First Course on Partial Differential Equations - II
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Lecture – 02
Hamilton Jacobi Equation

Welcome to the set of lectures on Hamilton Jacobi equations. So, we will be discussing some aspects of Hamilton Jacobi equations which have applications from calculus of variations, which is a pretty old subject. It has the more advanced topics like Hamilton Jacobi Bellman equation and Hamilton Jacobi Isaacs equation. We will be doing very little in this course, because he said nonlinear type equations has lot of literature and lot of modern techniques are available to study the general Hamilton Jacobi Bellman equations.

Our main aim is to derived 2 things, to study 2 things, one is representation formula called Hopf-Lax formula. And then we will do that of derive the Euler Lagrange equations corresponding to that and then we also connect to what are the 2 terminologies, what are Lagrangian and Hamiltonian these terminologies are drawn from the usual classical mechanics and we will give a connection between the Lagrangian and Hamiltonian why are that let that transformations?

We will also introduce the concept of some weak solutions and nonlinear equations we will give examples will not produce existence of solutions of times there will be multiple solutions. And then you are to choose when you have multiple solutions, you have a right notation of solutions and right conditions to pick up the correct solution. So, we will be doing this in 3 hours and 5 to 6 hours and we will do the minimum things, but we advise you to go through more about it. So, Hamilton Jacobi Bellman Equation.

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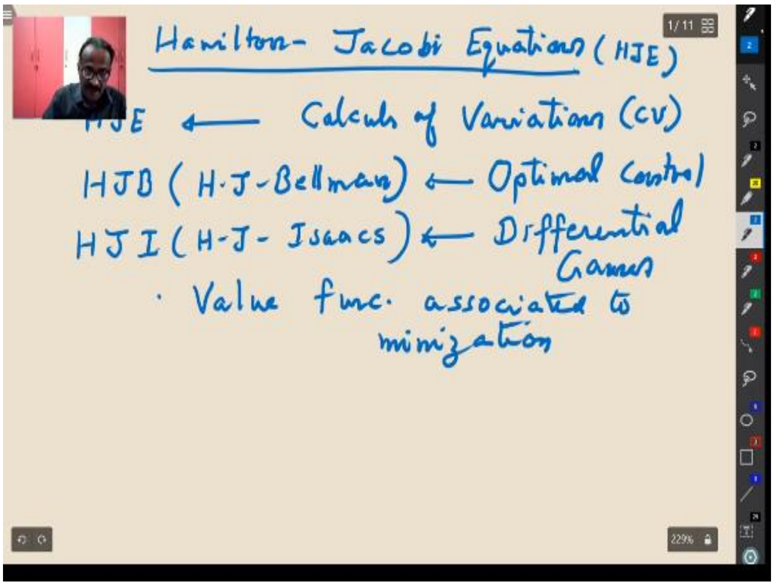

Hamilton Jacobi equations, so we will write a short form what is Hamilton Jacobi equations for as I said, Hamilton Jacobi equations is connected to what is called one of the most classical subject called calculus of variations. And as you know, in olden days, people found that many of the important problems comes from the minimization, you will see a lot of examples in this course, you minimize it and you study the minimization problems, it leads to HJE called Hamilton Jacobi Equation.

As I said, there is an advanced topic Hamilton Jacobi Bellman that is called Hamilton Jacobi Bellman. And this comes from what are called optimal control problems. So, as I said calculus of variations to minimize our trajectories, but in optimal control, the trajectories are determined by the control. So, you apply control and that trajectories are designed or can change the trajectories based on the optimal.

So, there is a kind of one player game and there is a more, much more general notion and called Hamilton Jacobi Isaacs equation, this comes from what are called differential games theory. As I said, there will be multiplayer game, so it is not just one person controls that trajectory. Basically, there will be 2 or more controls and there is a lot of literature in Hamilton Jacobi equation thing, the thing is that there is what is called an associated value function associated to minimization problems.

And this value function basically satisfies the Hamilton Jacobi equation or Hamilton Jacobi Bellman equation or Hamilton Jacobi Isaacs equation, as I said you will see some of these things. And so, as I said you will see more and more equations.

