First Course on Partial Differential Equations - II

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Lecture - 03
Laplace-Newtonian Potential

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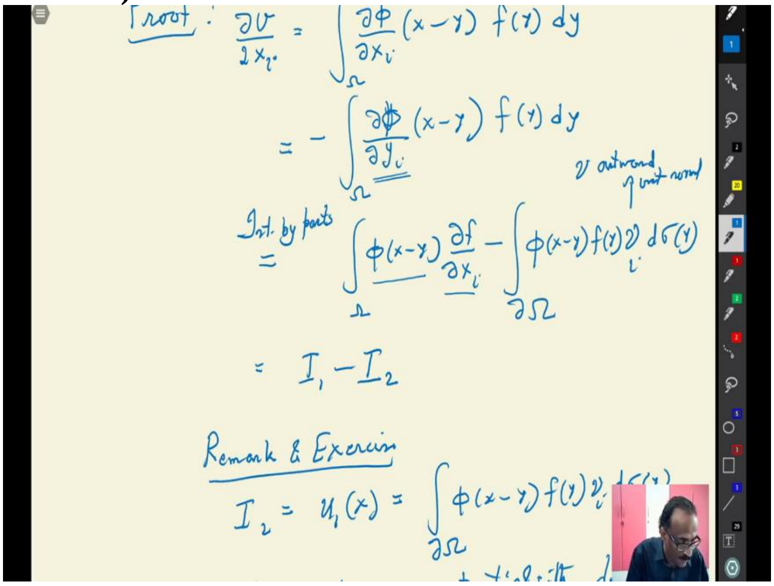

Good morning. So, in the last class regarding the Newtonian potential, we have seen that in general continuity is not enough to solve the problem potential theory problem mainly Laplacian of u = f and what we are going to see now, when f has a good smoothness like C 1 of omega bar then the v x defined there actually defines the it gives you the potential So, it gives the solution so, you can solve your thing.

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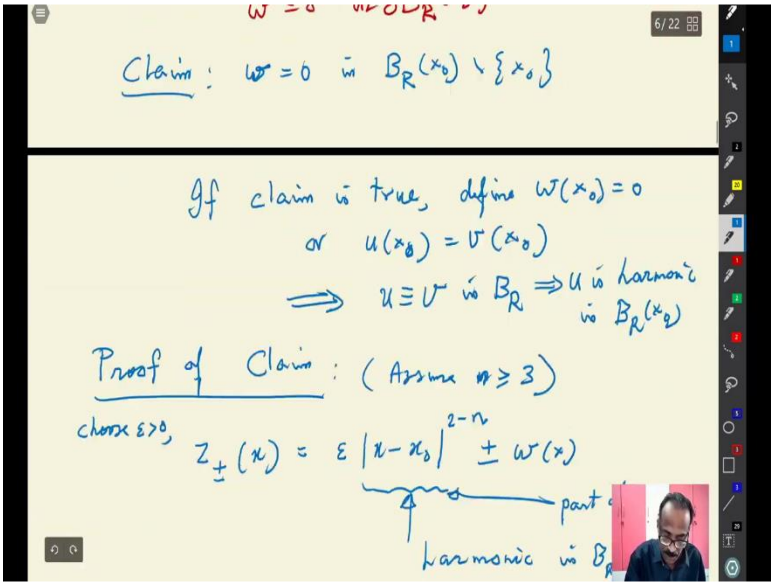

So, we have defined the let me recall that before going further that one your v x that is whatever is our v x is different is not seen here maybe v x is defined this is the Newtonian potential defined in terms of that. So, let me recall the first the Newtonian potential v x and then we will show that if you have enough smooth enough, so, Newtonian potential v x = integral over omega phi of x -y f of y d y.

So, we recall once again so, let me state the theorem and prove it today. So, that is the first job to prove these things. So, here is the theorem. Let f is in C 1 of omega bar, where omega is bounded open in R n, then v defined as above satisfies v is in C 2 of omega intersection C 1 of omega bar so and minus Laplacian of v. So, we will not discuss the boundary condition which will do it later. But we are saying that it satisfies the Poisson’s equation.

So, this is the Poisson’s equation as we know that we the difficulty is taking the second derivative, that is the first derivative, you do not have problem because the d phi f phi and d phi / dx i are locally integrable. But then to take a second derivative, it is not allowed because d square v by dx i dx j is not locally integrable. So, that is what we need to do a bit of final analysis. So, that local integrability lack of local integrability of the second derivative compensated by the regularity of f. So, let me give you a proof of this proof is interesting.

