First Course on Partial Differential Equations – II
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Lecture - 01
Laplace Newtonian Potential

Good morning and now in the next few lectures maybe 5 lectures or so, we will be discussing about the solubility of the non-homogeneous situation with the homogeneous boundary condition. And what we use is what is called the Newtonian potential and this is also known as the potential theory. So, before coming to a potential theory, let me spend some 10, 15 minutes maybe 10 minutes of what we done last year and some few examples we left out in the previous class about the Perron's method.

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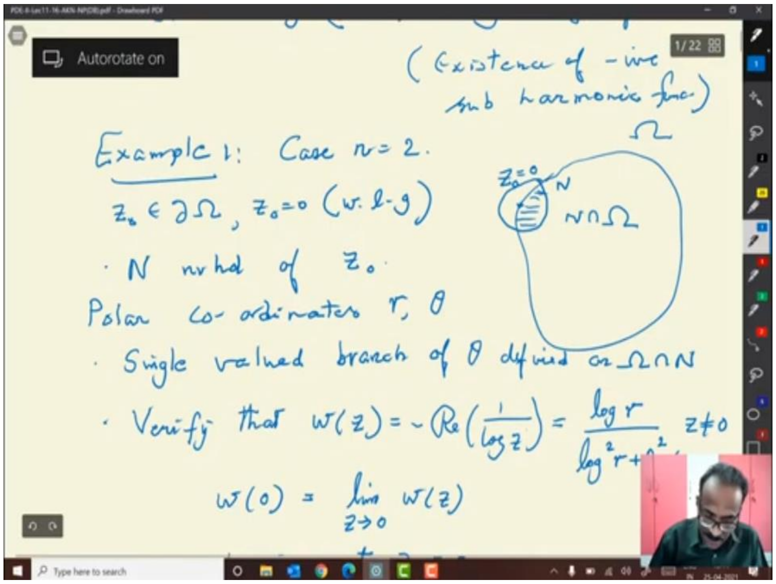

So, in the Perron's method we were studying this problem delta n = 0 in omega and u = g on d omega and you will see that the solubility is equivalent to some regularity of d omega, but then this regularity may not be it is basically an existence of some sign negative sub harmonic function. So, today, we will give 2 examples where you can verify this regularity. So, example 1 is that we need some sufficient conditions to verify that example 1, this is the case when n = 2 that is a pretty little bit easy.

So, you have a domain here n = 2 omega subset of and then you take a point Z belongs to that, I am writing it in this format Z not belongs to d omega. So, it will be of XY form and you without loss of generality, so, you have a point Z naught here and without loss of generality you take that to be 0. So, you can take that does not matter without loss because you can shift the domain if you like it, without loss of generality, you take Z naught 0.

And suppose n is a neighborhood of Z naught and look at this is the neighborhood, So, you have it a neighborhood of n and look at this part that is n intersection omega and then neighborhood of Z naught and then you use the polar coordinates r theta. So, I will define the expense in polar coordinates r theta and because theta is the angle it can have multiple value, so, we take the single value of the branch.

So, assume that is possible single valued branch of theta instead of boundaries such that you can choose this theta as a single valued branch defined on omega intersection N. So, I will exactly define you the thing then you verify I am not going to verify here, verify that in complex for my time writing it as Z verify that, so I defined wZ is equal to for Z in that minus real part of that is why you are taking some value logarithm of Z that is nothing but you can compute this.

This is nothing but log r, from taking already normalized with Z = 0. This is log square r compute these plus theta square where Z naught equal to 0. So, for Z naught equal to zero and as it is like a complex numbers on the complex notation and given define w at 0 is equal to limit Z tends to 0 that limit x limit Z tends to 0 of WZ at that limit.