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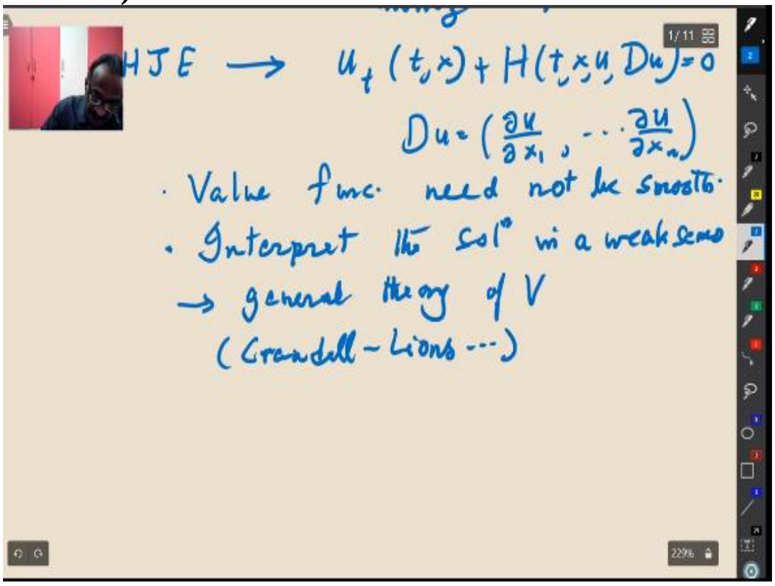

So, typically a Hamilton Jacobi equation is a nonlinear equation of this form u t of t, x + H of t, x, u is something like Du. Du is that derivative basically, Du is the derivative with respect to u du / dx 1 etcetera du / dx n. We will not as I said is equal to 0. So, we will not be studying in this generality and I said that quite often value function need not be you may understand what is value function; later, value function need not be smooth.

And you hence you have to interpret the solution in a weak sense and there is a very general theory developed in the 80s by Crandall Lions there are many people. So, Crandall Lions etcetera. So, they developed a theory of viscosity solution we will not discuss that here. And this viscosity solution is also extended to general nonlinear equations not necessarily Hamilton Jacobi equation.

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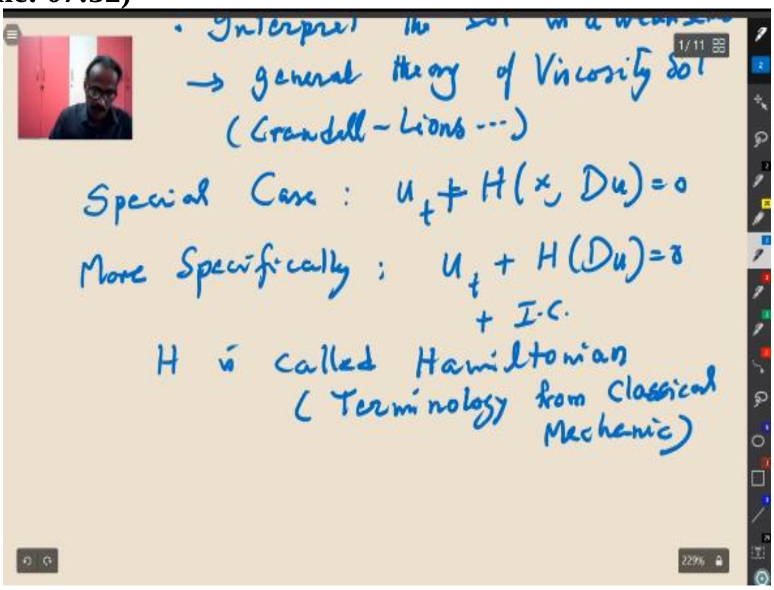

So, we will not be studying this equation study the special case, where this does not depend on u t + H of x Du = 0. So, there is no dependence explicit depends on dense of t or u there and we more specifically we will also avoid, so certain results we will do when there is a x dependence on this H that more specifically. So, we will be considering this very special case H of Du = 0. And or course there will be an initial condition and this H is called Hamiltonian, this is a terminology from Newtonian mechanics or classical mechanics.

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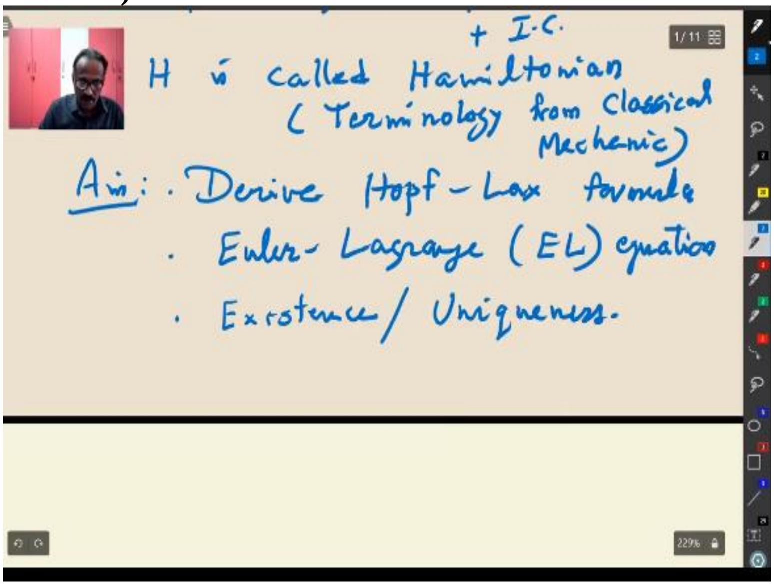

