First Course on Partial Differential Equations – II
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Lecture - 03
Laplace – Perron's Method

Good morning, we continue the lectures on Laplace Perron's method regarding the existence of the solutions of treasury boundary value problem. So, what we have introduced in the last class that we have defined basically the sub and super functions or sub harmonic and super harmonic functions for continuous functions, earlier the definitions for the C 2 functions, but of course we using mean value theorem you can define for continuous functions.

But last week or last lecture, what we have defined you for continuous functions using the concept of maximum principles. And later at the end of it, what we have done is that we have defined a class of a sub function and for such functions we are defined. So, let me quickly recall we are defined few properties regarding this continuous function, regarding the harmonic functions, we have seen a maximum principle.

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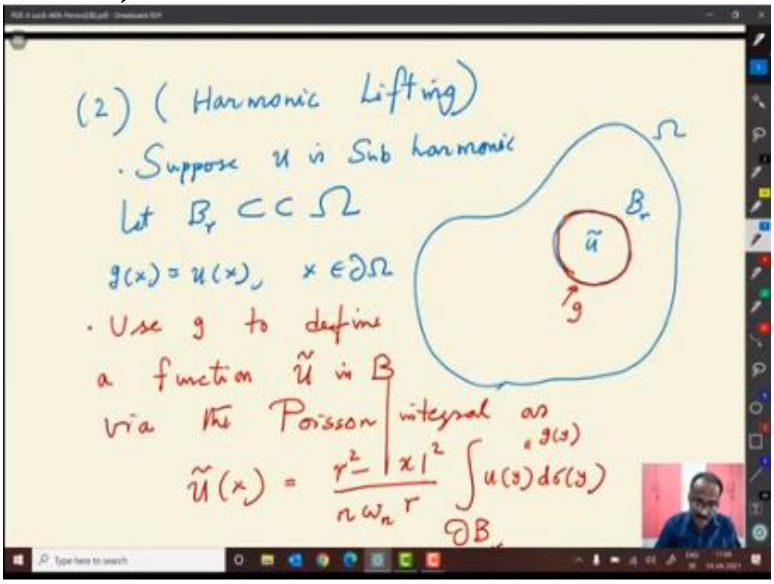

And then we have seen harmonic lifting, because if you have a sub harmonic function for any ball, you can redefine to be as a harmonic function using the Poisson integral. So, if you are given a continuous sub harmonic function and then you can use that continuous function value on the boundary, you can define the harmonic function using the Poisson integral.

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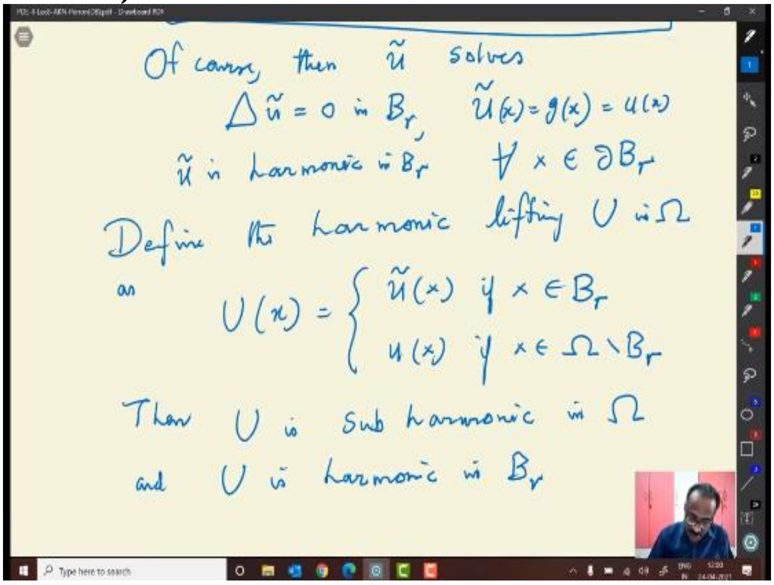

And you define in that ball this harmonic lifting outside it is a sub harmonic function this function as in omega, it will be some harmonic but harmonic inside the ball.

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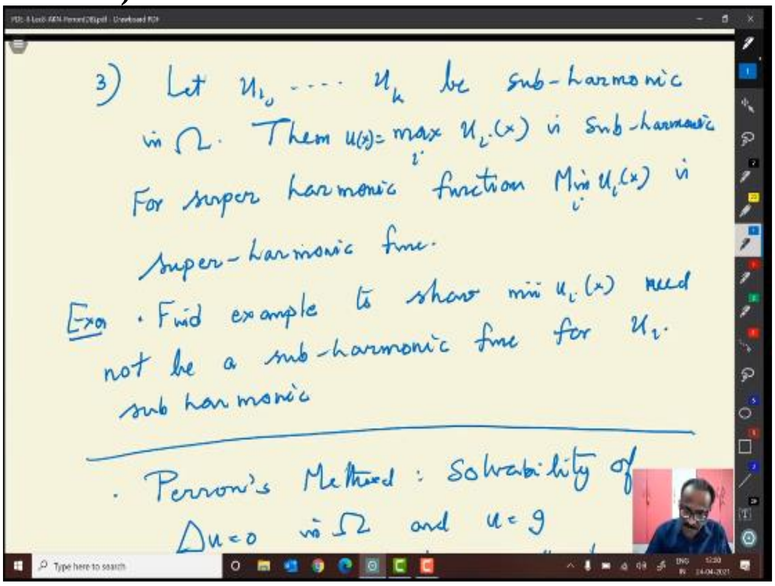