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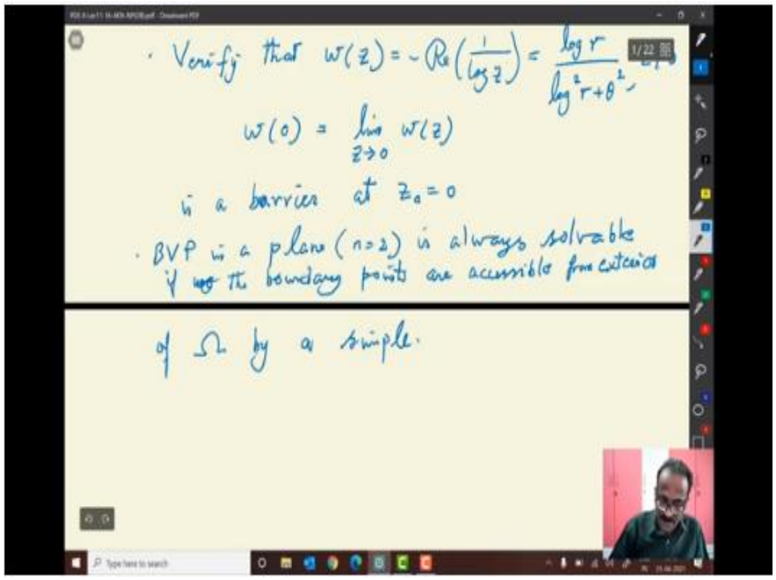

Then you can verify that is a barrier, that is what you have to verify irregularity, it is a barrier at Z naught not equal 0, that is what you want to prove regularity. So, whenever you have the single value branch in every neighbourhood, you can do that one. So, geometrically it says that so the boundary value problem in a plane domain, n = 2 is always solvable, if we can access if the boundary points are accessible from exterior of omega by a simple arch. So, whenever you can, so that R 2 more or less, it is easy to solve in a domain.

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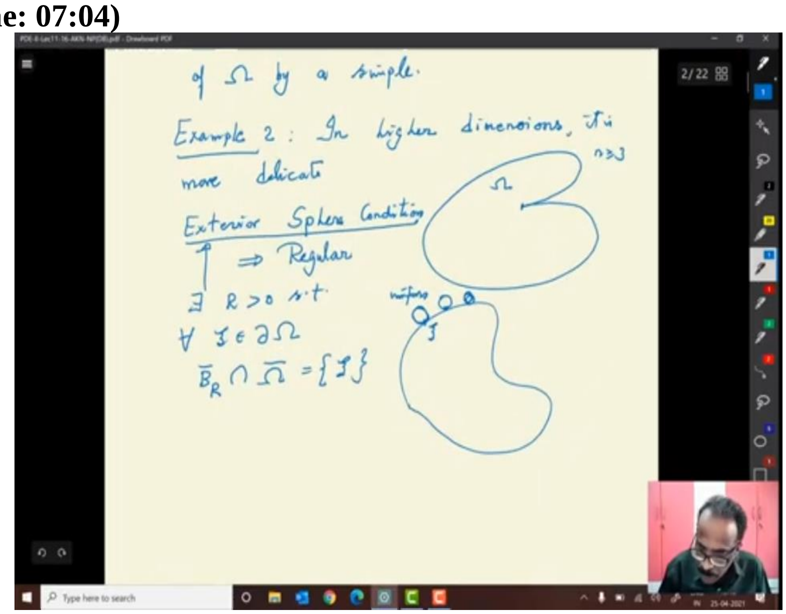

But when example 2 but when you go to higher dimensions example 2 in higher dimensions things are not that simple, one sufficient condition higher dimensions it is not simple, it is more delicate. For example, if you have a domain like this, if you have a sharp or something like domain is n greater than or equal to 3, solvability is an issue in such a kind of domains. So, one condition of regularity is what is called a exteriors sphere condition, this is much stronger for exterior sphere condition is satisfied then exterior sphere condition implies regularity.

So, this is a more regular so, the as I said there are various ways of defining regular exterior uniform sphere condition basically this regular. So, what is this exterior sphere condition geometrically means that if you have a domain you can attach a sphere at all points of the uniform radius that is it attached. So, you may not be able to do that for example, you cannot attach a sphere here at all it is exterior.