So, that is what we are going to be our main aim as I said there derive Hopf-Lax formula which will do it in next class Hopf-Lax formula and then we will also want to study Euler Lagrange equations. Euler Lagrange is called E L equation and as I said earlier, we will also see some sort of transformation which we will explain to you as we proceed further. And then maybe some indicate the existence uniqueness. So that is what we are going to see.

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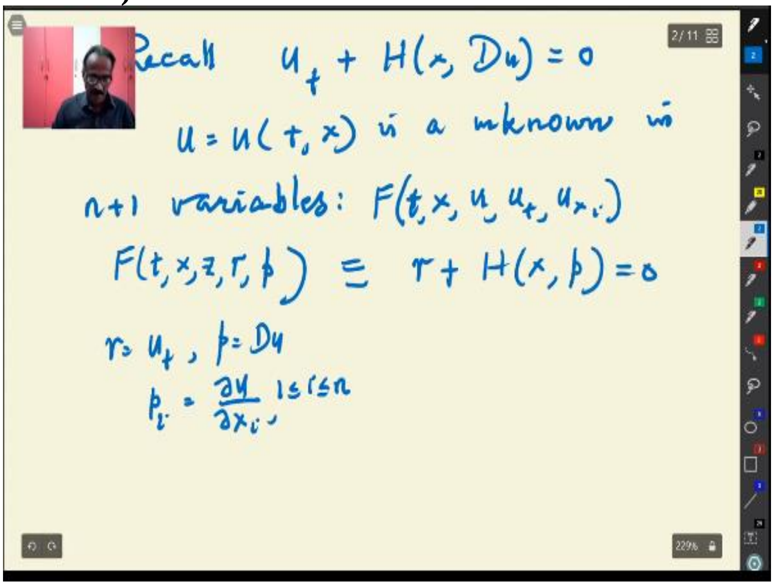

So, let us look at this equation now, recall equation u t + H of x Du = 0 this is in general is a nonlinear equation, but it is a special form of this nonlinear equation u = u of t, x is an unknown in n + 1 variables. So, if you put this equation in a very general form, we can write it as F of t, x. So, that gives you n + 1 then you will have u, u t then u x i for all i equal to x is also a vector, where x is also a vector n vector. So, x is equal to so, this is what you are having.

So, if I put the notations here in terms of my method of characteristic, so, if I introduce this notation F t, x, u, z, r, p this is where r = u t and p is equal to your Du that means, that is p i = du / dx i. So, this is a multi-index notation 1 less than or equal to i less than or equal to n. So, you see this will be of the form r + H of x p that is what x p = 0.

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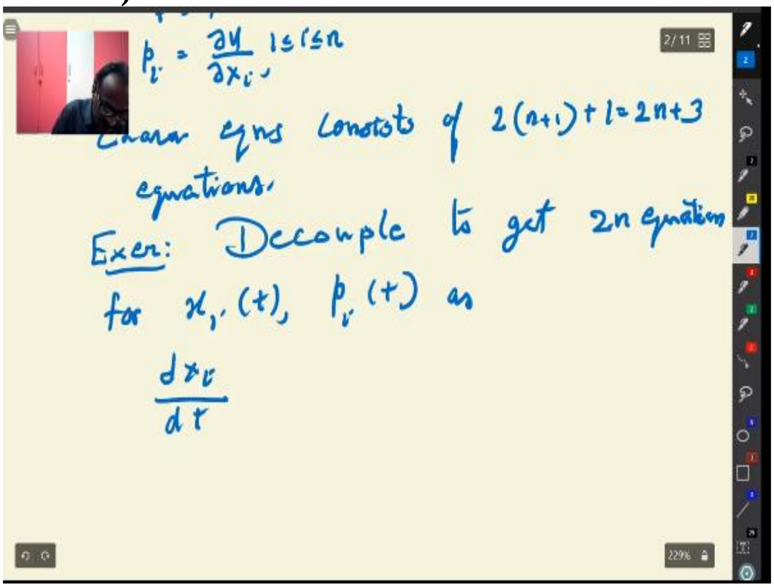

So, since it is n + 1 equations, the characteristic equations consists of 2 into n + 1 + 1 = 2n + 3, a characteristic consists of 2n + 3 equations but this is not very general second order equation, this is an equation you see in very special form you have that one equations. So, hence, you do not have the explicit dependence of all the variables. So, I believe it as a small exercise you can decouple to get 2n equations for x i of t and p i of t.