We also seen the another property with all that we will be using now, the maximize in sub harmonic function is a sub harmonic and minimizing super harmonic functions are super harmonic, but not the other way. So, the minimum of a sub harmonic function need not be sub harmonic. And then what we have done is that we at the end of the lecture last class, we are defined what is a sub function a function g is in given a continuous function.

We say that V is defined in omega is a sub function which is relating to this function omega and of course continuous functions of harmonic the important property we have defined is V also less than or equal to g on the omega is what we have defined there.

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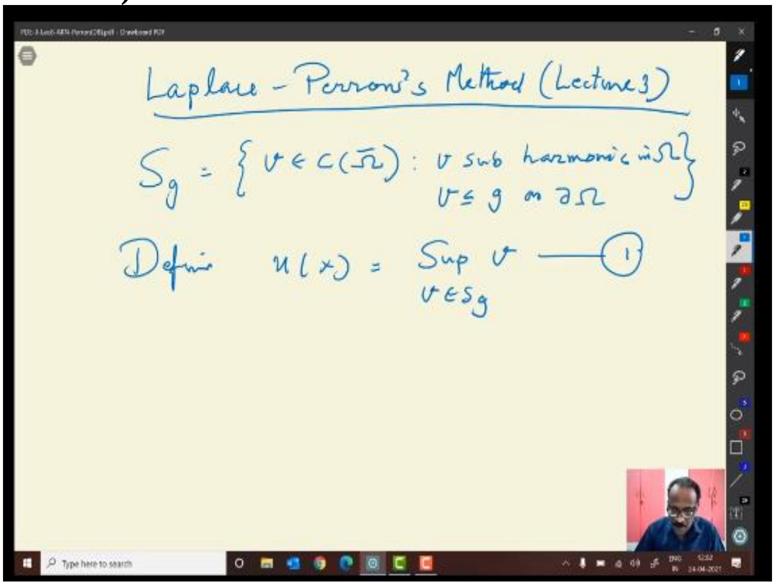

And we denoted S g is so let me recall that one S g is the set of all S g equal to set of all V in c omega bar at the set of continuous functions, which is V sub harmonic and V less than equal to g on the boundary harmonic in omega and V is less than or equal to 0 on the boundary of omega. And using this now, we are going to define what we have already defined a function ux which is what we are going to get the solution. So, let me define once more ux equal to supremum of V and V is the S g.

So, this is my definition of loop and what we are going to show. So, there are this Laplace thing satisfies basically 3 properties, it is a continuous function subharmonic and V less than equal to g and we are maximizing that u and you expect that to be a sub harmonic, what we are maximizing function.

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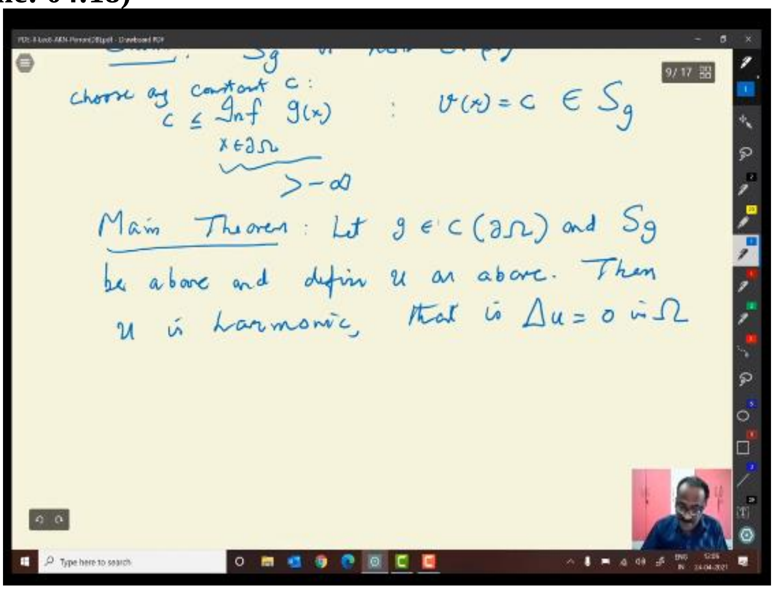

So, you have to prove some important thing this what about whether this S g is the so, the immediate claim, this is a trivial claim S g is non empty, that is very easy to prove S g is nonempty what do you do is that you take infimum of g over x in d omega, g is a continuous function d omega is comeback. This is a bigger than it cannot be minus infinity so you choose any constant c such that c is less than or equal to infinity, this is bigger than minus infinity. So, it cannot be the otherwise infimum can reach infinity so this is a constant function.

