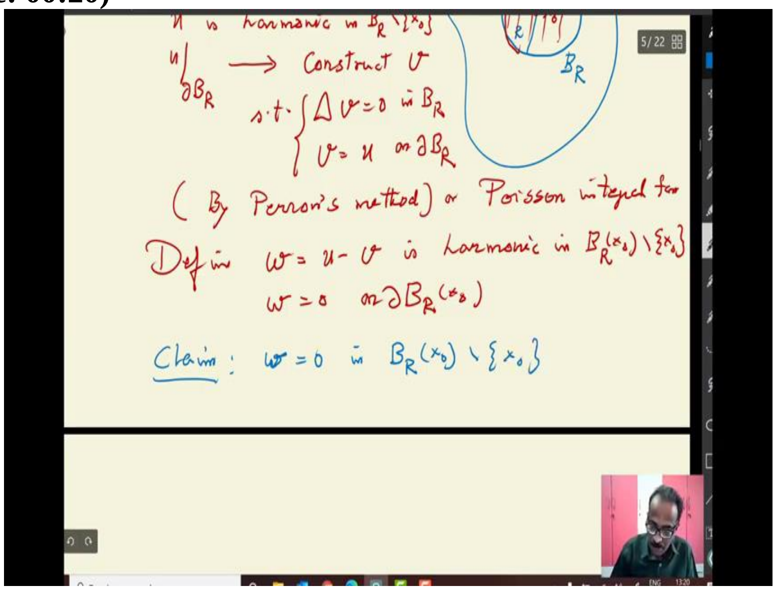
First Course on Partial Differential Equations - II
Prof. A.K. Nandakumaran
Department of Mathematics
Indian Institute of Science - Bengaluru
Prof. P.S. Datti
Former Faculty, TIFR-CAM - Bengaluru

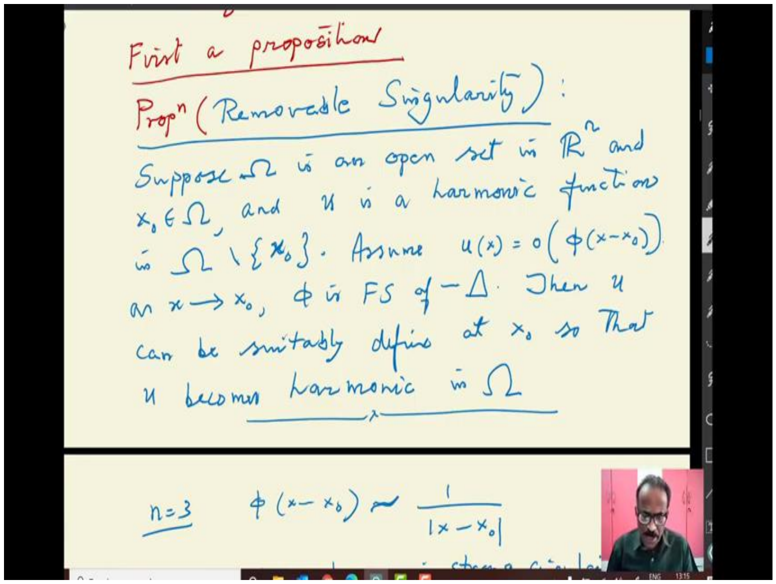
Lecture – 02
Laplace - Newtonian Potential

(Refer Slide Time: 00:20)



So, morning, so we will come back to the Laplace-Newtonian potential. So, last to class, we have introduced a proposition where when you can remove a singularity of a harmonic function. So, let me go and read the thing and then give you a proof. So, this is a removable singularity, suppose omega is an open set in R n and x naught is in a thing and u is a given harmonic function omega -x naught and your u x is in the small order of your fundamental solution that means, the singularity of u is in order smaller than the singularity of the fundamental solution. Then you can be replay a suitably define the x naught, so, that u becomes harmonic in omega.

(Refer Slide Time: 00:42)



So, let me give a proof of this theorem. So, we will present a proof of removable singularity. As I see, you can see that is a bit of a tricky proof, but it is not that very difficult, but the proof is difficult. So, we give the proof when n greater than or equal to 3 that does not matter, because the form of the fundamental solution is different for n greater than or equal to 3 and n equal to 2 for n equal to 2 use the corresponding fundamental solution then the proof goes exactly corresponding fundamental solution you will see where I am using the fundamental solution.

