First Course on Partial Differential Equations - II
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Lecture - 04
Laplace - Perron's Method

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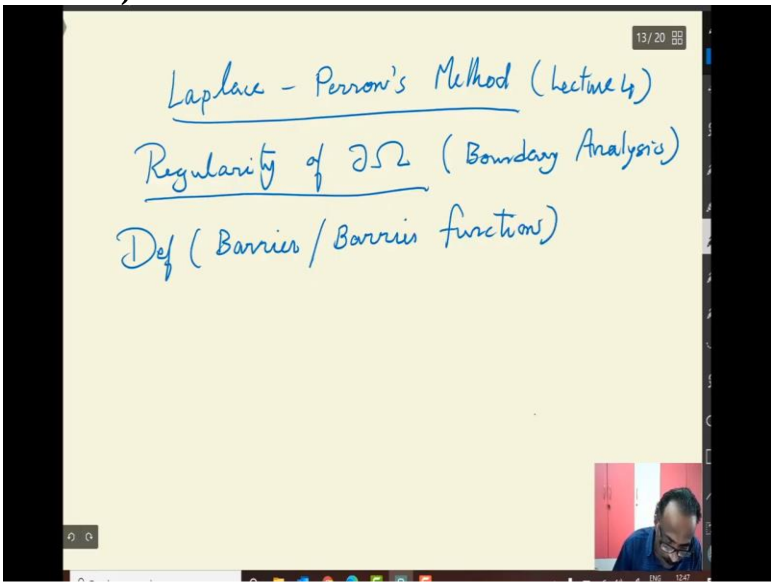

So, we will come to the last lecture of Perron’s method in the previous lecture we have seen the existence of a harmonic function and if you do not know whether that harmonic function will satisfies the boundary values G but we know that U is less than or equal to G but to prove that U = G requires some conditions on the boundary and that is what we are going to introduce now what is called the regularity of d omega.

So that we going to define what is the regularity of d omega today so analyze the boundary so this is basically the boundary analysis. Now so, let me define what is a barrier? So, I am going to give a definition of a barrier or barrier function can call it that barrier function.

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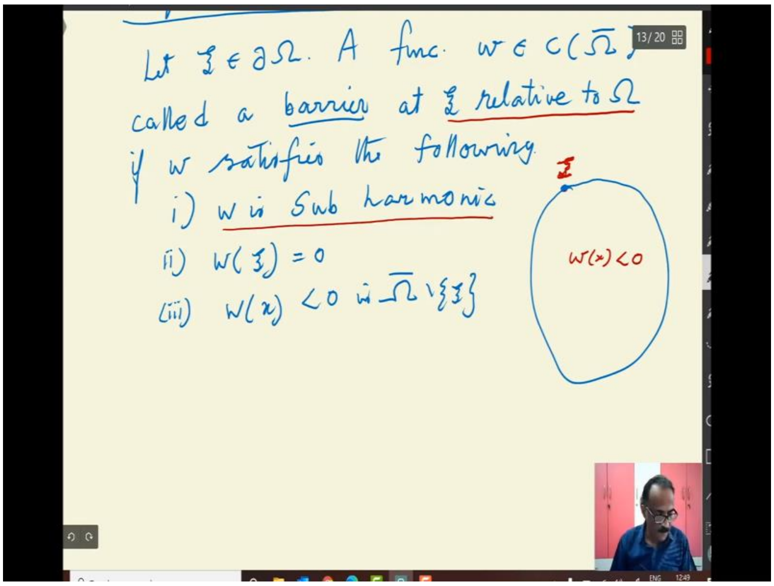

So, let psi belongs to d omega a function w in continuous function up to the boundary is called a barrier at psi relative to omega so this is what you understand relative to omega if w satisfies the following 1 w is sub harmonic it is important 2 w psi = 0 at that point w psi = 0 and in other points you want w of x less than 0 in omega bar minus. So, in your domain here is called a barrier at some point psi here is psi. So, at this point w psi = 0 and all other points where w x is less than 0 and is also sub harmonic.