So, if you define Vx = c which is a constant function then this belongs to S g. So, any constant function less than this minimum is in the class of S g, so S g is the nonempty function so, that is very important to begin with them. Now, we will state the main theorem this you what do we have defined now, going to be your solution to your destiny harmonic function not that boundary value problem it is going to be that boundary value problem.

But it takes some time to produce prove, the proofs which I am going to present is a bit of technical so, you have to follow it very carefully. So, let me state this thing let g belongs to d omega c of d omega c belongs to c of d omega and S g is the set of all sub functions be as above and define u as again as above. Then u is harmonic that is it solves the problem that is not the boundary value problem proving u = g on the boundary requires little more effort this itself the proof is a bit delicate proofs so you have to be carefully followed this one.

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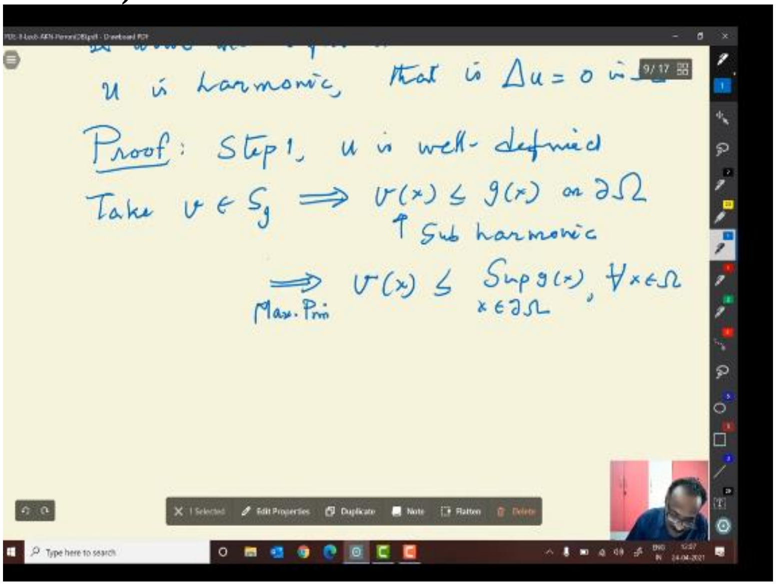

So, let me try to give the proof sometimes even if it is delicate, it is good to learn this proof so, we will prove in step by step, the step one is u is well defined, this is important for us to see that because u is defined as an supremum function, you see u is defined as a supremum.

So, it is possible that it can go to plus infinity here to show that that is not true so, you can define. So, take any V in S g that implies your V of x less than or equal to gx on d omega.

And V is also subharmonic great, because V is sub harmonic so, that implies Vx is less than or equal to supremum of gx where x is in the boundary of omega this is true for all x in omega, this is may not mean value property maximum principle. So, the maximum principle tells you that V is in fact less than or equal to the value c, V is less than or equal to supremum over g. So, any x is a supremum forget it so, in d omega the supremum of that one.

So, that implies that is a finite quantity is always less than so this is V is arbitrary so, taking supremum over V now, taking supremum V. This is for any V this is true taking supremum V you get the ux is less than or equal to supremum over g with x s in d omega, this is a finite quantity which is finite because it says so, therefore you cannot be plus infinity so, that is the first step.

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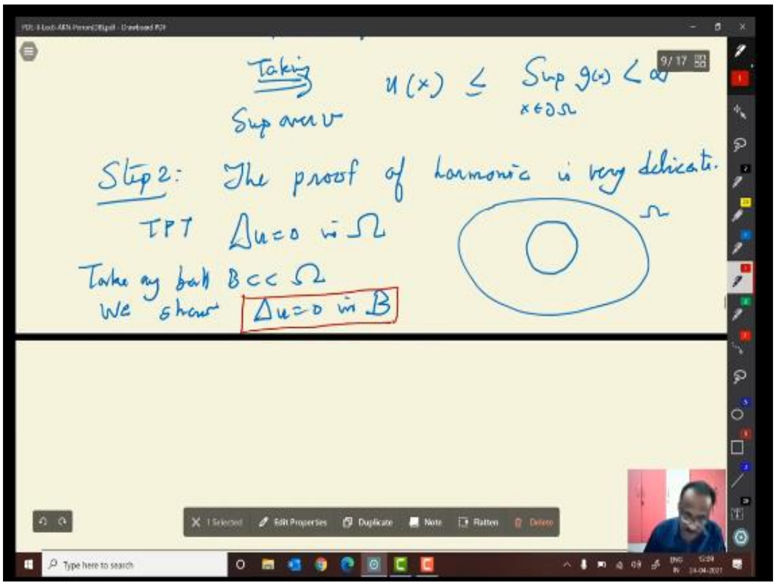