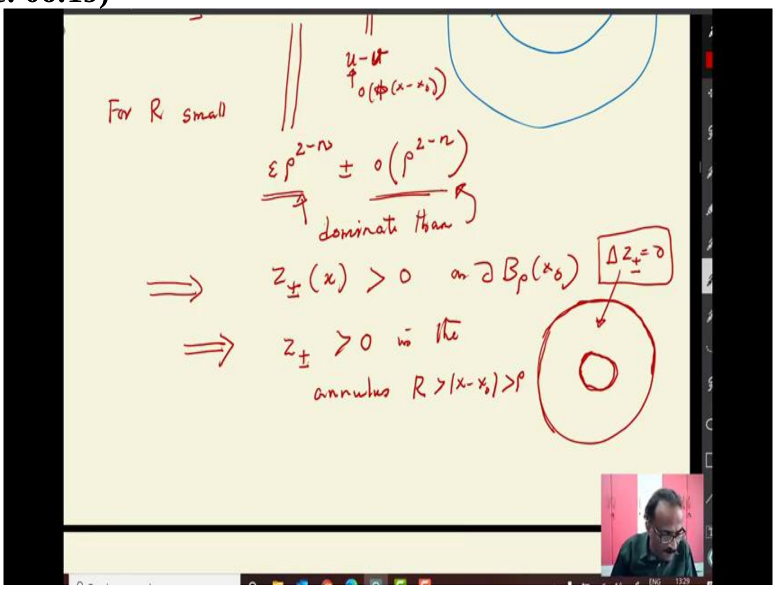
So, want to do this one. So, what do you there, let me start with that. So, you have a domain here and you choose an R positive. So, you have to choose R positive and consider this board in such a way that your board should be closed board so, that it is compactly embedded in containing omega. So, you want this one, so, this is your R. So, this is your B R the ball of radius R. Then I can consider this we have already used it I can trust it my u here. So, this is your x naught.

So, the probably x naught and the u is harmonic you will see that u is harmonic in u is given to you B R minus we are of x naught -x naught except at that point, but using the boundary values. So, you have you look at your restricted to the boundary of B R and then you can construct a v. How do you construct a V such that these harmonic in B R such that Laplacian of V = 0 in B R and V = u this is a boundary value problem which you have studied already.

So, this is by Perron's method you can use the Perron's method to solve since it is a board you can also use the earlier method with which we have introduced. So, you do not need Perron's method even for the solubility of a ball we have already solved earlier. So, using the by like all Poisson's Integral formula. So, you have studied Perron's method or Poisson's integral formula. So, you have that already. So, define v = w = u - v that is a thing you can do it u - v is harmonic.

Now harmonic in again u is only harmonic in B R of x naught - x naught not again because u is not harmonic and w = 0 on the boundary my claim is now. So, the claim w identically 0 so, w = 0 on the boundary What I am saying is that w = 0 not only on the boundary it is in the except that that wherever at because w is not defined that x naught because there may be a singularity of you, but then it is 0 this requires a proof I will do that. So, the main proof is proving this.

(Refer Slide Time: 06:19)



Once that if claim is true you are done now, if claim is true defined w x naught = 0 because W is not defined here at x naught, so, you can have you define u at x naught = v at x naught itself. That implies u is identically v in B R fully, so, I can be defined by my solubility of v, so, you can define exactly by u at x naught in B R implies u is harmonic in B R of x naught. So, you want to give the proof of w = 0, so that it remains to show your w = 0 in that so the proof of the claim.

That is what technical proof of claim. So, you are using whatever you have studied so far here. So, as I said we are assuming and assume n greater than or equal to 0. So, far we are not used that one. So, assume n greater than or equal to 0. So, now, I defined for n greater than or equal to 3, I defined something like 2 function z plus or - x is equal to choose epsilon positive arbitrary small eventually this will become small.

So, choose epsilon greater than 0 and you will define epsilon equal to modulus of x - x naught power 2 - n minus plus or -w of x. So, same w of x so, in here you see, this is the part of fundamentals except for the constant so, you do not have to include constant here, this is part of fundamentals solution. That is the main singularity part of the fundamental solution, whether you put constant or not it does not matter.

So, in the case of n equal to 2, 3, you are to work with the log mod x -x naught. That is all the difference here, then this is a harmonic, this is you know that is what it makes it this is harmonic in B R -x naught using except that deleted neighborhood deleted board it is harmonic similarly, w. So, that implies z plus when you use plus here minus here minus here. So, that means z plus or minus both functions are harmonic in B R -x naught. So, now, we have to play a little more game. So, look, let us look at the boundary.