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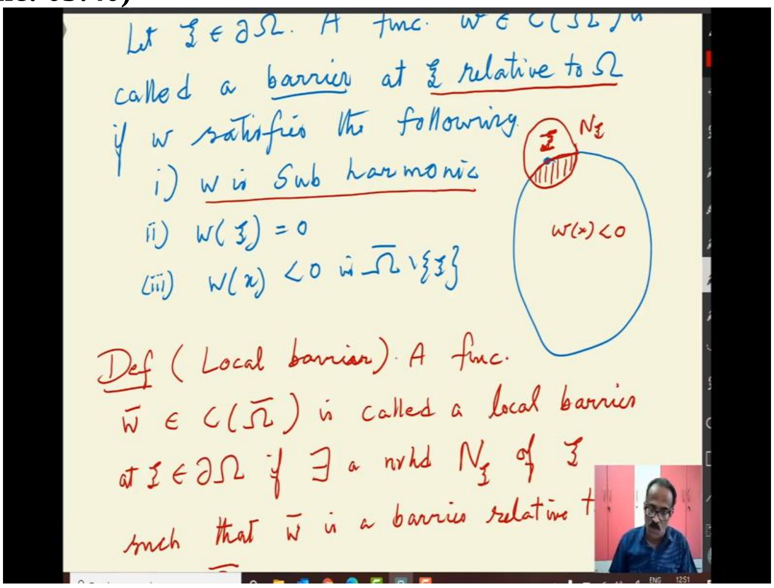

In fact this you will see that it is actually a local issue because you are basically defining the neighbourhood of that and so you so I will give a definition now and you will see that its existence or local barrier is good enough. A function w bar belongs to C omega bar is called a local barrier at psi. So, it is nothing related that called a local barrier at psi in d omega if there exists a neighbourhood N psi of psi such that w bar is barrier relative to this neighbourhood actually.

But neighbourhood may have outside relative to N psi intersection omega bar. So that is all you are looking at it so you may have a neighbourhood here N psi so you are looking this barrier relative to this neighbourhood. So, you are looking at that is enough to have a barrier here so every point this neighborhood can change then that is what is called the axis. So, you are not demanding the existence of w for all omega but this existence only in that neighbourhood.

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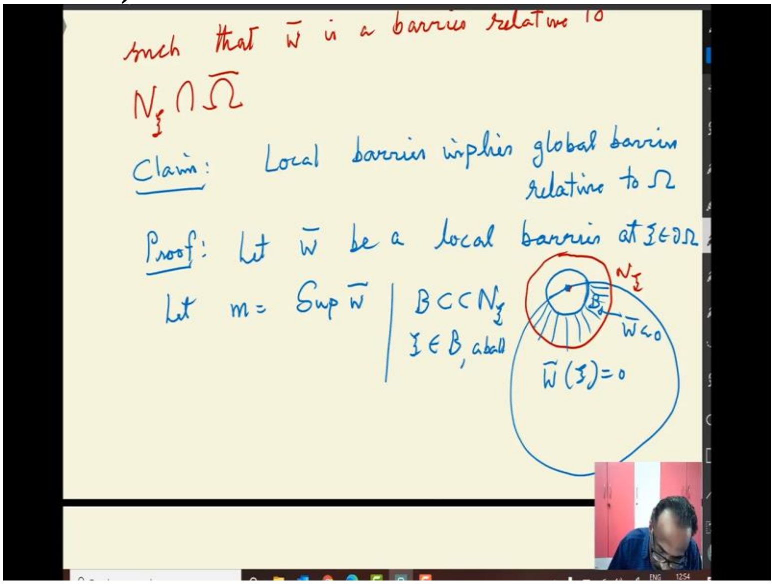

But what I am going to an immediate claim so the immediate claim so local barrier so it is a local concept implies global barrier basically so if you can find w bar global barrier but global barrier relative to omega you can say. So, the proof is actually easy we can just check this you can one can verify easily these concepts without any much trouble. So, you have this one let w bar be a local barrier at psi in d omega then what do you define let m is equal to the supremum.

So, I will not verify everything you can verify so what do you do is that so first you take in neighbourhood so this is your psi and you have your N psi here on that you take a ball here first this is a B. So, I will take so what you will do is that so your B is contained in N psi and of course psi is also in B a ball centered at ball centered at side. So, you can take this ball so you are taking supremum so to understand the supremum the w bar of psi = 0.

So, you will see the w bar of psi = 0 so at this point w is positive but then w is negative everywhere. So, in particular the w bar is negative here strictly negative there and you have your closed bounded set here your w bar is negative. So, I can take this supremum which is a finite quantity and I can define this supremum in N - B so I am defining this your N psi, N psi - B on that.