So, you compute this so taking derivative inside as I said is not a big issue dv / dx i, I can take inside, because this is locally integrable which we have already seen in the last class how to take the derivative here, x -y f of y dy, now what we do here, you do an integration by parts, because f is a C 1 function, you can take this derivative d by dx i to the other side, so you can immediately solve with the minus sign, you have phi of x -y df / dx i.

This you could do it only because f is a C 1 function and then there will be a boundary term. So, we will have a boundary term -df / dy, no there is a minor problem here, this is an xi this is a y variable. So you have to be bit careful in that you have to change the variable first, before doing this integration. So, you want a differentiation with respect to y, because f is a function of y. So, of course, this is a symmetry you have it is a modulus thing, phi is a modulus function.

So, I can change this function with respect to y, but when I change with respect to that function with respect to y, you will have dy into x -y so, because if I change this with respect to y a minus sign will be picked up. So, you do this. Now do the integration by parts so, this is equal to integration by parts that divergence theorem, you apply divergence theorem, because minus will become phi of x -y, these are all in omega. So, you can do these omega d f / dx i and then there will be a boundary too.

So that will an external downward derivative f of x - y f of y nu i, this is the corresponding outward nu is the outward unit normal which are all you are done many times outward unit normal and this is our d sigma this is our boundary of omega and I call this is an integral I 1 and this is I call it I 2. So, before proceeding phi further, let me make some remark, I will make continue here, but I want to make some remark and also an exercise remark and exercise, the I 2 this is I 2 term I 2 I will call it as a new function.

So, I 2 which is a functional, this is called some say u 1 x, let me denote by this notation u 1 of x integral phi of x - y f of y and then you have your nu i of d sigma y and this is d omega, this is called the single layer potential is called these are all physically relevant quantities, you see it is the boundary term boundary effect of your potential with respect to f the single layer potential and there is also with the density f and there is also a double layer potential, which will also you will come across I may or may not do it.

Let us see double layer potential which we denote by u 2 of x that is the with respect to derivative these are all locally integrable function and hence there is no problem d phi / d nu j so, you are differentiating this is the normal derivative at x - y. So, this is the boundary values with the nu i coming this is the grade phi dot nu so, you know that this is equal to grade phi dot I cannot call it because d phi / d nu j.

So, I am nu by so, you have to be careful this is with respect to y I am doing not nu j. So I am differentiating this is grade phi dot nu y, y is to represent that normal is with respect to y, x is the kind of parameter d sigma. This is called the double layer potential.

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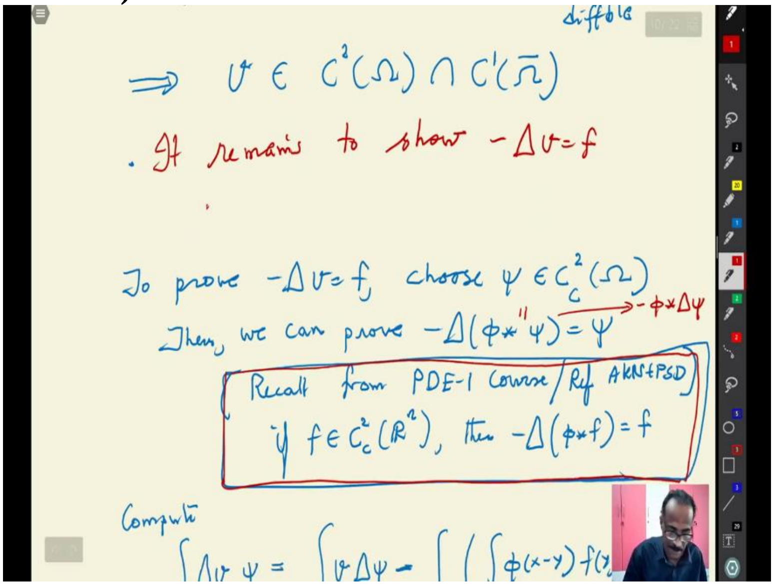

So, what is this is among and what is the exercise is these are all smooth functions C infinity functions, that is because these are all locally integrable and then y varies from the boundary and if x is not on the boundary, you can differentiate as many times as you want. So, if this is defined for every x. So, u 1, u 2 belongs to actually very smooth C infinity of R n minus d omega. So, that is it.

So, that is exercise and remark, so, we will come back to the earlier case. So, this is the single layer potential, this is a smooth function and phi is a again a locally integrable function. So, you see, so, now the derivative has gone there, the trouble was there when there is a derivative. You cannot take a differentiation in further differentiation second differentiation inside, but now there is no derivative here derivative is taken here, you can take the one more derivative here.