So, it should be uniform sphere uniform sphere condition. So, you will be able to attach a sphere that means exterior sphere condition means, there exists R positive such that for all Xi on d omega you look at the ball of radius R and then you take its closure. So, it touches at the closure intersection omega bar is the singleton that means only at this point xi you can put this ball and this ball is only from R works for everywhere.

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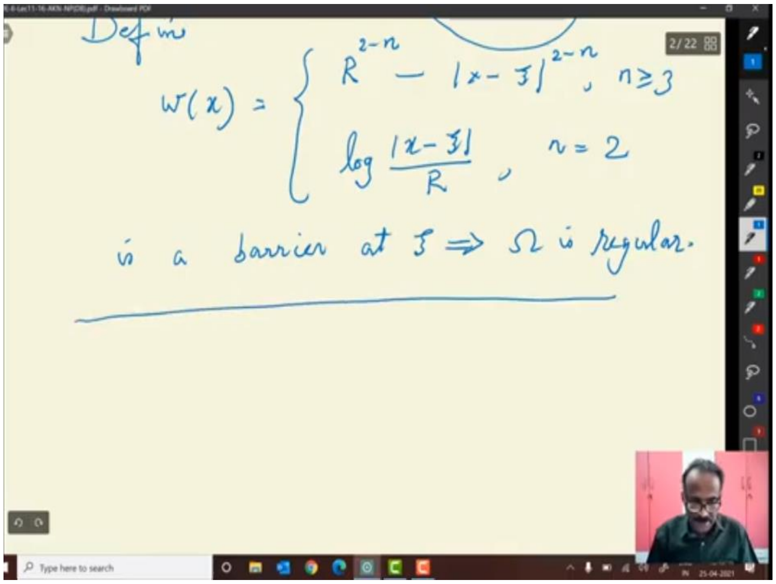

So, in this case, you can define W of x equal to with this condition. So, you can prove all these R power 2 - n, this is n greater than or equal to 3 minus modulus of x - Xi power 2 - n when n greater than or equal to 3, any condition you can define this is another way of giving the condition. This is more stronger condition, x –Xi by when n = 2 is barrier that means that implies omega is regularity. So, with these 2 conditions, I will move on to my next Newtonian Potential, maybe we will start with a new page.

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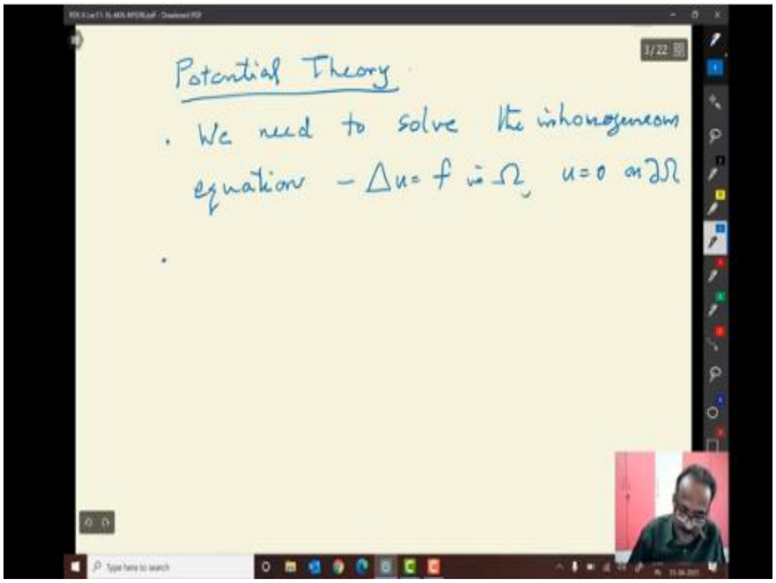

So, we will go to the potential theory, so, we will go a little bit slowly. So, what to do? So, let us see. So, let me give you a brief idea about what we are going to do. So, we need to solve the homogeneous equation, minus Laplacian of u = f in omega and u = 0.