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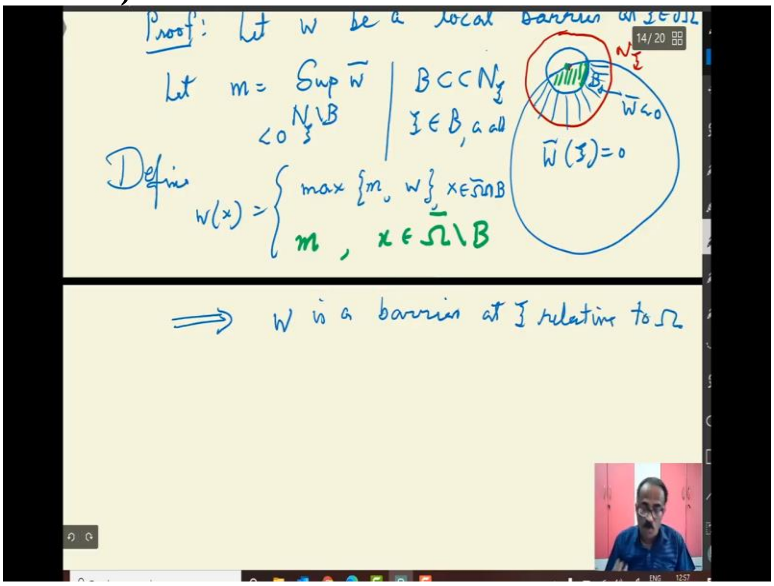

And then you can define you just define so I do not know things that came up here. So, define your w of x is equal to maximum of so I am defining by a maximum so you do not lose your sub harmonicity w see this is a negative quantity you have to see that research to clear negative quantity because w bar you are defining here and it is a negative in that thing. So, the supremum achieve as a strict negative quantity so that m is negative.

So, I will define this a continuous way of defining something so I will define maximum of N. So, when at w at psi the w psi = 0 so m strictly so this you get immediately w psi = 0. All that verifications you can easily do this I will defend x is in omega bar - B and omega sorry inside part. So, I will define this omega intersection B and at psi your w psi = 0

So, N is to play negative you get w psi so you can verify all this property and outside that I will define to be m, m for all x in omega bar minus B. So, I can extend this function as a continuous function and taking this w so that maximum and then defining this way I will retain my w as a negative function and you also get this is a sub harmonic is easy to prove it is a sub harmonic because I am taking maximum of a constant function.

And a sub harmonic function and then the defining this value and w psi = 0 is trivial. And then by maximum of whatever it is you can prove that the m is less than 0 every where w less than 0 you can also show that w of x is less than 0 for elsewhere. So, this implies w is a barrier at psi relative to omega. So, whenever you have a local barrier that gives you automatically a global barrier so it is enough to see whether thing.

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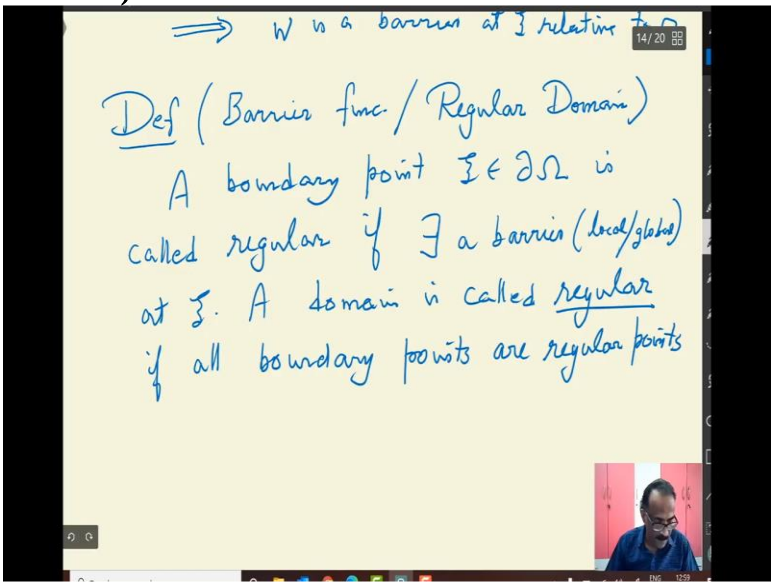

So, this gives my definition of regular thing. So, you have your definition barrier function you can call it existence of barrier a barrier function exists is called a regular domain. So, if such a barrier exists you call it regular barrier so a boundary point psi in d omega is called regular if there exists, a barrier it does not matter local or global now. Because the moment we exists local you barrier at psi.

So, whenever there is a barrier function exists that is a negative sub harmonic function basically then it is called a regular point if all points a domain is called a regular if all boundary points are regular points so you see that is my definition of regularity of the domain for the type.