The characteristic equation you introduce a variable t and previously you have a parameter and then you can consider that the tau variable will become the t variable itself equations as dx i / dt this is your.

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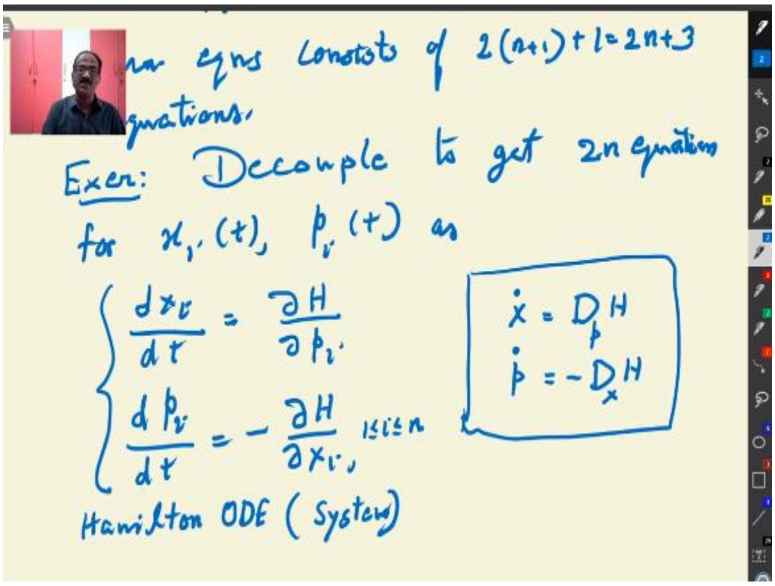

So, this I will leave it as an exercise to derive is equal to your equation you have dx i / dt is equal to dH / dp i and then dp i / dt = - dH / dx i. So, this is a 2n set of equation and this is called Hamiltonian ODE or Hamiltonian system. This is again have importance from classical mechanics, what we are discussing is a very general case but in when you have a special Hamiltonian which is nothing but the Hamiltonian in classical mechanics is nothing but your total energy we making that example.

So, in compact notation we will write it in this form, so we may not write it so, these are all for 1 less than equal to i less than or equal to n so, you have 2 n set of equations. So, eventually we will be writing x i dot is equal to the derivative let me not even. So, we will write x dot is x dot means x 1 dot etcetera x n that is equal to the derivative of H with respectively. So, we are going to eventually use this notation so, you will see it is a function of x and p.

So, you are using that one and you are p dot = - D x of H so, this is a compact notation for the same representation. So, we will be using this notation throughout this class. So, I suggest you to go through these equations and derive this using the method of characteristics which we have explained to you in the previous class in the previous PDE 1 course or you if you will be familiar with already such equations, you may easily get this equation. So, you can derive that equation.

So, with that, now we let me before going to these equations that connection etcetera will give I want you to give some examples, where you may not get existence or uniqueness, and then we will also explain to you one simple problem of simple problem from the calculus of variations, if I have time this lecture or we will continue with an in the next lecture so, these are the things that you should be able to derive easily.

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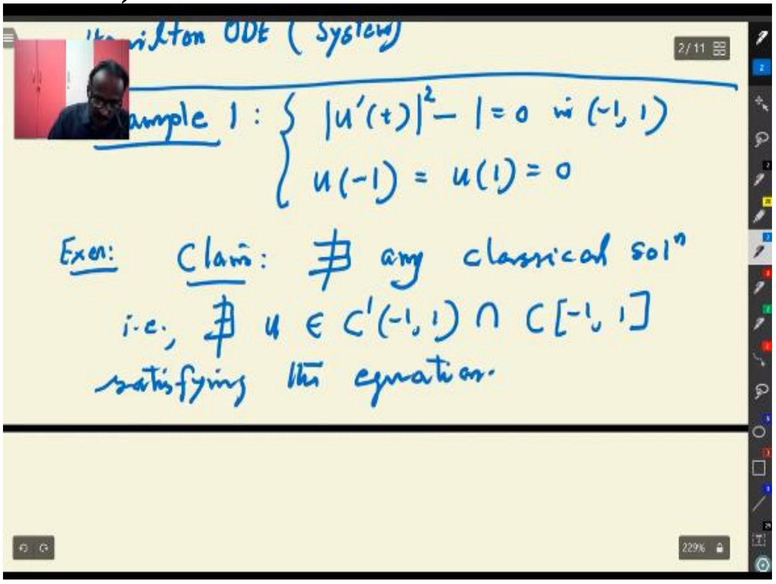