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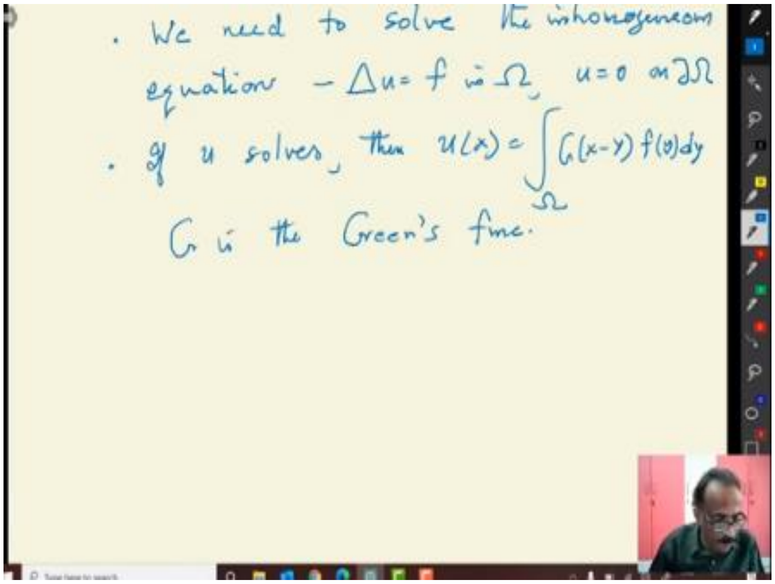

And you will know that from the representation formula if u solves then the representation formula tells u of x is equal to using the Green's function G of x – y, f of y dy. This is on omega and there would not be the second term on the boundary which thing that and G is the Green's function. And right now, we are because we have already done by Perron's method

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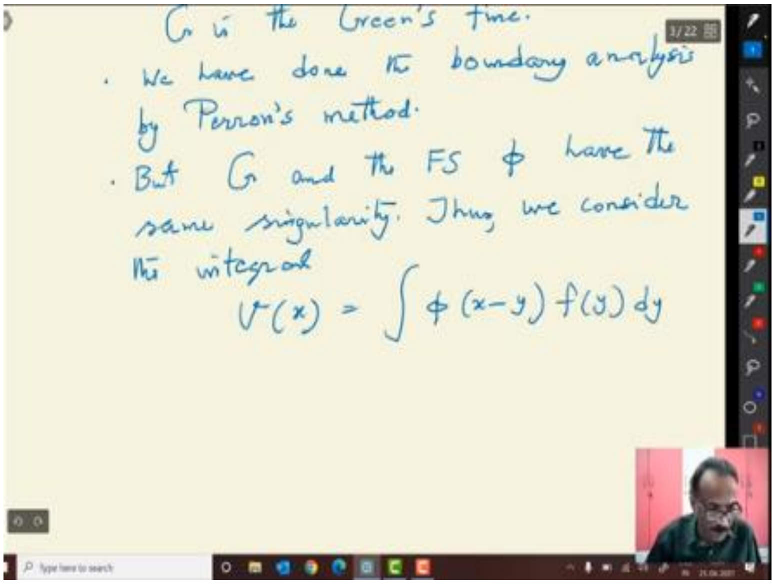

We have done the boundary analysis by Perron's method. So, we will not worry about the boundary condition now. So, the difference between the Greens function G is only that G take care of the boundary condition also. But, so, but G and phi the fundamental solution phi have the same singularity and the singularity is the one it is recovering your source term. So, far you have seen the earlier lecture on details from PDE1 and other things you can see that that singularity of that is what eventually recovers yours F the source term.

And the same singularity that is what we have same singularity because we do not impose any boundary condition on the fundamental solution. On the other hand, G has satisfies the 0 boundary condition which you have seen it. Thus we consider that it is enough to consider; thus we so we do not worry about the boundary now. So, we consider the integral Vx equal to with respect to the phi of x - y f of y, f is given to you, we will talk about more about it. This is over omega.

