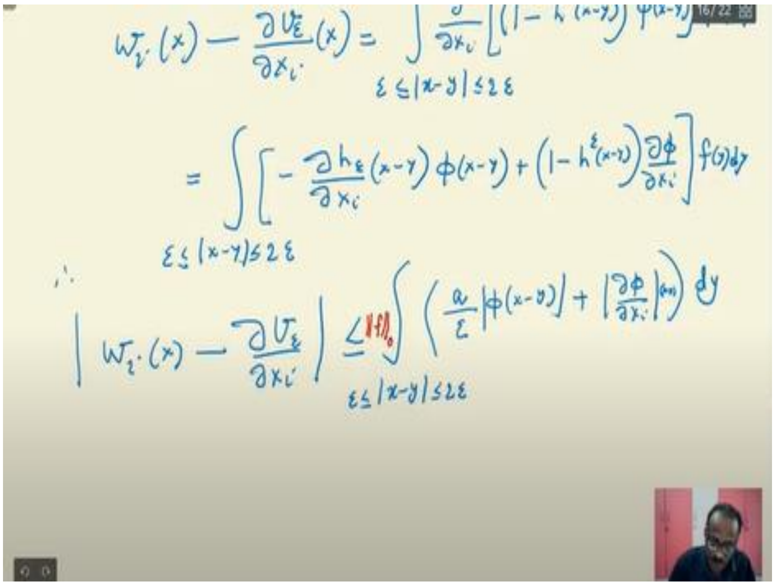
So, you have to understand that h epsilon is equal to 1 epsilon of x - y if I compute this will be greater than or equal this is equal to 1 if mod x - y greater than or equal to epsilon and that is exactly the integral in your w i of x. So, you look at your definition of w i of x is d phi by dx i f of y is that one. So, if you look at it this case this equation so if you work it out, so h epsilon will be one with which is the same as w i.

So, if I do the subtraction for mod x - y greater than or equal to epsilon it will get cancelled because h epsilon equal to 1 which is the same as w i. So, if you do that one so I will get to only this integral over epsilon the singularities is removed away from this also removed less than equal to 2 epsilon this is less than or equal to epsilon. So, you will get this thing only inside that one.

So, you can I can write down so this is exactly equal to d by dx i here it is one only in this interval h is let us says as a as other than 0 or 1 value. So, you will have 1 - h epsilon of x - y.

So, under this this is the little trick you do it regularization phi of x- y this is integral integration is with respect to a differentiation is with respect to x and dy.

(Refer Slide Time: 20:19)



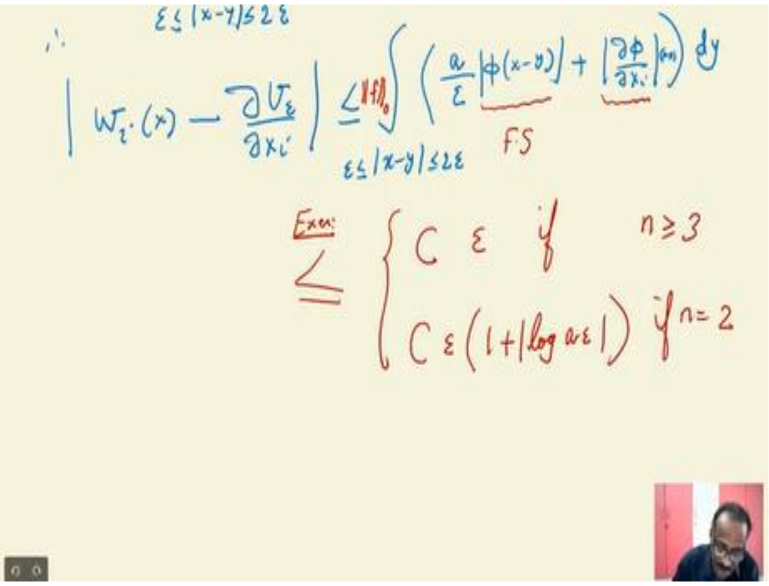
Now we will do an integration by parts here not integration by parts. I will expand that one. If I expands that one of course this, is I will write only this second part this part in weight is 0. So, whether you write it or not it does not matter if you want you can write it but this is not important to write this for because there is no singularity. Now look at here, so I integrate here so if I differentiate this one you will have minus dh epsilon by dx i at x - y into phi of x – y.