So, let me start with an example, this is a very simple equation mod Du square not mod Du square so, let me do it in 1 dimension so that is better easy. So, you have a u prime of t is the value we will use either prime or dot or du / dt etcetera things. All represent that are waiting, it is one available thing du / dt square - 1 = 0 my domain is minus 1 to 1. And I want this conditions boundary value problem u - 1 = u of 1 = 0.

So, that I will not be doing all the details so, you can actually work out as an exercise, I will give you the reason it is a trivial exercise, it means the claim is a claim that does not exist any classical solution. So, you see, this is a very simple equation and even such a simple equation, you are unable to prove the classical solutions a typical nonlinear problem you see that is that does not exist u in C 1 of minus 1 to 1 and since you are to interpret away thing and you need continuity of to the boundary satisfying the equation.

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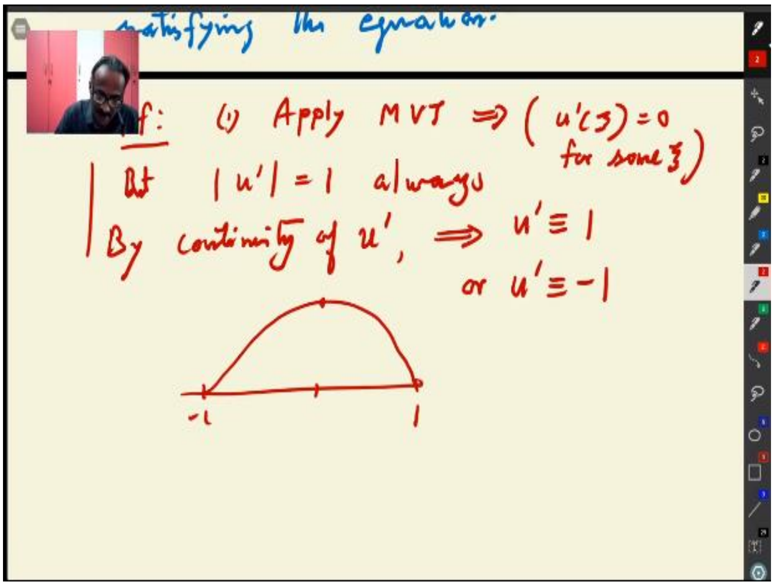

Proof is almost trivial so, I will not give the proof you can maybe you can prove it in different ways. You want it one way can apply if you want the Mean Value Theorem. Because mean value theorem implies you see u - 1 = u1 = 0. So, there will be a point activity u prime of xi will become equal to 0 for some xi. So, you can use continuity and all that results and you can prove that if you want to apply this theorem and but we have u prime =1 always, but u prime = 1 always.

So, what basically can prove is that by continuity, this is one way, other way you can actually see that by continuity you can actually see that this is another way. You can u prime is a constant continuity of u prime, because it is a C 1 function, you can actually show that. So, this is another way of showing it, implies u prime is either identically 1 or your prime is either identically minus 1. So, you see, so cannot get a solution.

So, they typically says that you have a function, you want u -1 here, you want here and you want something like that. So, you will have somewhere you will there. So, you cannot have, it means the slope is 1 all the time, so you can have. But interestingly, you see supposed to relax, so this is fine so if I relax a bit relax.