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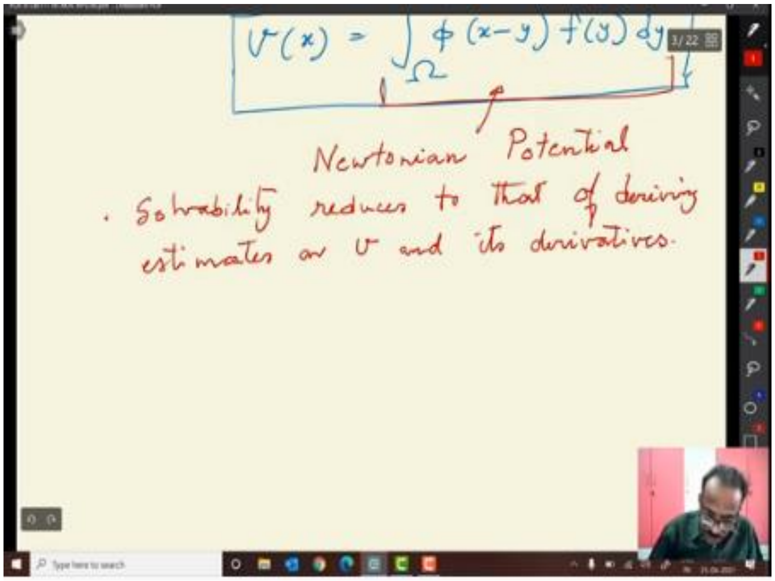

So, we will talk about, so we want to understand this boundary condition. So, you want to so and this is called this integral is called the Newtonian potential. So, solubility reduces to that of deriving. So that is what because this is a singularity for V phi reduces to that of deriving estimates on V you will see that in the next few lectures and its derivatives.

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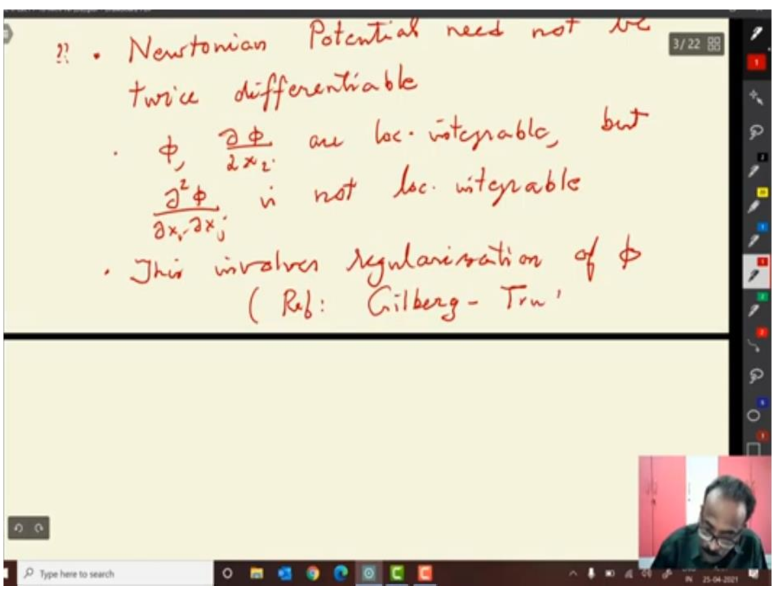

So; the Newtonian Potential one of the major things that Newtonian Potential need not be twice differentiable? That is the trouble. So, the issue the question, the difficulty is here, Newtonian Potential need not be twice differentiable. So that is the issue, because if you want to compute your Laplacian of phi, you have to take that Laplacian inside the integral, but then phi and its derivatives are locally integrable, but d phi / d square.

So, the problems are that phi d phi / d x i, say, you have seen this in the PDE1 course are locally integrable, but that is where the problem, but d square phi / dx i dx j is not locally integrable. So, you cannot take the; differentiation under the integral sign not locally integral. You see the trouble comes there. So, this involves regularization of phi, appropriately. So, what we will do? We will do a very little analysis, we will not do it.

If you want more detailed analysis, just refer the Gilbert Trudinger as I said earlier, it is a complete analysis on elliptic equations not only potential theory, a lot of analysis there in the Gilbert Trudinger book about the Shrouder theory. So, you can if you want to have mastered these things, you can refer this book.