So let us look at the boundary suppose x is in boundary of B R. If x is on the boundary of B R, what happened to z plus or -x? x -x naught is R. So, this is epsilon power, R power 2 – n. But the w is 0 in the boundary, since w of x = 0 on the boundary, so you have epsilon power 2 - n and that is positive, it is a positive quantity. So, on the boundary you have a harmonic function not fully. So, you can apply maximum principle etc, but you cannot immediately apply for a maximum principle.

Because that plus is and minus or harmonic inside is not given, but you can take unless now, so, if you have a ball here, this itself is in a bigger domain. So, you can be and then this is your R this is your x naught, so I can construct a ball here, any ball here we like it. Eventually you can choose you are interested in only near the neighborhood of x. So eventually R will be also small. So, I can choose a neighborhood here R in a rho.

So you can choose R greater than rho greater than 0 and then you can actually show that u x as you look at the boundary of that one. So, x is in the boundary. So, if I choose x, so, you know that this case, this is the one if x is on the boundary of B rho then this will be then what is your z plus or - x then = epsilon you will get it epsilon rho power 2 – n. Instead of R on the boundary again x - x naught and plus or minus w of x.

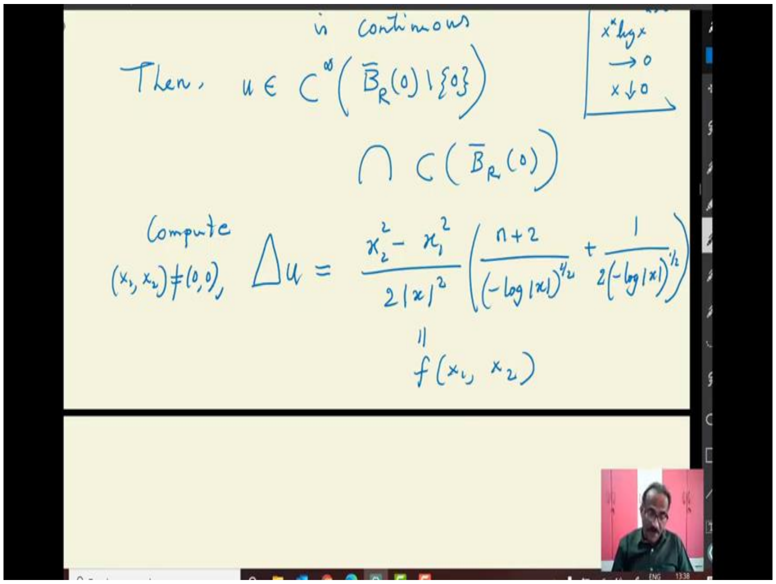
So, now, look at these things, so, this is where you are coming here w x is u - v and v is about quantity and this singularity, this is something order of mod x minor order of singularity of phi x - x small order. So, for R small, so, you choose R small if you write down this thing, so, x is on the boundary, this will become epsilon power rho 2 - n and this singularity is more order of rho power 2 - n.

So, that immediately tells you that if you have this commodity you see this is rho 2 power n - 1 and this is small order of 2 power n -1 that means, this will dominate as rho tends to small or rho becomes smaller R become smaller rho becomes smaller and this will rho will dominate then the second term then this term. That means this will be a bigger quantity as rho becomes R becomes small rho becomes small this will become small quantity then irrespective of plus or minus there that will immediately imply your z plus or.

See, you see why beautiful way of applying your theorems. So, this will be greater than 0 for on B R of 0 also on this boundary. So, on both boundaries this boundary you already got on the boundary of B R you got this one positive and on this boundary also. So, if you have 2 domains here and so, you have a Laplacian, so, this is a boundary. This is positive, so, this is boundaries also you are on this domain, your z plus or minus is a harmonic. So, you will see, so, you have a harmonic function in an annulus whose boundary values are positive.

So, that will immediately by maximum principle and other things that; you will immediately get that your z plus or minus is strictly positive on in the annulus. So, you get that in the annulus R greater than mod x - x naught is annulus. This is what the derivation you will see. So, you have a beautiful way of deriving that it is positive z by plus or minus is positive on this annulus.

(Refer Slide Time: 15:07)



Now, we will complete the proof chosen. Now you want to show that w = 0, so you do not forget your aim. So, you want to prove that w = 0. So, in other words, you want to show that w = 0. So, our aim is to prove. So, please recall once again our aim is to show w of x 1 = 0 for all excellent in B R -x naught. So, choose fix x 1 any x 1, so fix x 1 in this annulus, so you have a here and then you have x 1 here, you choose x 1 here, R is already chosen by small satisfying the earlier thing.