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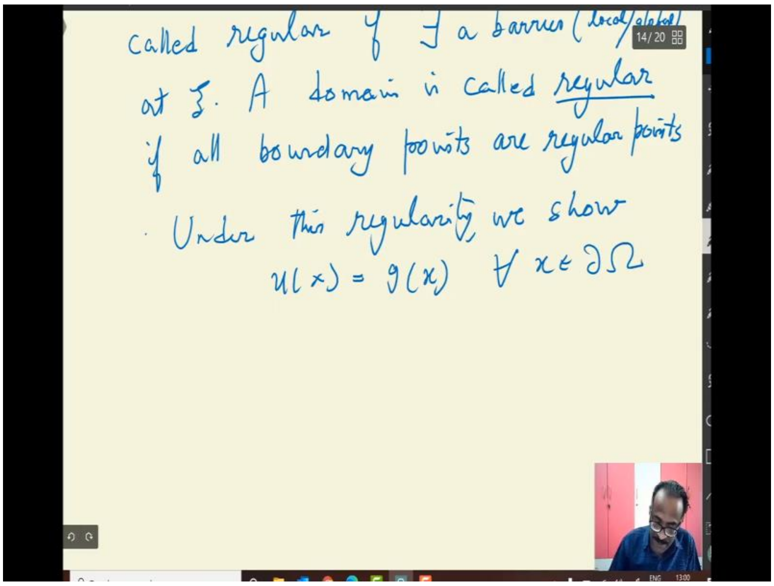

Of course you may be able to give us sufficient conditions for such regularity if you have other kinds of good definitions of the regularity of the domain that will imply some not all but one sufficient condition maybe I will look so under this regularity we show now ux = gx that ux defined earlier in the last class equal to g x for all x on boundary.

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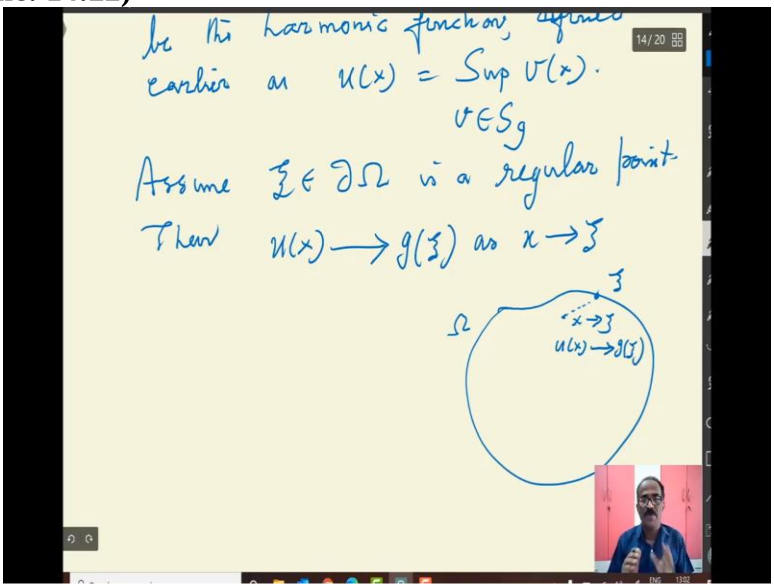

So, let me write it as a theorem and then start proving it again it is not very difficult but you need to cleverly use the theorem. Let g belongs to a continuous function on the boundary. And u be the harmonic function defined earlier which we already proved it defined earlier as how did we define ux = supremum v of x where v is in S g is assume psi belongs to d omega is a regular point then that is how the boundary values are always defined why are the limiting process.

Because u is defined inside not at the boundary necessarily that is enough then ux converts to g psi as x tends to psi so you are basically you have a domain is how the boundary value problem always indicator so you have a psi if psi is a regular point you take x converges to psi. So, for all these x interior your ux is defined. So, you proving that ux converges to g psi this that is how your boundary values are always interpreted.