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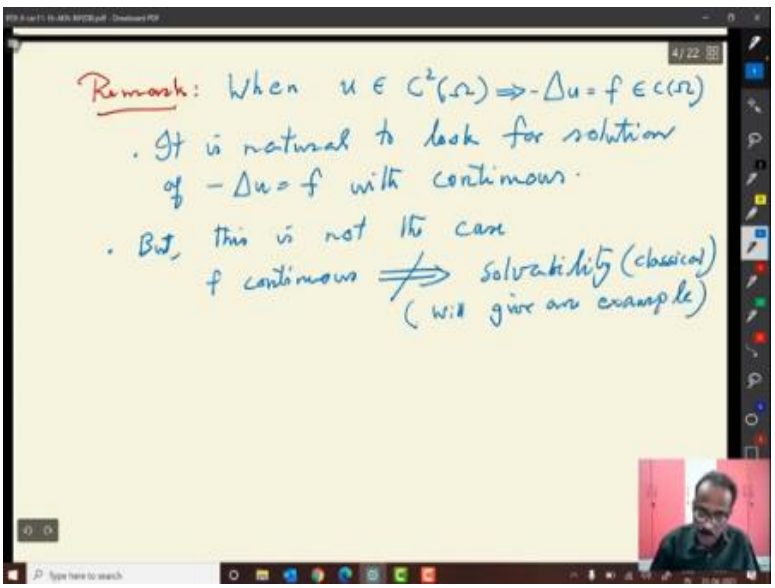

So, now, another remark which you want to make it here mark this is where the troubles now issues. When u is C 2 of omega when u belongs to before proceeding, let me tell why we want the difficulties in studying Newtonian Potentials. When u = C 2, then this immediately implies, my Laplacian u, in fact minus or plus does not matter supposed to take f. This is a continuous function. So, whenever you have a u is a C 2 function which is then that classical solution which we will longer than Laplacian u equal to.

So, it is natural to look for solubility, this is given u you think so, naturally when f is continuous, you expect to have solution u in C 2. So, it is natural to look for solutions of minus Laplacian of u = f with continuous f, but the unfortunate issue is that when f is but this is not the case in other words, f continuous may not produce may not imply classical solubility. We will give an example classical, may not; it does not imply classical solubility. So, just so there is some trouble so, I will give an example. So, you see, so you have trouble.

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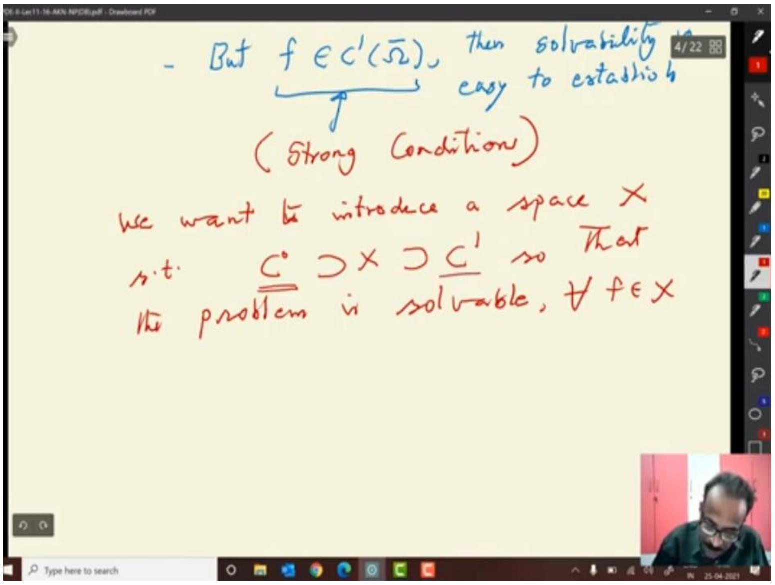

But when f is supposed to have f is C 1 of omega bar then solubility is not a difficult then, solubility we will do this also solubility is easy to establish, but this is a very strong condition now, you are assuming a evenness of that strong condition we want to get rid of that we do not want to for just strong condition. So, when f is continuous need not be solubility. So, you have contiguity which is quite weaker and so but when it is C 1 you have the solubility, but strong condition.