So, we will do the second step, step 2 the proof of harmonic is very delicate so, to prove very carefully. So, what we are going to do is that so, you have a domain omega, you want to show that you are to prove that Laplacian of u = 0 in omega. So, what do we will ? You take any ball inside take any B take any ball B contained in omega and we show that is enough we show Laplacian of u = 0 in B. Since these arbitrary we get Laplacian u = 0. So, we are going to solve Laplacian of u equal to 0 in B that is what you are going to show. So that is what so, we will show this is your B.

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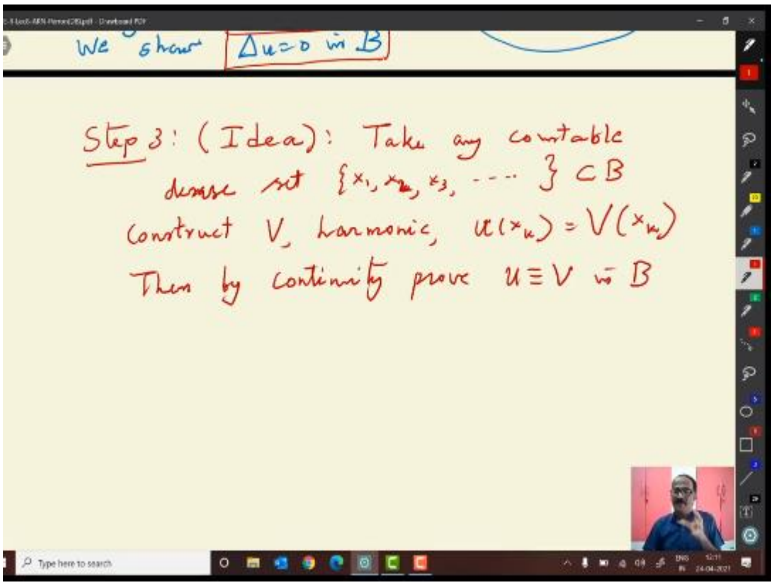

So that is what we are going to do in step 3 slowly maybe step 4 so to show that for any domain in a ball, you take it you show that Laplacian. So, what we will do so, let me give you an idea and then we will prove it, the idea is take a pieces in Euclidean space. So, take any countable dense set. Since it is countable, the dense set, you can write it like this x 2, x 3 like that in B and construct these ideas, construct V which is harmonic, this is what we are going to show harmonic and u of x k = V of x k.

Then by continuity, you do not know even u is continuous, that is a different issue, then by continuity prove u identically equal to V in B. Since V is harmonic in V, V is harmonic in B, you show that you conclude that u is harmonic in B.

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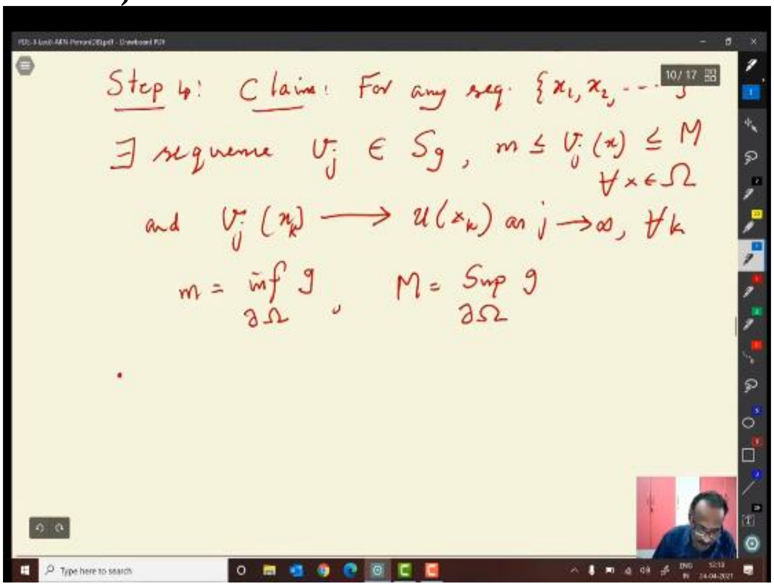

So, let us go to the step that one, so, take any that step so, first the claim is you want to do how to construct this B, so that is step 4, step 4 is a claim. So, we will do a claim now first claim for a sequence, I do not say that it is dense set. So, take for any sequence we need this to be proved not just for dense set. For any sequence x 1, x 2 etcetera. So, this is a sequence x k, there x is a sequence V j in S g.