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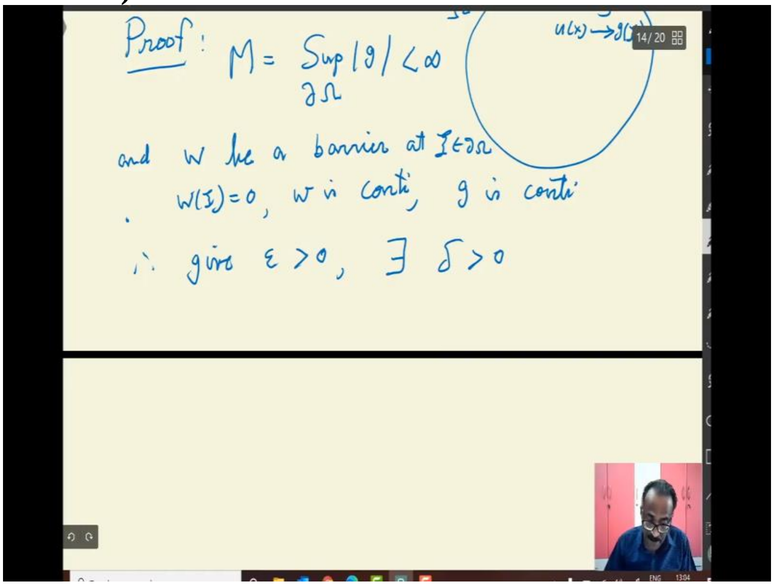

So, let me try to give a proof of this fact now not very delicate not very difficult but as I said the proof involves some amount of work. So, let me call this d omega is a compact set you have your M = supremum of mod g where here on the boundary this is a finite quantity. So that is not a problem and w be a barrier I would say because your given site to be a regular point there may be many barriers that is not a problem barrier at psi in omega.

So, to prove that one so you know that w is since now w since you know that w psi = 0. That in fact you know that g and w is continuous so you have all these factors g is continuous these are all at psi continuous. Therefore for given epsilon we use this part E given epsilon greater than 0 there exist is delta positive such that w psi = 0.

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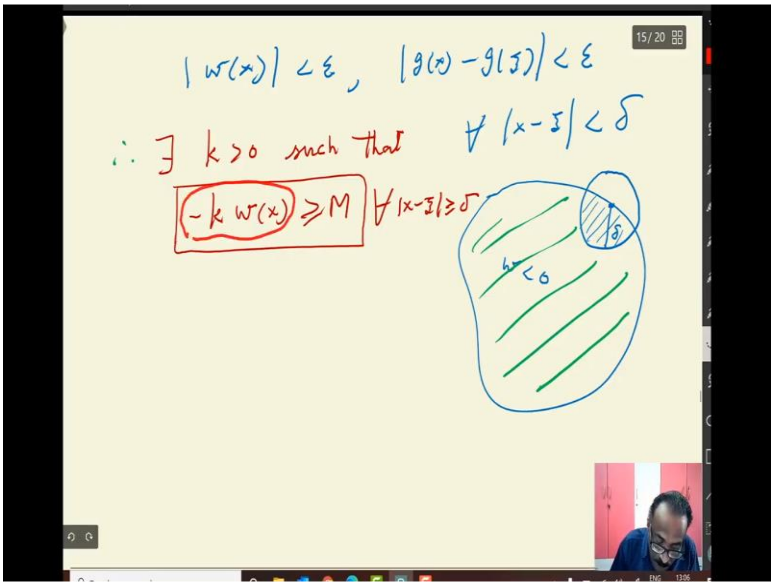

And w is small enough so w x is small enough such that modulus of w of x less than epsilon less than or equal to does not matter for mod g x - g psi less than epsilon this happens for our whenever a mod x - psi is less than or equal to delta you see that is what so let me put this figure once again here. So, you have a psi and then you have an x - psi this is your see this is your delta.

So, whenever x is here so this is where your x here x belongs to the set and you are g x sub g psi and w psi = 0. So, this you can make a small enough now w is a nice continuous function on here w is negatively bounder g is less than 0 on the set. So, here you get the w is negative so it will have a negative maximum occur because it is a bounder and in a bounder compact domain on here.

So, it will have a psi write that you can prove that w on this domain mod x - psi greater than or equal to delta you can actually say that w is less than or equal to some minus c with the c positive I will write that in a slightly different way. Therefore you can also see these so therefore their axes I write it something like that k this is a different way of writing that but you can prove this thing because I need the k w of x is greater than or equal to M for all mod x - psi greater than equal to that is trivial.

Because I am only demanding w of x see this is a minus sign. So, the w is the negative you see so that if I divide this you know that w is less than or equal to some minus c. I am rewriting that in this form that is all you can choose some case sufficiently large enough so that the w is negative so this is a positive quantity. And you can make choose large enough so that it is that greater than equal to M.