So that means you can actually show that the second derivative exists, so if you differentiate, so, the I 1 is also differentiable that is what you are trying to say that so, therefore I 1 is differentiable and by exercise so you see why that differentiability coming because there is no derivative now. So inside you have only one derivative to take here you can take the one more derivative to talk about the differentiability we could do that because of the differentiability of f.

So, I 1 is differentiable and by exercise I 2 is also I 2 = u 1 is also differentiable and first derivative is already there. So, this will imply your v the Newtonian potential is a C 2 function. Now, of course, you have the boundary up to boundary. So, you will see your C 1 you will have it up to the boundary. So, that is not a problem after the boundary in first derivative you can show that so, it remains to show now.

So, let me it remains to show minus Laplacian of v = f this requires so will go back this requires regularization, because phi has singularity. So, to do that I will do that and here the exercise has something more you want you to belongs to and they are also harmonic that is what Laplacian of u 1 = Laplacian of u 2 = 0. Of course, in this space only then you can do that. So, it is all harmonic functions in R n - d omega. So, you have some extra things to be done.

So, now, we have to recall an earlier result. So, you have to prove - Laplacian of v = f, choose phi C psi choose psi to which is C 2 function or C infinity function C 2 in a C c with a compact support in omega. Why did I choose this one then we can show that this is a result which we have actually proved in the PDE 110 it is similar proof you can work it out as we can prove it solves this problem mine the convolution of if you take phi with respect to star is equal to psi same proof.

So, you recall from PDE 1. So, follow the same proof recall from PDE 1 course or from reference our reference AKNTPSD power book; see that. If f is in C c 2 of R n so, we have proved if this problem when omega is in R n, the similar problem if you work we have used only 2 facts, we have not used any property of our whether it is bounded or unbounded, but we have proved that efficiency C 2 of R n essentially f is supported in a bounded combined set.

So, you use the twice differentiability of this is a more so, the result we proved in PDE about the existence of solution with the compact support and twice differentiability what presently we are proving with C 1 of omega bar now compact support nor only it is twice differentiable, so, that is why but, for this I am choosing as a test function for which, then we have proved that minus Laplacian phi star of f is solves the PDE this is what we have done.

So, we have done actually this in the last class when you have a, the same proof instead of R n you work, we are not used any special property of R n or something like that, you need compact support and twice differentiability. With this now we so you choose this function. Now we do a simple computation. So, you now compute because you are to show that Laplacian of v equal to - f or - Laplacian v is equal to f, so I act with this is always a trick omega.

Now, look at this one, this is compactly supported, so, you can take differentiation to other side as many times as possible, because it is twice differentiable. So, 2 times you can take it here, because psi has a compact support. So, it will not affect anything. So, you will have and 2 times you are taking this will be Laplacian of psi over omega now you replace this v. So, you have omega and v is integral over omega phi of x - y, f of y dy, by the definition and Laplacian of psi of x dx.

You apply Fubini theorem interchange the integral everything is in a good condition. So, you do this one so, I interchange this interval so, f i will come out and you will have integral of this will come out f will come out this will come so, x this also has x so, it will be phi x - y Laplacian of psi of x dx of f of y dy, but what is this one. So, this is nothing but Laplacian of this is nothing but phi star of Laplacian of psi of x and f x this is up to y.

So you will get because you are integrating with respect to x so, it will be y f of y dy, but this is psi itself that is what you say that you look at here Laplacian of phi and look, let me do this one these are all we are done here, this is same as because on the convolution, it is enough to differentiate on one of them. So, this is nothing but Laplace - phi star of Laplacian of psi, because the differentiation only takes in one of them for the convolution.

So, this is nothing but psi of y this is equal to integral of psi of y f of y dy. So, now, you see that what is the left-hand side, left hand side is Laplacian of v psi over omega. So, this is true for so, this is f of y dy, so, that immediately implies this is true for all psi in C c 2 of omega that implies Laplacian of v = minus Laplace so, there will be a - psi here so there is the - psi, because earlier in this case, there is a minus.