So, you can choose R small enough you like it then you if you have an x 1 so it is not x 1 is different from x naught so there will be a distance. So, you choose your rho in this way. So, choose rho such that, if x 1 is in that annulus x 1 belongs to the annular. What is the annulus? R greater than mod x -x naught that is greater than rho. So that implies your w x 1 here. So, you have your z plus or minus x is positive and what is w z plus or minus, so you recall this now.

You will see. So, use this, this is positive and this, so this can be estimated by this one immediately. So, if I go there, so use that. So that implies immediately, if I compute my model so w x 1, I can estimate that that is equal to plus or minus does not matter which side it is, that is what either plus or minus does not matter and that will be less than or equal epsilon because it is positive x 1 - x naught x 1 belongs to their x 1 - x naught power 2 - x 1 is fixed.

Now, as epsilon is arbitrary, implies that my w x 1 = 0 and that implies w is identically 0 in B R of x naught - x naught and then a v we already seen that, if that v that is the case, we concluded the claim w = 0 then I can define w of x naught = 0 and u x naught = x naught and you can x naught, so, that is a important proof of this and now, what we will do we will construct an example of a continuous function example.

That is what we are going to do example a continuous f such that minus Laplacian equal to f has no classical solution has no simple solution that is what we said in the beginning or in the previous class. So, we will do that. So, that I will not do all the constructions, but so, I will request you to do it. So, consider a ball of radius R. So, we are giving just an example counter example 0 in R 2, where 0, maybe less than R so you can use it, less than we use that R less than 1 and define u by u of x.

This is some formula so I will write it u in x 1, x 2 so do these computations, the x as that cleverly x 1 square and do the computation so that there are no error, but I am writing here into - log mod x. So, x is less than 1 because it is in the ball of radius less than 1. So, this will be negative so that is why put a negative sign here is equal to half. So, you will see there is a singularity here, but there is modulus here.

So, this is a continuous function if you know that we will even if the log mod you put some power anything x power it will become continuous. So, singularity of logarithmic singularity is smaller than this kind of x what our power of x what our power any small thing for example, x log x, x log mod x goes to log mod x goes to - infinity, but x slope mod x goes to log. So, you know such kind of things happen.

Any power x log x goes to 0 as x decreases. For any power here if you take it positive. Of course, negative power it will be full of singularity. So, it is continuous you can check it so, do this verification. So, I am not going to do all the verifications here and then you compute then you can actually so, the singularity is there otherwise these are all very nice functions including thing, so, that then you can compute then you compute this one.

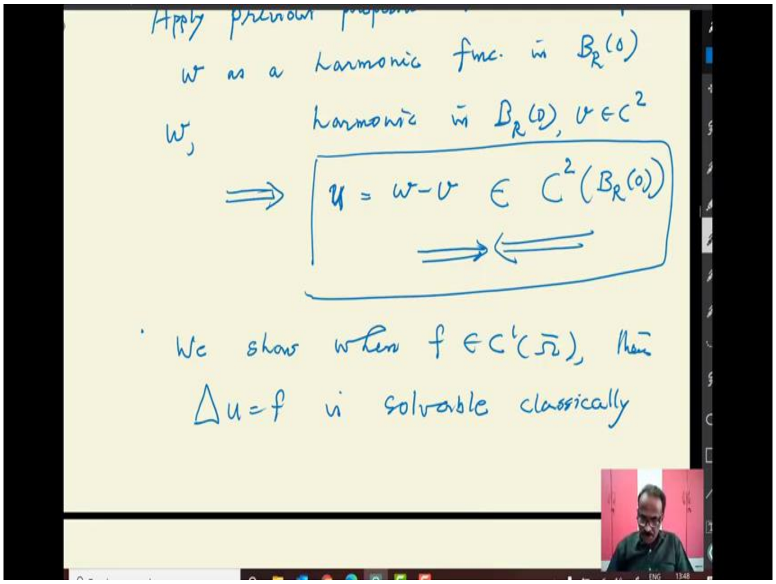
u is in so the continuity is there everywhere, but you do not have much differentiability but you can so, but today you have seen infinity instead B R you can go up to be a boundary. So, that singularity is only at the origin. So, you can have the differentiability but when you differentiate to further you do not get that the weight is continuous because it will produce more singularities. So, the only at the continuity level you have a so, see u is in continuity you have up to the above thing. You have that. There is no problem.