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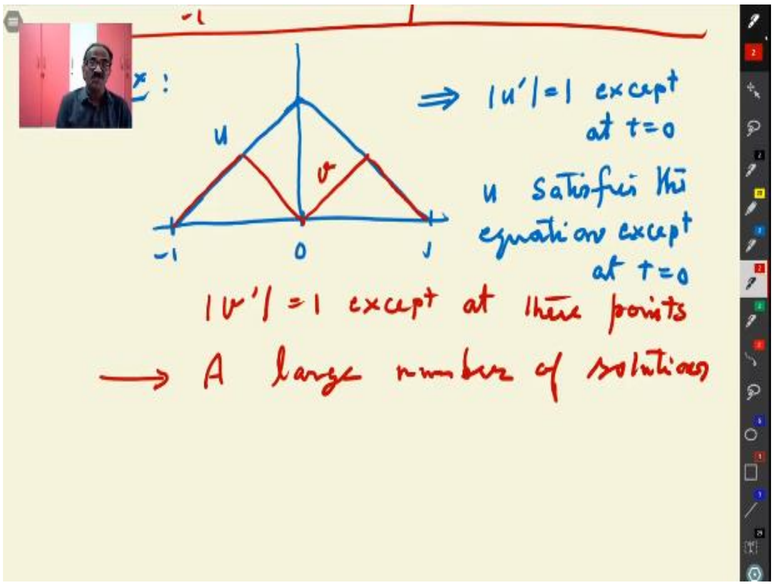

Suppose if I relax. What do I do look at this function now? So, I will slightly prove something these are the concepts which we are going to do it, but 1 here, so you have here so we will put this thing so, that is 1 and minus 1 here. So, you will see so here u prime = 1 here, u prime = -1, so this modulus of u prime for this figure my u prime = 1 except at t = 0. You will see does this function u satisfies the equation except at t = 0.

So, you are getting a solution. So, if you relax a little bit, that if you do get away that demand that u prime should be one all the time, modulus of u should be prime, then you have one solution. So, in fact if I relax a little more if I go like this, up to half and 1 like this, this is my u and then if I do this way, and then suppose this is my v then mod v prime is also equal to 1 except at 3 points.

So, if I try to explain a solution that and this u prime is definitely Lipschitz continuity etcetera, which is going to come. So, if I am trying to relaxing my demand for a solution, that the solution is the equation should be satisfied except at finitely many points, then I will get a large number of things. So, the relaxation produce a large number of solution immediately so, this is the one way of relaxing in fact, if I look asking for a solution, which satisfies the question solve most everywhere.

Then you may be able to produce infinity many solutions you see. So, the moment you try to relax a little bit on the concept of a solution, it is you are immediately lose you are getting a solution. So, that existence is available, but it is possible that you may lose uniqueness. So, let me know look into an example 2.

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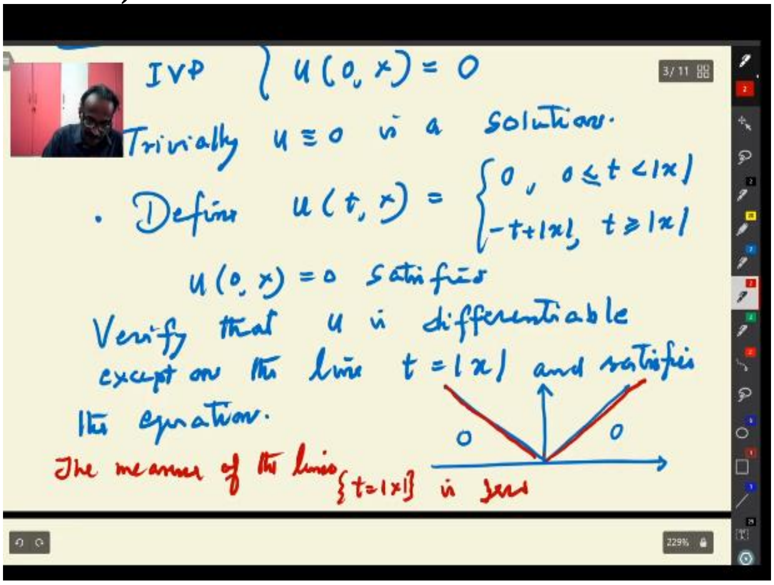

So, let us look into another example 2, here I am looking for my u t + u x square. This equation is a little general equal to 0 for t greater than 0 and x is in R and I will have an initial condition u 0, x the initial value problem. So, this is the initial value problem I am putting u 0, x = 0. So, trivially you identically 0 is a solution but I can define u t, x = 0, 0 less than t less than mod x and I define minus t + mod x if t greater than mod x this is t less than mod x and the other way t greater than or equal to 1.

So, can immediately see that 0 less than t less than mod x you have the solution so, t greater than or equal to mod x and you can show that indeed u 0, x. So, this we can even define less than or equal to so, when t = 0 this is always less than or equal to mod x = 0 satisfied. And so, I will leave it as a small exercise verify that u is differentiable u is indeed the differentiable except on the line t = mod x and satisfies the equation, that means it is a line so it is a 2 dimensional t and x problem.

So, you have, you are studying this problem here, this is your x and t positive, you will see and you are defining something so, you get some t = mod x. So, you have your solution 0 here, so and the line so, this line has the line more equal as line is a measure of the lines segment of the lines t = mod x is 0.