So, there x is a sequence V j in S g such that m less than or equal to V j of x I will write down what is a less than or equal to m for all x in omega where and V j of x converge x k converges to u of x k as j tends to infinity and this happens for all k. So, all these you are able to construct V j in S g of course, you can always construct fixed k. So, you look at this function you given you look at so for this is for all x. So, once x is fixed, you can always find a sequence in S g which converges to that.

But you want that sequence independent of x basically that is what it shows here. You want that for a given x k, that is what you have been given x k you should be able to find V j k. So, let me before that let me write down what is your m, m is equal to your infimum of g over d omega, and M is equal to your supremum of g over d omega so that is what you are going do it.

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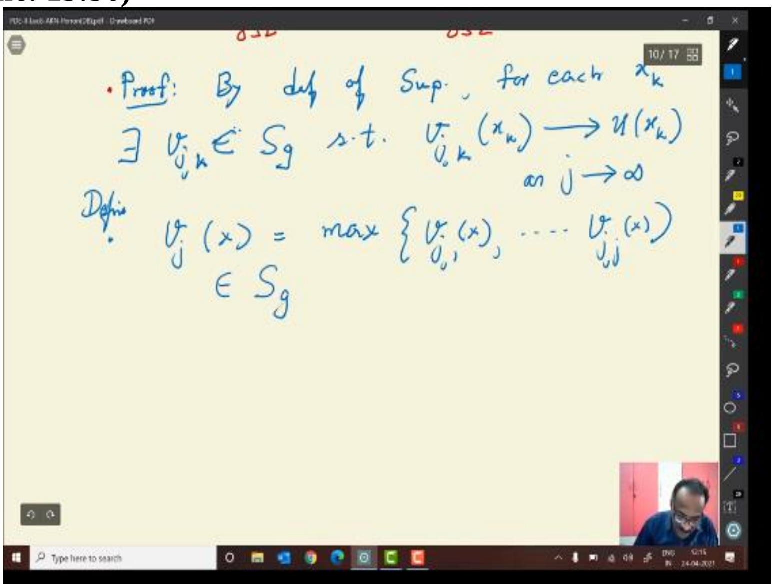

So, first observe so your let me go to this, so you need a proof of this claim so that is what we are going to do a proof by definition of supremum because ux is the supremum for each k for each x k, there x is V j k so constructing a sequence is C c V j k of x or V j k belongs to S g you can find V j k belongs to S g such that V j k of x k converges to you it this is just infimum. So, u of x k as j tends to infinity.

So, obtaining a sequence which depends on j in k is not that difficult, and what is more difficulties that thing, so you do not exactly diagonalization precede your, so but we will do something slightly different. So, you define V j this is a V j k is in S g so, it all you get is less than or equal to get all that properties for V j k you get it. So, we will define V j of x is equal to a maximum of V j 1 of x you do a maximizing because while maximizing the sub harmonicity will not change it. So, you will get it V j of j of x and then this will be belongs to S g by this maximizing thing and because each V j i belongs to S g.

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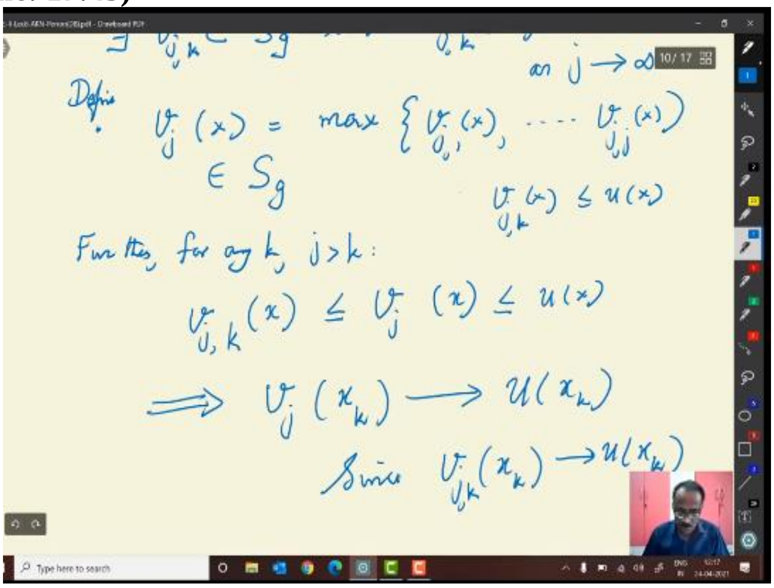

Further so, you see immediately further if for any k and if you choose j large enough j greater because you are trying to see what is the limit of V j? So, j greater than k but you will get it V j of x V j is the maximum. So, V j of x less than or equal to V j k of x that is of course less than or equal to ux, these are all less than or equal V j is also is the supremum from here also you get it all these V j k.

j because it is a supremum and you are in supremum you have this V j k is always less than or equal to ux, because ux is defined as the supremum of such quantities. So, this you have it ux. So, now, V j k of x converges to u of x k that immediately implies your V j of x k also converges to u of x k since V j of x k converges to V j k of x k converges to u x k, you are already known that since this converges and it is bounded with that one.