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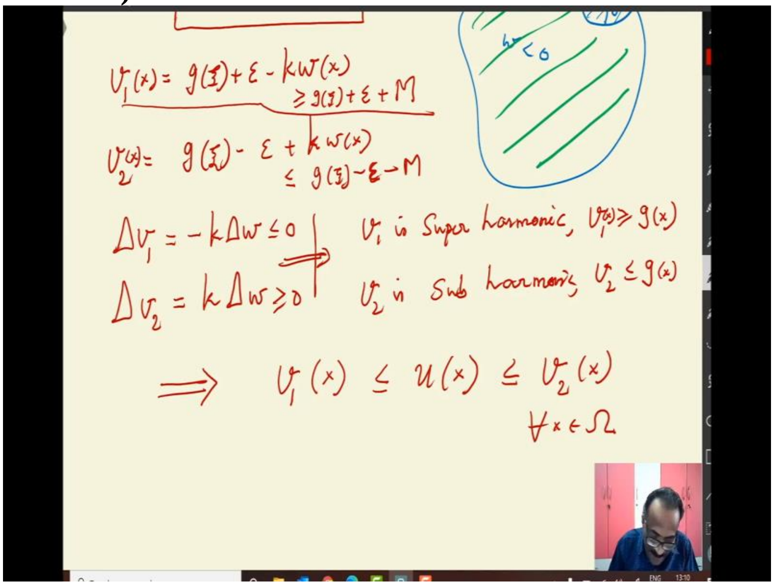

Now I construct 2 functions so let me construct 2 functions a V 1, V 1 of x is equal to so using that is a just a small trick it is not big trick is equal to minus k with that k w and I also define V 2 g x g + epsilon not g x so sorry x comes here this is a fixed quantity psi - epsilon + k these are all when you such tricks are used when you deal with sub and super harmonic function so this is V 2. So, you can immediately see 2 things from here if you compute your Laplacian of V 1.

Because these are all constant you will immediately minus k Laplacian of w and w is sub harmonic. So, this is positive so minus of that will be negative and Laplacian of V 2 is = k Laplacian of w that is greater than or equal to 0. So, this implies V 1 is super harmonic and you can also immediately verify that V 2 everything because I today will finish you can actually show that V 1 this is very easy to show.

So, I will get a V 1 is greater than or equal to g x similarly V 1 of x on the boundary and then V 2 is that is true because of the m you will get it. So, V 1 x is basically can show that this is actually let me do maybe if you want to do this is g of psi + epsilon this is greater than or equal to M. So, you will get the this is equal to M greater then equal to M and this is also less than equal to you can show that this is less than equal to minus M g psi - epsilon - M you can use this.

So, this is you see because k w is less than or equal to minus M so you will have these things use this one because M is he supremum you can get this fact. So, V 2 is sub harmonic and V 2 is less than or equal to g so V 2 is in s g, minus V 1 is also in s g that is what you are fact you are using. So, it is a super harmonics of minus V 1 will be some harmonic and minus V 1 it will be less than or equal to minus g so that fact we will not use it.

It is a super harmonic function so V 1 is the sub harmonic V 2 is the sub harmonic function V 1 is a super harmonic function and we also can easily imply use this fact to show that V 1 of x. So, V 1 is this one we can actually show that V 1 of x is use these facts all that what are described here you can show that this is less than equal to ux and less than or equal to V 2 of x this is easy to show for all x in omega. So, now you write down your V 1 from here so we V 1 less than or equal to this. So, you can take your g psi here and this you can estimate exactly.

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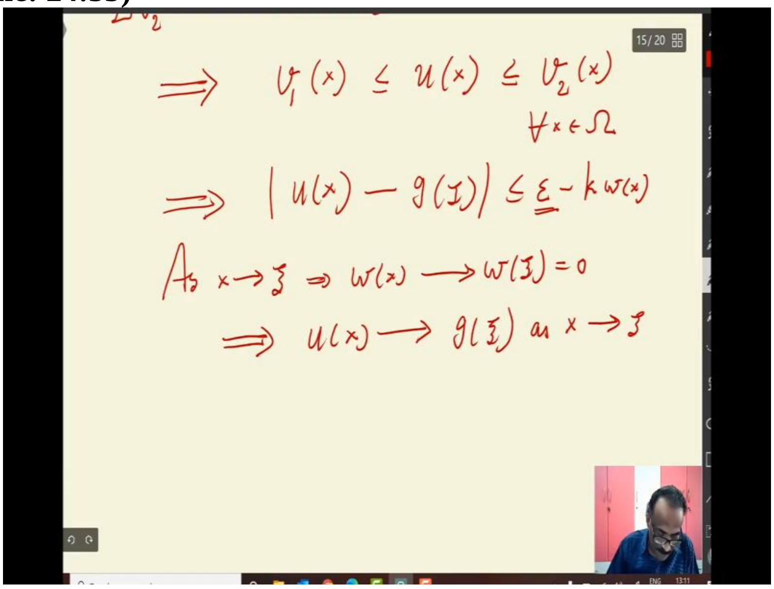

So that will imply immediately my V 1 u of x please write down these steps I think my g psi is less than or equal to epsilon - k that w m you do not need for these so that is it. So, you have just taken g psi here from V 1 and V 2 so that you want. Now look at it as extents to psi epsilon is arbitrary and w x goes to w psi which is 0 as w thing w x goes to w psi = 0 that implies ux converges to g psi and that is what you want to be proved as x tends to psi yeah.