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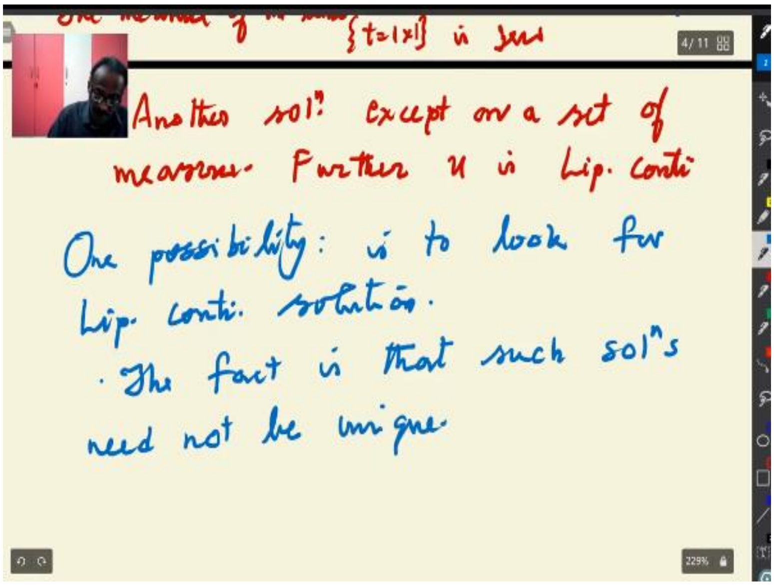

That means that you get another solution so, that gives another solution except on a set of measures, further u is Lipschitz continuous. So, in fact one will be able to produce many many solutions later, not just one solution, you will be able to we will see that your existence of many solutions actually later. And this is a completely a different solution from the trivial solution u = 0. So that is one possibility is to look for Lipschitz continuous solution that fact is that such solutions need not be unique.

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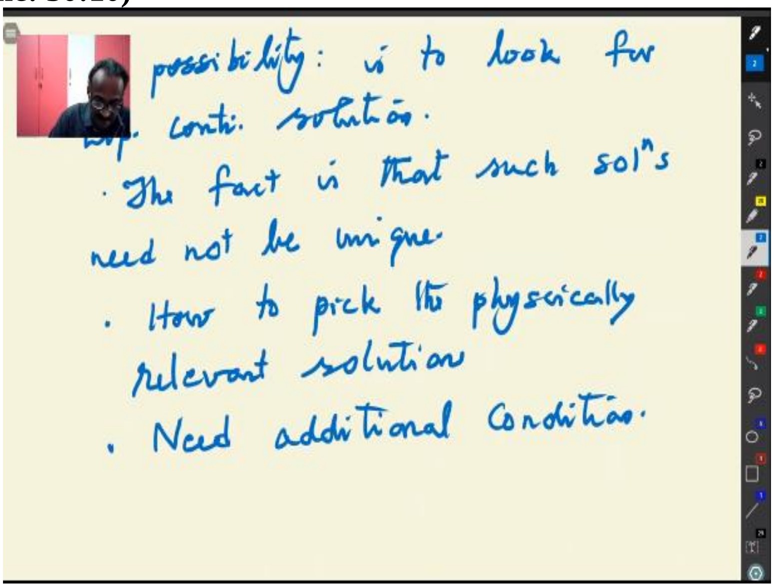

So, the problem is that how to pick the physically relevant solution and need additional conditions. So, we will see something of this sort in the, we will see some equations in this setup. What we are going to do a third example which will do so, I will start that I may not be able to complete that 1 and example 3 and then we will continue in the next lecture.

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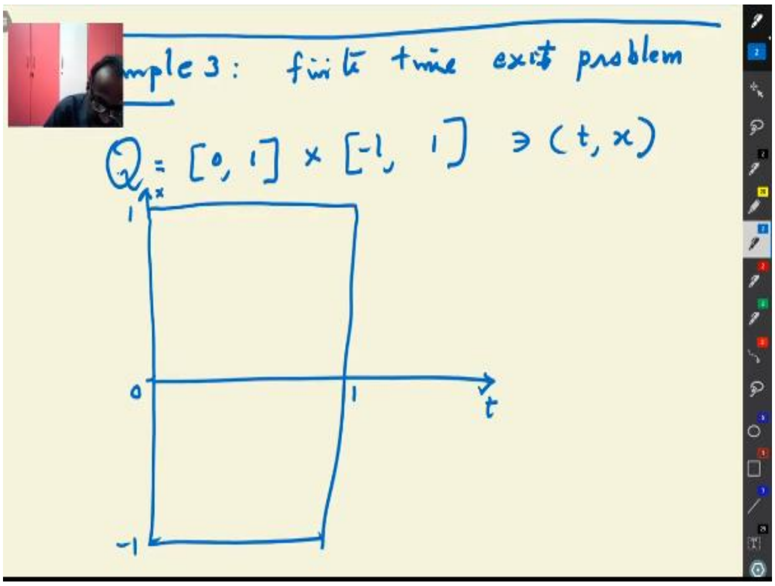