So, what we are going to do that we want to weaker condition, so we want to introduce a space X such that it is smaller than continuity C naught but bigger than C 1. So that the problem is solvable is a solver for all f X. So, you do not have solubility here in general, you have solubility here, but you want.

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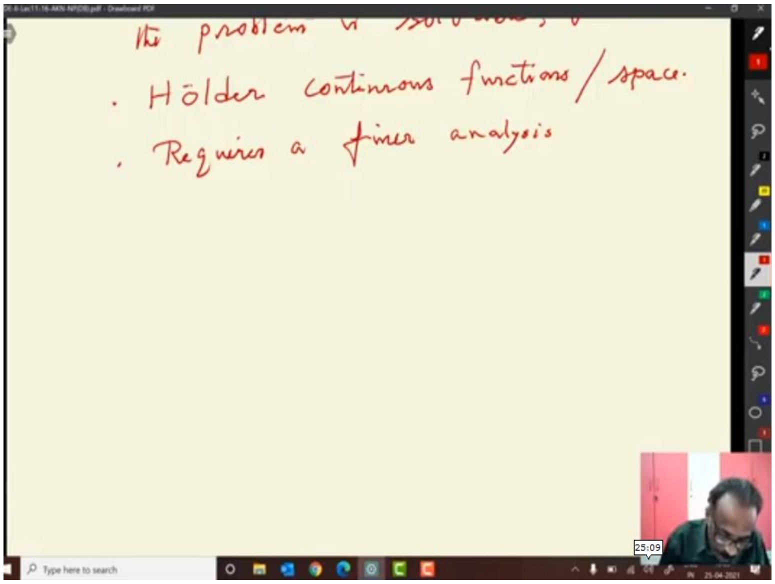

And this turned out to be what I called a Holder continuous functions we will introduce in the process Holder continuous functions. We will introduce in the next lecture function space continuous functions and space hence it requires a finer analysis who do not have the require the final analysis because you do not have the differentiability. So, you cannot do easily the integration under the integrals because you cannot differentiate f anymore precludes a final analysis.

So, if you recall the PDE1 course when f is in C 2 and compact support you have seen you can get the solution by differentiating f if you go back and see that analysis and so we will start with the proposition. First a proposition.

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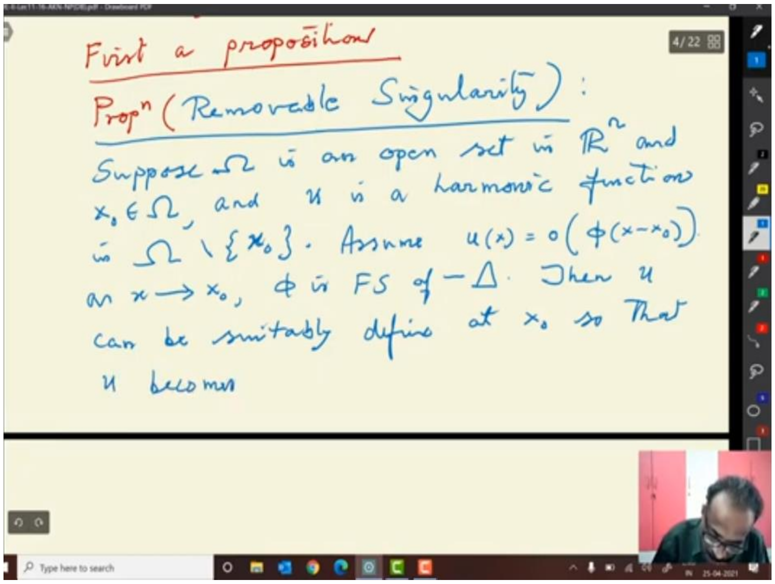

So, this is something different first you need it, you need first a proposition. So, I may not be able to prove it here for maybe we will prove it in the next class. So, let me at least try to give a proof, let me try to explain to you what it is and then we will continue this one. So, let me write that proposition. What is called a proposition? What is called the removable singularity? So, I will explain this. And then maybe we will give a proof in the removable singularity.