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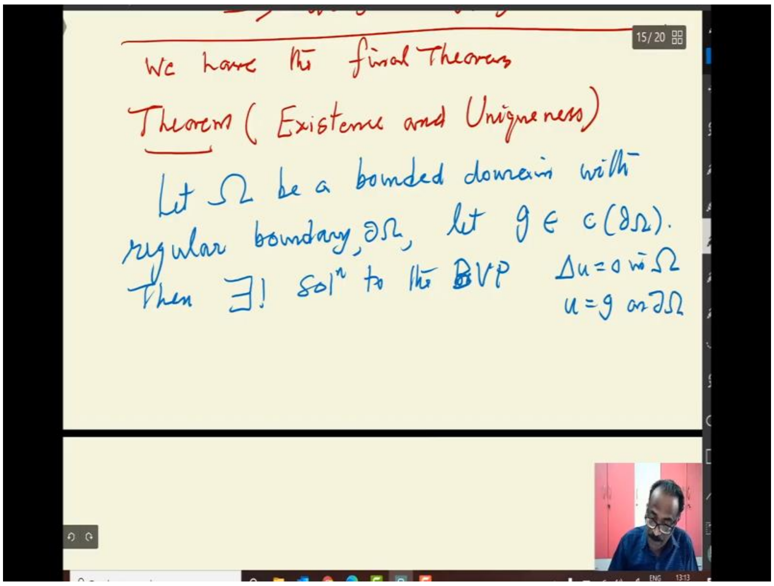

So that is we have so that proves the boundary where you think so that is we have the so that is prove so we have the final theorem basically existence and we have the final theorem existence and uniqueness unique enough we already proved now existence is also proved existence and uniqueness. So, let omega be a bounded domain so let me write it in this land let omega be a bounded regular domain so every point is a regular point bounded domain with regular boundary d omega let g belongs to c of d omega.

Then there exists a unique solution to the boundary value problem BVP minus Laplacian of u = 0 in omega and u = g on d omega. So, whenever you have regular thing but then there is a converse part that is what I say that regularity which we have defined is also thing.

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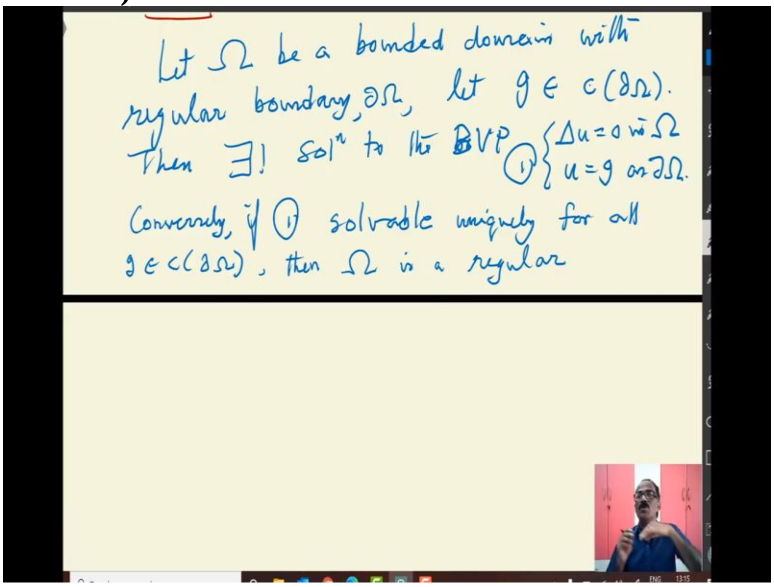

So, conversely if the above; if so let me call it 1 if 1 is solvable for all if this 1 is solvable the converse is easy to see solvable for uniquely for all g continuous for all g for some g may be solvable suppose you want a solvability for all g continuous then omega is a regular domain you see so the regularity we define is an if and only if condition for solvability for every g is a regular domain of course even if it is not regular for some g you may get it you will take any solvable it is a problem if suppose you take a any solution of Laplacian you can do 0 it will give some boundary condition. So, naturally it satisfies such equation so that.