So, this is minus Laplacian. So, therefore, minus Laplacian of v is equal to - f in omega that is the proof. So that proof is immediate that is, so, there is a fact we are using and this part we are actually using it and this requires a proof which we have done it anyway, in our earlier PDE course otherwise, it is a regular one thing that regularization is not here now, we do not need any regularization. So, let me move this, so, now, we do not need a regularization will come. So, we have done when f is continuous, so, let us slowly go f is continuous, because that may not be true, but f is C 1 result is true,

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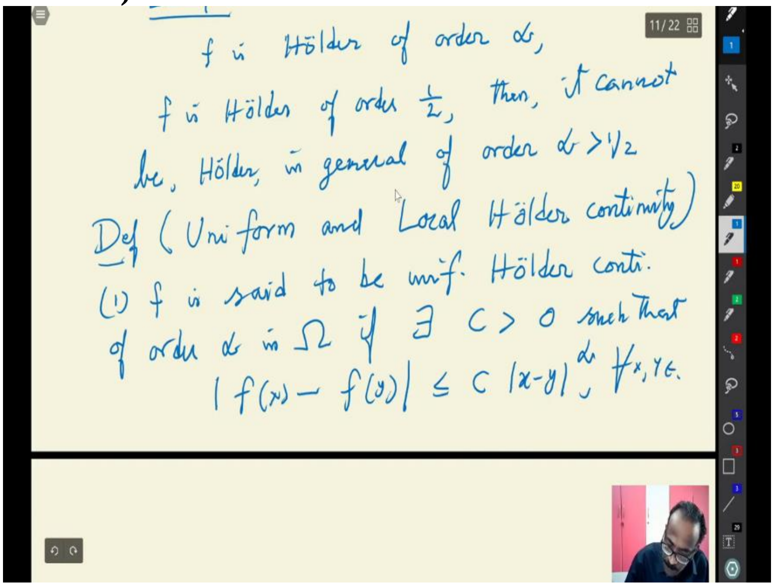

Now, we are going to have an in between what is called a Holder continuous functions. So, we are going to define what are Holder continuous functions?. So, you want the smoothness not as strong as C 1, but then not conduct a little more than continuity, that is provided by Holder continuous function. So, I will define that I will give you the definition and then we will see why a heuristic R n why this will work? What is the reason for looking for Holder continuous functions?

So, the reason is the continuity of f is not enough, this is again I am repeating many times repeated continuity of f is not enough not enough, but differentiability is too much differentiability is too restrictive. So, therefore, looking for smoothness stronger than continuity, but the weaker than C 1 that is fully and I will know how do we look for it will be motivated to you later, but before that, let me give you a definition of continuity.

Let 0 less than alpha less than 1 and we are always omega bounded open in R n. So, you have your definition Holder continuity a function f from omega to R is said to be Holder continuous at x now, said to be local Holder continuity uniform builds defines said to be Holder continuous at x 0 Holder continuous at x 0 in omega, if there exists a positive constant such that f x - f x 0.

You would have seen this in a special case already called Lipschitz continuity less than equal to constant into mod x - y power alpha x - x 0 power alpha for all x, this is for all x of omega, if alpha = 1 this is defined for alpha less than 1, but you can define for alpha = 1 then f is known as f is Lipschitz then you see this is something weaker than even Lipschitz in general. So, because when alpha becomes smaller and smaller, you know, this is a weaker quantity that because we are more interested in nearby thing.

So, when alpha =1 we call it a Lipschitz continuous function. So, the typical example we are looking this type of thing so you see f x = mod x power alpha then clearly F is Lipschitz of order alpha suppose 0 less than alpha less than 1, then f is Holder of order alpha, but it cannot be Holder with bigger alpha. So, suppose f is Holder of order half, then it cannot be Holder more than half then it cannot be Holder in general I am telling at this function Holder in general of order because it has more smoothness Holder of order of alpha greater than 1.

So, for this case so if you have order but there is no differentiability here and you have seen that when you have a Lipschitz continuity that means f x = mod x with alpha = 1 and then it is Lipschitz but still it is not a differentiable, but you have already seen in our earlier lectures when a physical process it is differentiable almost everywhere, but for order Holder with order alpha with alpha less than 1 you will not get differentiability more than half I said it is not the 1 it is half. So, these are the typical functions of Holder continuity.

So, you need some sort of a growth like mod x power alpha that is what you are looking you can control your function with the functions of type mod x power alpha. So, it is definitely a little more than the continuous functions because continuity will not give you any estimate because it cannot be so, you have other definitions uniform Holder continuity definition, uniform and local Holder continuity, f is said to be uniformly continuous in omega.

f is said to be uniformly Holder continuous of order alpha in omega if there x is C positive such that modulus of f of x is 1 - f of y is uniformly unique less than equal to constant into mod x - y power alpha this is true for all x y in omega.