So, we will start the situation from here example 3, this is more relevant in the context of our analysis which we are going to discuss it here it is that finite time exit problem. So, let me describe the domain here and then we will do so, let me set up the thing first so, you have a domain Q which is 0 to 1 cross -1 to 1 because understanding this problem is very important. So, you have a domain here this is 1 here so, you have a domain here this is your axis here, this goes to this axis here.

So, you will take 0 here 1 here -1 to 1 so, this is your domain. So, the points are represented by t x inverse of the t in this direction, this is the t variable, this is the x variable. So, you have you look at any point here, so fixed x so typically fixed x so, I am going to describe a admissible set first of all admissible set.

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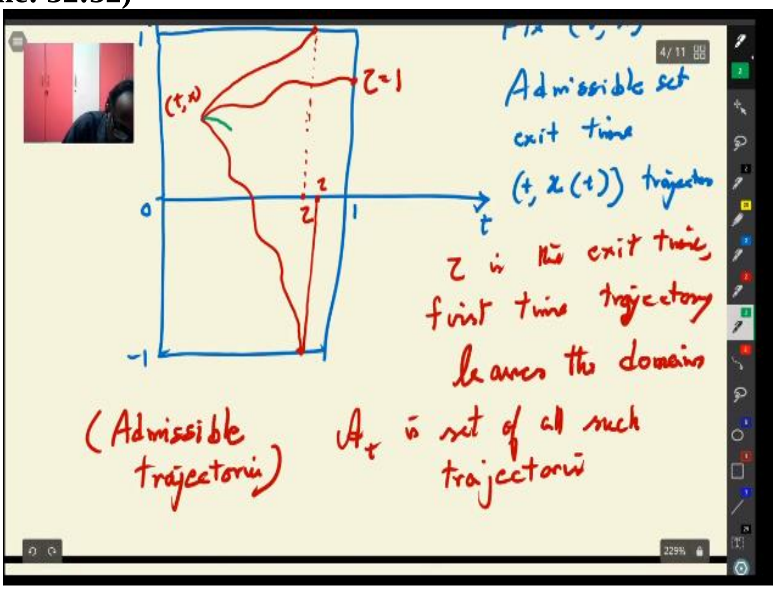

So, I will do this one today and tomorrow we will see the problem admissible set. So, you will start with the let me want to what is the meaning of exit time so, you are looking for a trajectory x t. So, looking for this trajectory t xt, this is what you are looking for a trajectory. So, you have a point here, you fix a point the x here.

So, now, you have made let me use that so, you have a point t, x here and you look for it trajectory this trajectory may go something like that. So, it will reach here so at t = 1 you exit if that trajectory. On the other hand, if you go here and if you reach here, so you are not reaching t here, so, you have the exit time is your tau here. So, it is the first time it is disappears, the trajectory we can move something like that here and here. So, you will have an exit time here. So, it clearly explained it in the book.

So, tau is the exit time that is the first time that trajectory leave's the domain and A t is the set of all such trajectory and this we called admissible trajectory, on this thing we will step so, for this case tau = 1. So, if any trajectory comes and leaves here, so tau = 1 so, that is a time so that it completes the path up to t because t =1 is the maximum time. So, up to t = 1 it covers everything so, in that case you are exit time is 1, but it is possible that trajectory will leave before that and hit the boundary.

So, you are looking for the first time tau for with that trajectory t xt is within that up to t = tau and tau is the first time so, tau x tau will be on the boundary. So, I will continue I will describe a minimization problem on this domain, what is called first exit time minimization problem and we will continue this example and with that respect to that example, we will actually solve that problem and see that actually we introduce a what is called the value function which I described that is a minimal value with respect to t, x.

And then we will proceed that that will be satisfies Hamilton Jacobi equation, whenever it is differentiable. The issue is that that value function or minimal value need not be differentiable. So, I will stop at this stage and we will continue the lecture in the next class. Thank you.