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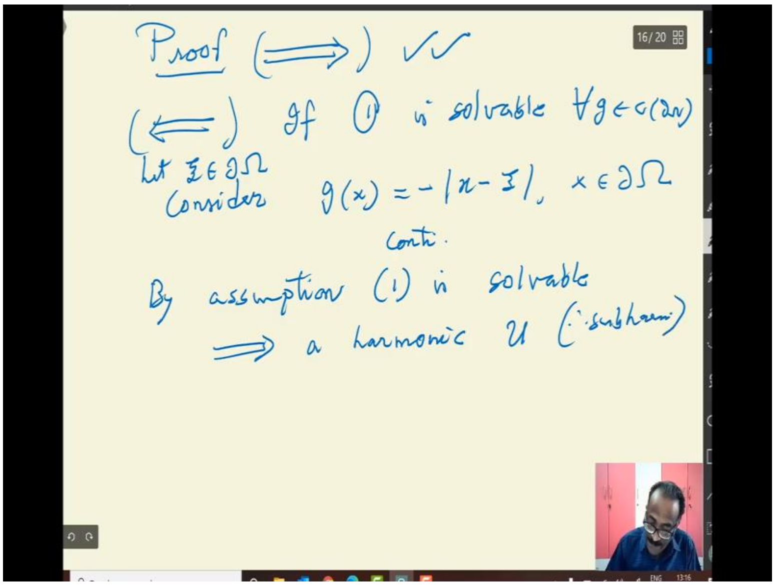

So, proof is trivial now 1 part is proved. So, assuming that omega is a regular domain this is what we have done now conversely if 1 is solvable for all g in continuous functions then take consider this particular g consider g x = - mod x - psi. Let x is in psi is in d omega you want to show that omega is regular. So, you want to show that every point psi in the omega is a regular point.

So, start with to psi and d omega considers this function you see this is a for x in d omega. So, g psi = 0 you know that immediately you get here for x in d omega and then g is a continuous function. Once it is continuous function so by assumption one is solvable by so that gives you a harmonic u hence it is sub harmonic a harmonic u. So, therefore sub harmonic.

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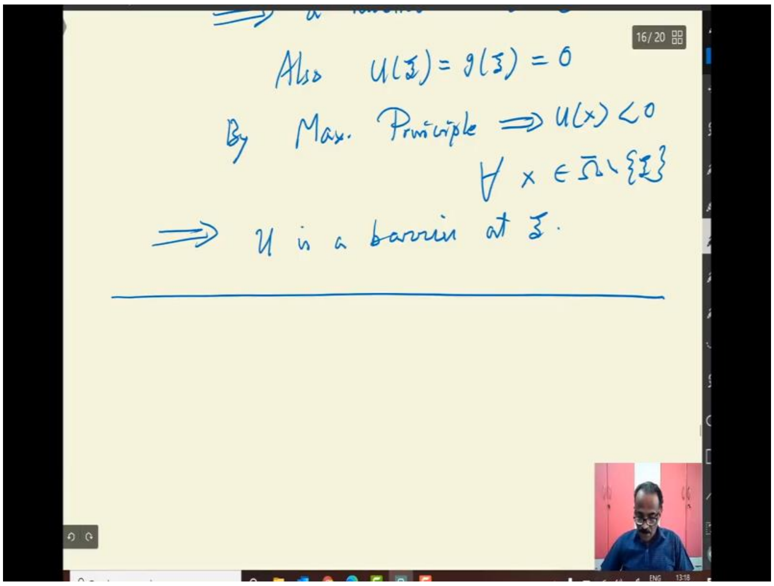

And then of course u psi = g psi in the limiting sense that is equal to 0. So that condition is satisfied sub hormonicity satisfied and by maximum principle which I will not state here maximum principle g is a negative function you see maximum principle you get u is negative for all x in omega bar minus that implies u is a barrier at psi. I thought of giving some examples but it may take a little more time.

So, maybe I will do that in the next class even though we are going to have a Newtonian potential to some remarks or 5 to 10 minutes I requires. So, maybe the selected you will not be able to do it so the but the main idea from next lecture onwards is to study the case of a Newtonian potential because Laplacian u = 0 with u = g is completed except from remarks I will do it here. But then Laplacian u = f with the u = g and which we will start studying from the next class on another 4 or 5 lectures we will do this. So, we will stop here, thank you.