So, you can differentiate this one and then of course, in general when you differentiate you do not get continuity, but this example is in such a constructor, the Laplacian of u some singularities will get cancelled and you can compute, so, I will leave this thing compute Laplacian of u. So, I am not saying that first derivative or second derivatives will be continued, but you can compute the Laplacian of u thing.

So, let me write down x 2 square - x 1 square. So, let me write down one thing 2 mod x square, please do the computation so that there are no mistakes what our I have written n + 2 because this is computer, but I hope there would not be any error, but still check it log mod x power half this is square root of that, so half + 1 / 2 times - log mod x. So, the interesting thing is I defined this to be my f of x 1 x 2.

So, though the derivatives need not be continuous, but I am saying that this Laplacian u is a continuous function, I have not defined this one, this is defined for compute for x 1, x 2 not equal to 0 0. So, I have computed only that one, but because I know that their weight is can define because of singularities, so I can do only at that point.

(Refer Slide Time: 24:42)



But what is interesting is that then by defining, so you can verify that this has a limit, basically, by defining f 0, 0 = 0. We can see that so I will leave these computations to go, we can see that f is a continuous function. So, our analysis is not our continuous in B R of 0. So, you have a function, what about u whether u is, we are not we have said that, it is only continuous, you do not know about its derivative, but the first derivative second derivative are all continuous, but that is not true it is discontinuous. So, only you get for the Laplacian.

And it can do also show that for I will leave that also to you show that you compute the only so, the kind of the cancellations may be taking place, if you compute d x 1 square and if you take any limit mod x tends to 0, you can show that this is equal to +infinity. Therefore, that gives you that you have therefore u is not a classical solution of Laplacian u = f, but this does not contradict the only the u you have constructor with that f is not classical, but what about does it have any other classical solution.

So, that claim is that claim Laplacian that does not exist any classical solution v that is the thing any classical not just you, that does not exist any classical solution v such that the Laplacian v = f. How do you prove this one and this we have seen if not, so, this is by contradiction, if not assume there exists v a C 2 function C 2 B R of 0 of course such that Laplacian of v = f then define your w = u - v. So, if you define w = v u is then implies Laplacian because Laplacian u = f Laplacian v = f and E where in the inside thing, so, v is harmonic in B R of x naught -x naught.

So, you will see and then u is a you look at here u is a because there is a limit here is that does extend to is a continuous function, you can define extend the continuity that is what we have done that is a continuous function again, you should also define u is a continuous function since u is continuous, there is no singularity at all this is a bounded function. So, no singularity at all bounded is also bounded, because it is harmonic given hence, no singularity that immediately implies it is much nicer function.

So, it is there is no singularity. So, definitely it will be like this. So, I said it can even allow some singularities there is no singularity here. So, apply previous proposition to w not to u previous proposition the proposition we have just now proved to proposition to w to extend to define proposition to w define w 0 you can extend that as a 0 function. So, you can define w to be 0 functions. So that implies as a harmonic function you do not even have to be so I do not say that w = 0, it can extend to defined w as a harmonic function.

You can extend suitably defined harmonic function in B R of x naught, B R of 0 rather. So, I am taking everything 0 or there is no x naught here x naught is 0 here. So, no problem. So, you have any full you can defend yourself. So, w is harmonic w, v already constructed as harmonic function R harmonic in B R of 0 that implies w is equal to u - v. So, u = w + v is harmonic which is a contradiction which we already thought harmonic in B R, this is a contradiction, because we already show that u is not a classical solution.

So, u is not C 2. So, you will see u is not a C 2 function and you cannot do that one. So, it shows that is a w in fact it is a C 2 function which you are showing it which is a basically not harmonic. So, I have mentioned something that only w is harmonic. So, what I stated is wrong, what do you know that w is harmonic v is not harmonic. So, that is why I made a mistake, w is harmonic, but v smooth harmonic is w and v is smooth C 2.

So, therefore, w is harmonic hence, it is C 2, v is C 2 but these are not harmonic, so, that w is smooth, v is smooth and you need only this one it is smooth in. So, it is not harmonic for the mistake which is the contradiction is not harmonicity, Laplacian v is not given to be harmonic because there have to v is the solution to the Laplacian v of equal to u. Similarly, u is not harmonic because, so, but you show that this is a harmonic function which w is harmonic. So, you have this C 2.

So, that essentially it shows that there are continuous functions for which you are Laplacian of u = f if not solid. So, that is in C 1 will so, what we are going to show in the next class to begin with the next we show when f e C, C 1 of omega bar then then Laplacian of u = here solvable. First we will show this and then we will develop our analysis for the solvable classically. So, I will stop here now. Thank you.