And then you will have 1 + 1 - h epsilon of x - y into d phi by dx i there is no singularity. This is equal to your f of phi. Now I will not do you can do a bit of an analysis. So, you use the bounds now see this is a bounded quantity the derivative bound with respect to an epsilon. So, there will be an epsilon coming in the denominator. So, you can estimate the recent epsilon coming on this case this is less than or equal to 1.

So, if I compute this one modulus of therefore modulus of w i of x minus the there is an epsilon over here dv epsilon by dx i if I compute this one this will be less than or equal to integral over mod x - y less than or equal to 2 epsilon you want to put it this also here. And this can be bounded with an epsilon n a because the derivative is bounded thing but when you differentiate it will bring up an epsilon.

So, that will be less than or equal to before that there is also a boundary you can do this one there will be an a over epsilon is and this will be less than or equal to one so it is phi of x – y modulus does not matter because it is anyway non-negative number plus this is less than or equal to 1. So, you will have dv by dx i and then dy this is also x – y. And then f you can take it as a bound inside so I can bring it here that one norm of f at 0 that supreme norm know supreme norm you can bring it here.

(Refer Slide Time: 23:09)

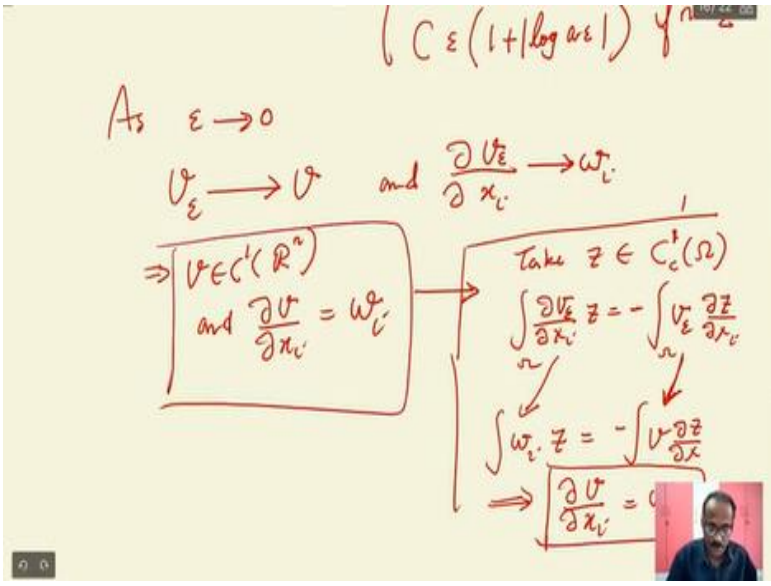


Now look here you know this is the fundamental solution and it has its own estimate and this will have an estimate here. So, this will have an extra epsilon in a small epsilon when you estimate so you have to estimate only in this ball. When you estimate in this ball it will bring up an additional not only epsilon, for example n greater than or equal to greater than 2 it will bring epsilon square.

So, one epsilon will get cancelled. On the other hand, this will bring only one epsilon. But then there is nothing epsilon to cancel, so this will bring thing. So, this I will leave it as a small exercise for you which we have done it essentially in the PDE course you can prove this is less than equal to a constant this will be a constant. So, this will bring in epsilon extra. This will also bring this will bring in n greater than or equal to 3.

It will bring some epsilon square so that will bring epsilon and this will bring in epsilon if n equals or greater than 2 but n equal to 2 it will be something like a logarithm. So, you will have epsilon so there will be little less than epsilon. So, because one plus log but this is a 0 quantity which is log a epsilon you can exactly calculate this if n = 2 here you see.