So, I have written in a wrong way V j is the maximum, so the case should be here as I wrote here and V j of x so, V j k of x converges since this is convergence, so this sandwich functions also converges to that one. And so, that is hence the claim so, you have proved that, so, this is the claim. So, you keep this claim now for any sequence so, you take your each.

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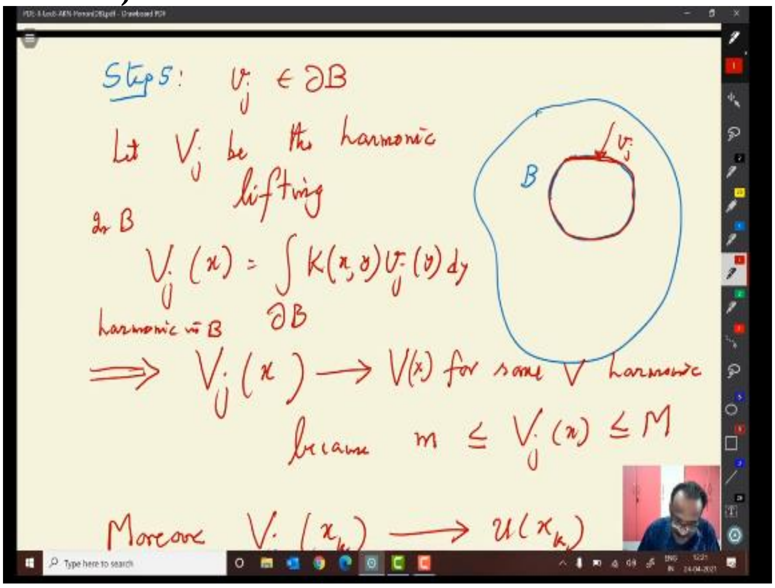

Now, you use the harmonic lifting next step, step 5 you take that step 5 now you look at the ball here, so you have your domain here and you have your ball here this is your B any ball here, so look at this boundary values. So, you have V j everywhere for you take your ball here. So, you have your ball here, so use this value here at V j and when you have for you consider V j constructed to the boundary of the ball and let V j be the harmonic lifting.

So, you already seen that when you V j is given using this you can define the V j inside so and you can define V j by the harmonic lifting because you know that how do you define this V j of x in B you define V j of x is equal to the boundary of B you are the K of x y and V j of y and which also gives you immediately you can because x cases sequencing V j is everything. So, that immediately also implies your V j of x k you will see converges to this integral V j x k converges to this one.

And you will get immediately and V j is bounder converges to V for some V. So, V j not x k V of x since for some V harmonic because V j is harmonic in V so understand this harmonic lifting, so, you have your harmonic in B. So, you are working only in B, so, you have a harmonic lifting not anywhere else and V j x harmonic for some V harmonic because V j's are bounder because this boundary is always preserving an all that V j of x less than equal to M. So, you have your boundary here. So, you have your harmonic lifting immediately.

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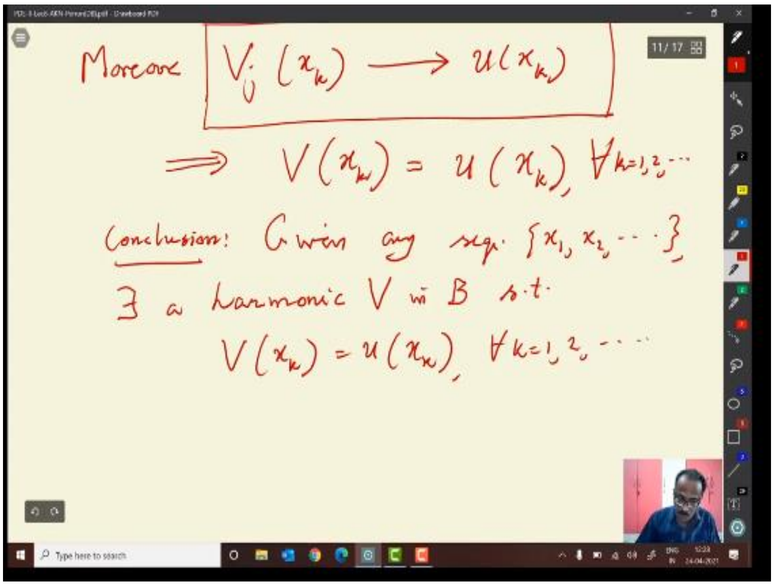

So, in particular, so, and you can show that to be V of x k, so, because but V j of x k is converges to, you have a moreover, you have this also, you have V j of x k converge to u of x k that follow second from the definition, because we do have x k is k of x k y V j of y that will converges that x k y V j of y that is nothing but your u of x you get immediately but so you have because you are V j of x k converges to u of x k.