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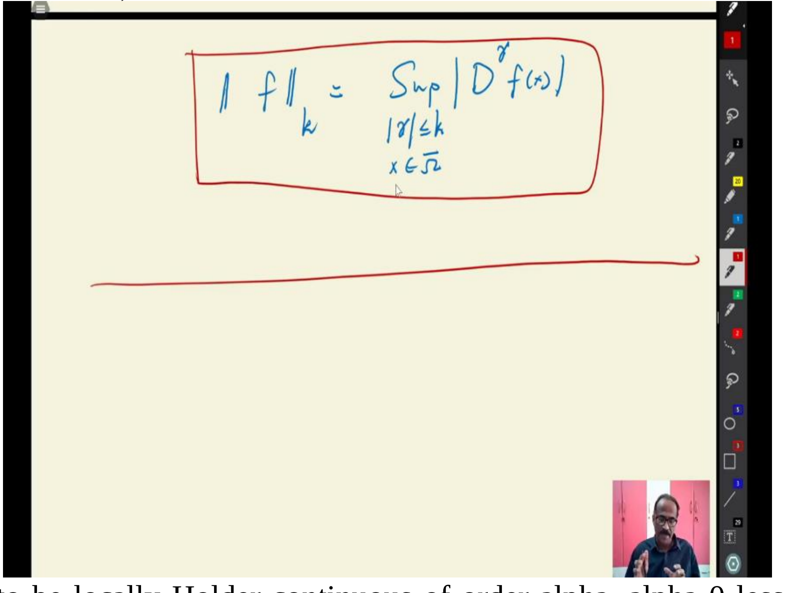

And 2 f is said to be locally Holder continuous of order alpha, alpha 0 less than alpha less than 1 that is what of course alpha = 1 also, you can define Holder continuous of order alpha if f is uniformly Holder continuous of order alpha in every compact not in omega, but in every compact subset of omega. So, that is what is called a f is uniformly. So, you so, if you have a domain omega, so, you take anything so, you have the same constant uniformly.

So, it is each compact set you have the Holder continuity, then it is called the local said to be locally Holder continuous. If it is unique this is uniformly Holder continuous, the first one is the definition of so, now, we will denote this one. So, notation is important, these are some important spaces notation. So, you can take even 0 equal to 0 also get it is a better alpha you can also take less than or equal to 1.

So, you get Lipschitz continuity. So, see, so, in general here we take 0 less than alpha because, more stronger when alpha = 1, so, you do not so, you have the continuity so, that is a first remark every Holder continuous function is continuous and every f is in C 1 of omega bar then the f is Holder continuous. So, you see that is just a mean value when it is in C 1 of omega bar, then d f by dx i is Holder and you can apply mean value theorem to get estimates in get a you get Holder continuous.

So, you in fact, you can get the Lipschitz continuous that you already proved that Lipschitz continuous, so, have more strong results for that. So, now, what do we so, we denote C 0, this is to represent continuity this is to represent the power of omega bar be the space of all uniformly Holder continuous functions and then you can make it a Banach space I will not prove the Banach space?

So, introduce the norm to be denote by norm with 0 alpha if you want but norm for u in f in C 0 of alpha norm of f in 0 alpha is by definition denoted by you take it supremum node, that is the continuous norm and then it has an extra property f x - f y is bounded by x - y for all x and y and x 0 = y. So, if you divide this one this quantity is a boundary quantity by a C, because by definition this one so, you take for all supremum but x naught = 1.

So, this is norm f where norm f at 0 is equal to supremum of modulus of f of x, where x is in this domain omega bar. So, you have that quantity and then the actually this is a Banach space. C naught of alpha of omega bar is a Banach space. So, I will not prove here Banach space actually you can define other spaces in fact, you can define a more general higher order Holder spaces C k alpha k is an integer where k is a integer 1, 2, 3.

So, you look for all functions in C k functions for which that C k derivative is Holder continuous? So, C k alpha get familiarize with this before we proceed for the set of all f in k times continuously differentiable function, but then omega bar. So, but then you need the kth derivative or totally kth derivative D power beta. So, let me write it D power beta of f is in C 0 alpha of omega bar for all mod beta this is multi-index notation.

So, these are not all the kth derivative or Lipschitz continuous and you can define your norm f in k alpha is equal to so, you define the kth norm of that one so, this is the norm k and then you define summation and you have D power beta of f. So, you are 0 alpha it is thing for all mod beta = k. So, these are all the Holder norms of our order alpha not infinity order alpha and this is for all kth norm. So, what is your kth norm? You can define f at k is all the derivative the supremum of this is multi-index notation D for alpha of f of x.

For all mod alpha less than or equal to k and let me know to use alpha here mod beta or gamma not gamma less than equal to k and for all x in omega. So, this is called the kth Holder spaces all these spaces are useful in the study of potential theory and later Shrouder theory. So, at present I will stop here and then next week we will continue with the main result which to be proved. So, we will try to understand the derivative there are 2 major theorems which will eventually prove the solvability. Thank you.