(Refer Slide Time: 24:42)



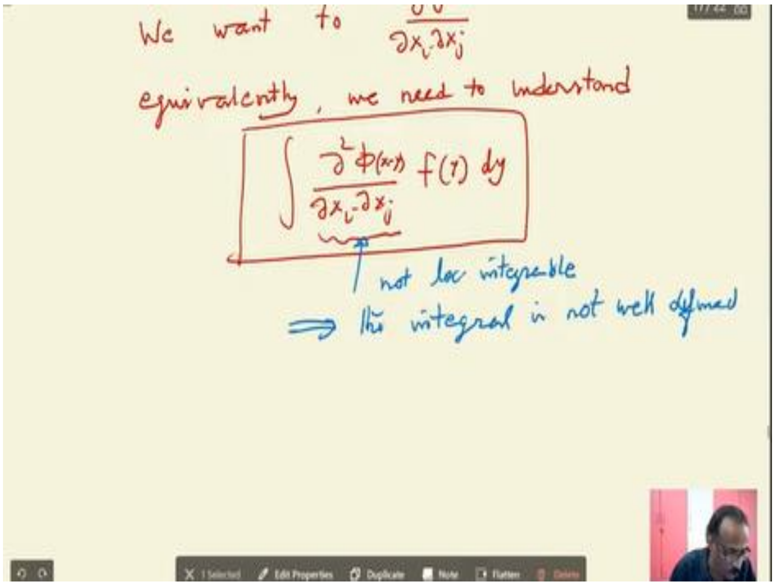
So, as epsilon goes to 0 so therefore you have the correct estimate these are the estimates. So, as epsilon goes to 0 you already know that we epsilon converters to v that we know it and now you are also proved that you are dv epsilon by dx i will converge to your w i. This will be from here you can actually get it maybe I can do a little thing from here it this will imply your v is in C 1 of R n basically in omega a whatever it is.

And that you already know and dv by dx i is nothing but w i to prove this what you do is that take a say some test functions and workout. You will take some what you say z in d omega this is how you prove that things. You will take z in one thing and then you can compute your so this is dv epsilon, so you compute your dv epsilon by dx i z this is by integration said this compactly supported this will be v epsilon d sub by dx i.

Because it is a d omega function or this, I have not introduced is a seen C 2 function you see. You do not need this let me not introduce this one. So, you have a choice continue once is will do even one will do once can compactly supported function in omega. So, you have this one so there is no boundary through but v epsilon converges to V so this converges to minus integral over v dz by dx i and this you have shown that this converges to integral lower w i z.

So, you have this equality which will imply dv by dx i = w. So, you have proved the theorem here. So, what we proved is that basically we have proved via regularisation w i = dv by dx i that is nothing but dv by dx i you can take the differentiation. So, it is only the kind of regularization that we will do. But then the problem with the second derivative is not locally this we could do because it is our own local integrable function. So, there is no issue, so the just regularization you could manage it.

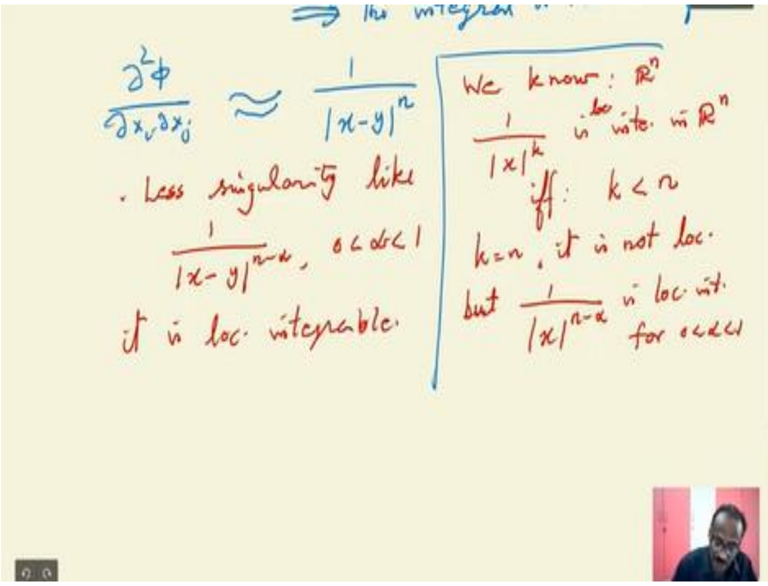
(Refer Slide Time 27:40)