So, you see so you have because of this convergence V j of x k you will get your V j of x k also converges to u of x k just check if you are not convinced please check. So, that implies your V of x k, so this is the aim. So, V of x k so u of x for all k = 1, 2 etcetera. So, what do we conclude? So, what is the conclusion? Conclusion given any sequence x 1 etcetera x k, x 2 etcetera x k there x is a harmonic in B such that V of x k is = u of x k this happens for called k = 1, 2 etcetera.

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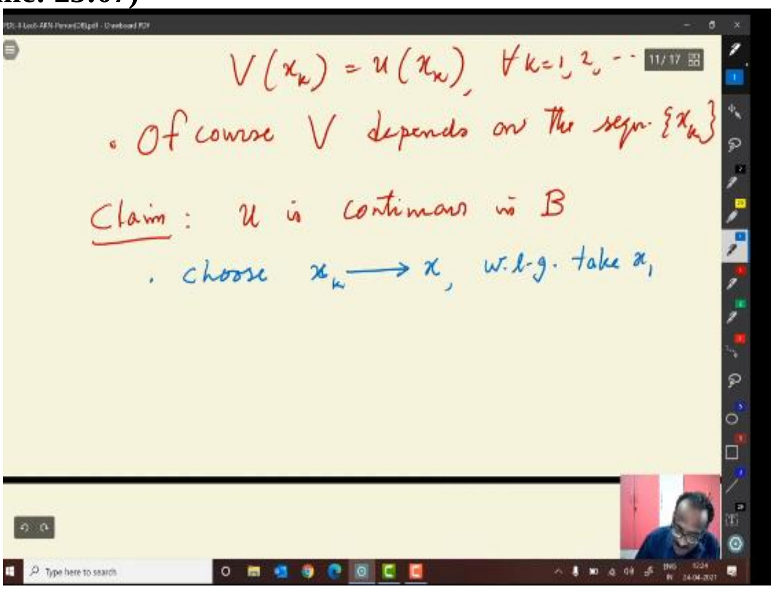

The important point is that of course V depends on the sequence x k, so immediately you can of course V is harmonic, but we only coincides with sequence x k. So, immediately you can conclude about the harmonic seeking of u immediately. So, what we need an important thing is what we call it continue of u. So, the next big claim u is continuous in B that is all we are working with only in B.

Let me give the proof that so choose, so these are the choose x k you want to show u is continuous. So, for any converging sequence, you are to show that u of x k converges to u of x. So, without loss of generality, after taking this without loss of generality, take x 1 = x that will not change the convergence x 1 = x. So, if you take any sequence x k converge to x you to replace that x the first element by x or you add the first element as x that will not change the convergence at all.

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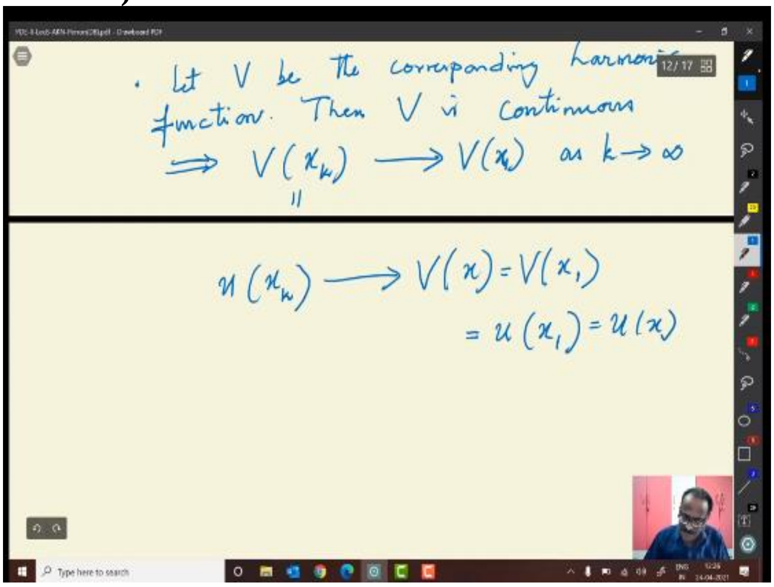

So, corresponding to x k let V be the corresponding harmonic function. So, every sequence you have seen a harmonic function satisfies this property, let V be the corresponding harmonic function, this is a small trick harmonic function. Then V is continuous that, you know because it is a harmonic function V is continuous that implies V of x k converges to V of x as k you know as k tends to infinity because it is with respect to k.

Now, V of x k u x k so, this is nothing but u x k so, u x k V of x k V of x so, u of x k converges to V of x but V of x what is x = x 1 we have taken, so V of x is nothing but to V of x 1. But V of x 1 is nothing but the u of x 1 by the construction of V. But then now, again replace x / x 1 / x this is u of x, it is a very small trick plane. So, you implies that implies your u is continuous. So, now, we are proof is almost done this is one important thing proving so, after the lifting using the lifting we proved u is continuous.