So, first understand that one, and then the proof will be given here. Suppose omega is an open set in R n. And x naught belongs to omega and u is a harmonic function in omega - x naught. So, x naught is a possible singularity of u. So that is what it is. So, C naught u is a harmonic function in omega. So, you can have a singularity, but then I do not want a singular, that is what very delicate issue I do not want a singularity which is worse than my fundamental solution singularity.

I will explain a little bit here and then we will proceed here. So, assume the singular I will explain that soon, the singularity if at all singularity if you have no singularity no issue, but if at all if there is a singularity, so by singularity of u of x should be something like a small order or that is the meaning of smaller order of x – x naught singularity, as x tends to x naught. So, the singularity we do not want a strong singularity then my fundamental solution where phi is a fundamental solution to my Laplacian.

Phi is the fundamental solution of minus Laplacian, then if the singularity is more essentially it says that you cannot have a singularity or you can remove the singularity basically, so if not given there is a singularity or not, if the singularity is less than that you can remove that singularity, then u can be suitably defined at x naught. So, that u becomes harmonic in omega.

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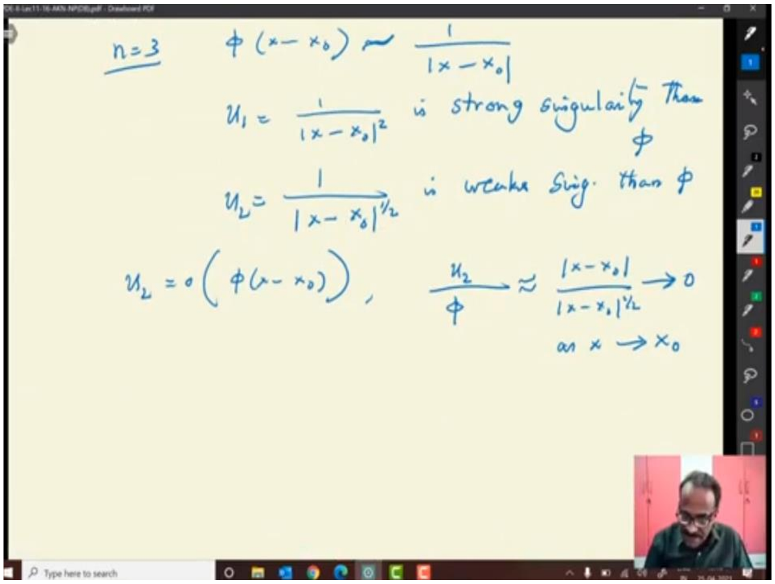

So, let me explain a little more about the thing. So, we will prove this in next class. We will give a proof of this one. So, but what does this mean? So, if you look at it the phi of x - x naught say you take n = 3, for example, the singularity is something like 1 over mod x – x naught, that is a strict singularity. So, if this power increases here, so, for example, 1 / x – x naught whole square is strong singularity then that is it right? It is strong; it goes much, much faster to infinity.

The strong singularity then phi on the other hand, if you take 1 over mod x - x naught power say half, then it is a week of singularity then phi, if I defined u 1 here, u 2 here, then you can see that, u 2 is smaller thing order of phi of x – x naught. Because, if you look at it here, if I take u 2 / phi, and you will have something like this mod x - x naught, this comes at the top singularity by mod x - x naught and this goes to 0.

So, that is what you want to show that as x tends to x naught. So, it is a smaller order of that, so, you want to understand that small order and u 2 by that function should go to 0 as x tends to x naught. So, that is why so, if you have and this is assumptions need not be in this form, it can be the singularities can come in different form and different functions. But it is basically is that a singularity cannot be that strong it should be in order, it should be strictly less than the normal singularity.

And that is what we want to do it. So, if that is the case for is essentially says that if you have a harmonic function in an; except 1 point, and it singularity is in order smaller than the small order of the singularity, or the smaller does not matter, it goes to 0. And then that singularity essentially can be removed. That means you do not have really the singularity it is a removable singularity which you can do it we use this concept to prove something first of all, in the next class, we will prove that this removable singularity. So, we will stop at this stage now. And then we will continue in the next lecture. Thank you.