So, let me give us a heuristic argument before concluding like this lecture some why Holder continuity now. Why? So, I will just give why Holder continuity and then I will stop here Holder continuity a heuristic argument. So, what do we want to do it? So, we want to understand this term. That is just like previously we want to get d square v by dx i dx j equivalently we need to understand second derivative insight.

That is what we need to understand integral d square phi by dx i dx j f of y x - y f of y dy. So, this is where you want to understand. But this is not, so if use this one not locally integrable that implies the integral is not meaningful, the integral is not well defined.

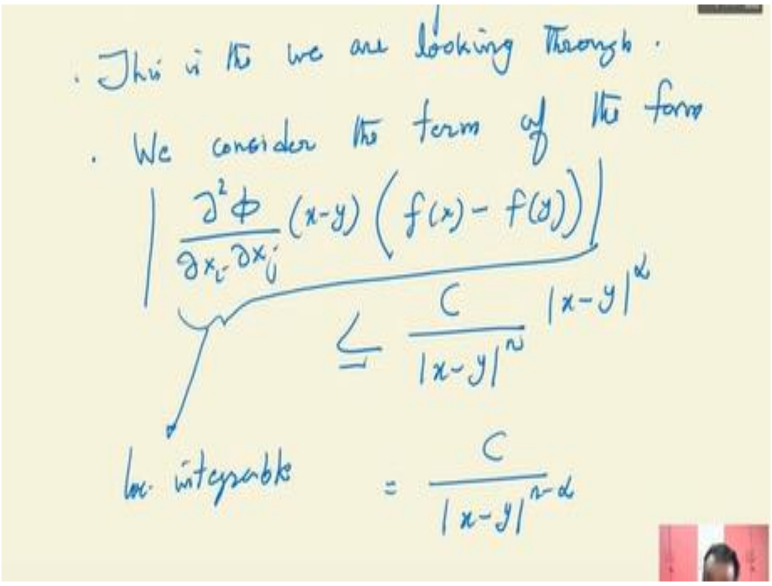
(Refer Slide Time: 29:50)



You will see that is where we have an issue because we know that d square phi by dx i dx j behaves like singularity is behaves like one by mod x - y power. And you know already that this is where you are to make a remark here, we know one by in R n one by mode is let me take the singularity at the origin. One by mod x power k is integrable in R n or locally integrable only singularities integrable in R n.

So, let me take local integrability if and only if this is a different only if condition when k is strictly less than n. So, k equal to n is not locally integrable. But if you anything less little less suppose one by mod x power say n minus some epsilon some alpha, so is locally integrable for integrable for 0 less than alpha. So, if there is a little less singularity anything less singularity like 1 by x - phi power n minus say alpha 0 is less than alpha when it is locally integrable.

(Refer Slide Time: 32:02)



And this alpha is what this is the alpha we are looking through this is the alpha because we cannot remove the singularity from phi. So, this is the alpha we are looking through, so we basically saw what we have to think so we considered that term basically so to play with it. So, we consider the term of the form d square phi by dx i dx j at x - y into f of x minus f of y. So, if you try to estimate this norm this will give you some constraint by mod x - y power N.

And this will give you some mod x - y power alpha which is same as constant by one buy of course you have to apply with it mod x-y power alpha. So, the extra singularity just at the boundary given a produced by your fundamental solution is what is to be compensated from this source to have eventually the existence of the solution for your potential theory. So, this is locally integrable, so you see so this one together is locally integrable.

So, we will do more on this one and we will eventually prove your final theorem in the next class. Thank you.