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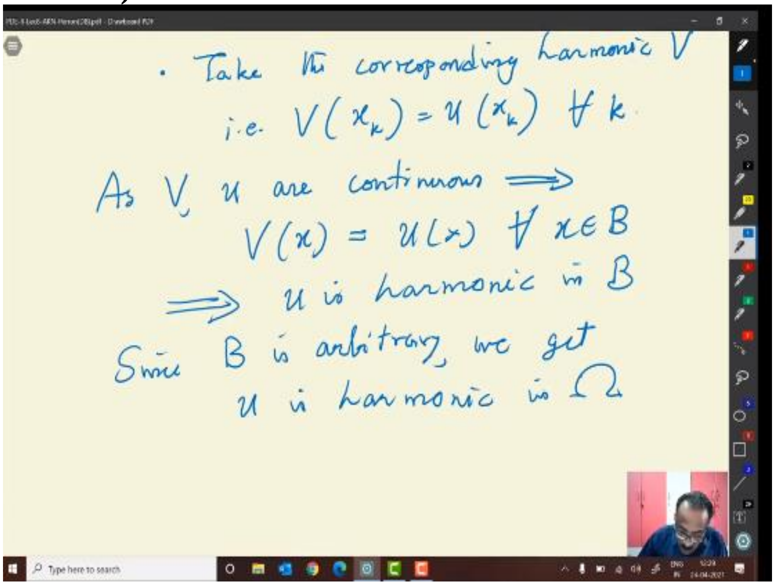

Now, the final part now take x k sequence which is dense in B which you can do it you can always take because B is a subset of Rn okay. And you know that you can choose dense set like are you have the rational which is countable, which is dense set. So, even in Rn, you can always choose the density which is countable. So, sequence which is dense in B. For this now take the corresponding harmonic V what it was that is V of x k = u of x k for all x k for all k.

Now u is continuous earlier we could not conclude because there is not continuity. Now as V, u are continuous conclude that V of x = u of x for all x in B because it is dense you said very cute nice proof that bit in a delicate way we are proving that implies u is harmonic in B. Since B is arbitrary we get u is harmonic in omega. So, that proves in a sense the solution existence of so that proves the theory completes the theorem existence of that.

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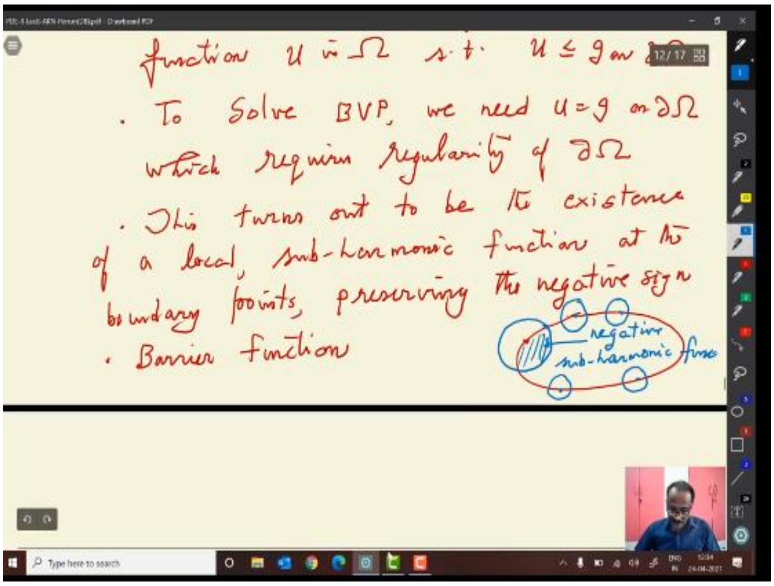

So, that proves existence of harmonic function u in omega such that you have only this one u less than or equal to g and d omega, you are not proved that u = g that requires some regularity to see to solve boundary value problem we need u = g on d omega which requires regularity in general you may not be able to show that you will see in fact you will see some necessary and sufficient condition. So; that of the regularity for solubility for all regularity of d omega. What we are going to introduce this actually.

So, this turns out to be the existence of a local subharmonic function at the boundary local subharmonic function which reduces everything to that minimum thing local, sub harmonic function at the boundary points preserving the negative sign that is very important preserving the negative sign such a thing is called Barrier function. So, we prove the demand that Barrier function.

So, basically you have a domain here and then every point you are looking for some neighbourhood on these you will look for a negative, this may not exist all the time. So, that is going to be the assumption negative subharmonic function and this should happen for every point in some neighbourhood and such things are called Barriers. So, you need some negative preserving harmonic negative side preserving subharmonic functions something and tightly defined to be as the regularity of this function.

So, we will proceed this definition of Barrier function and then define the regularity of d omega and using that regularity we will show that u = g on omega if every point satisfies that regularity assumption. So, I will stop here and we will continue as in the next lecture. Thank you.