First Course on Partial Differential Equations - II
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Lecture - 02
Laplace Perron's Method

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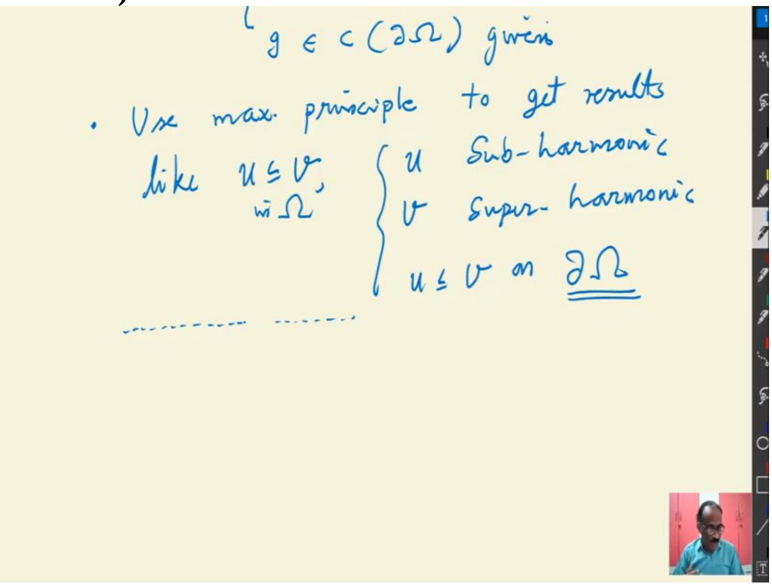

So, we will slowly introduce the Perrons method. So we will develop the machinery now required for that. So, do know that it is easy to construct in the Perrons method, easy to construct sub and super solutions, easy to construct sub solutions and super solutions, I will tell you that sub solutions and super solutions so, that is the one thing. So, you say again, we will start with omega in R n. So, assume the bounded say we do that one bounded and then your problem is - Laplacian u = 0.

So, you do not even have to use this Laplace u = 0 in omega and u = g on d omega. So, the given assumption is basically g is a continuous function on the boundary. That is what we are taking it. So, this is the problem we want to construct g is d omega given. So, what we will be doing is that, it will be maximum principle to get the easy results that it is a basic idea about some of the motivations about the harmonic functions.

If you use maximum principle to get the results like we will do that result like u less than or equal to v for u sub harmonic v super harmonic, we will see these things now, v super harmonic and u less than or equal to v on d omega, that is what the whole idea. So, if you have a, so, there is basically an arrangement, so, if you do this one all your sub harmonic functions u less than or equal to v on the boundary, you see, and you have your thing, then you will get u less than or equal to v.

So, you have a all collection of sub harmonic functions basically on one side and you have super harmonic functions basically on the other side. So, the idea the first step to take the maximum of all the sub harmonic functions and the minimum of all super harmonic functions and u expect that u = v to have that supremum an infimum essentially coinciding. So, we will not do exactly the same thing.

So, the basic idea is to look for all sub harmonic functions, satisfying conditions something like u less than or equal to g, which is the boundary value and then expects a maximising thing will become a harmonic function not only a sub harmonic function, it becomes the harmonic function satisfying the boundary conditions.

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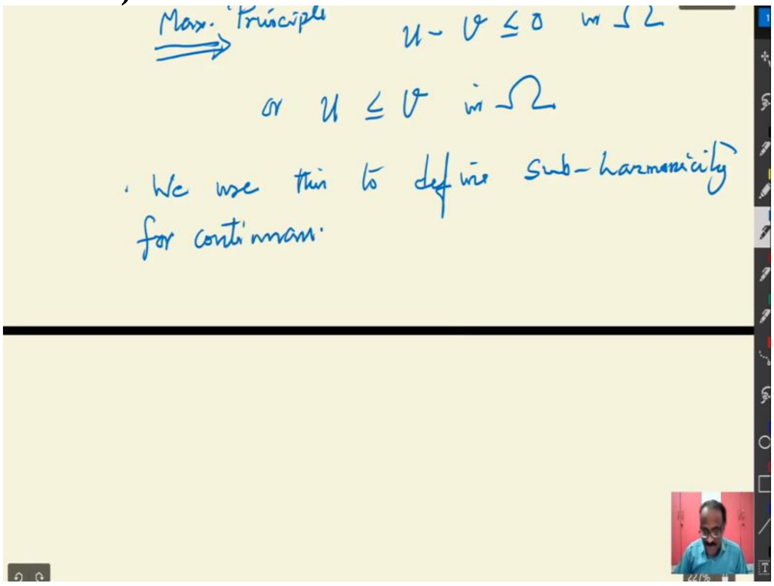

So, the first step in this thing define in this direction define sub and super harmonic functions, sub and super harmonic functions for continuous function, you already know that by our mean value theorem we can do it, but we are going to do it is defining via maximum principles now, motivation by we will motivate you how can you define a sub and super harmonic functions for continuous functions via.

It will be it will coincide with mean value definition via super harmonic functions for continuous functions that what we are immediately going to do continuous functions via maximum principle this is what we are going to because these are all some of the defining properties of harmonicity maximum principal. So, let me give you a motivating argument and then we will use it as a definition motivation.

Suppose, u is in C 2 of omega twice continue, it is a smooth function we want to define for continuous function, but you start with the that is what we will always do for smooth functions and but continuous have to be in sub-harmonic that is classical way because it is smoothing this means that Laplacian of u less than or equal to 0 in omega because it is twice differentiable, you can do that one.

Now if v is harmonic in omega, both v is harmonic in omega v also belongs to the same space C 2 of omega intersection c of omega bar is harmonic in omega and u less than or equal to v on the boundary. So, u is a sub harmonic function classical way and v said sub harmonic function u and v is a harmonic function which satisfies this inequality it satisfies this inequality then what can you tell about it by a maximum principle.

So, that implies immediately what about u - v u s sub harmonic you think if you take Laplacian, Laplacian of v = 0 because harmonic this implies that is u - v is sub harmonic and u - v less than or equal to 0 on d omega. So, the maximum principle immediately implies you, you apply a maximum principle because the maximum cannot be on the boundary implies u - v less than or equal to 0 in omega.

That is what we have discussed recall the result or u less than or equal to v in omega, see you get it. So, you see, so, if you have a sub harmonic function and the harmonic function satisfying this inequality, immediately you have the we use this definition this has the dividing property we use this to define sub harmonicity for continuous functions. So, this similarly for super harmonic function.

So, you have your definition a function u belongs to C goes up to omega bar is said to be sub harmonic. A new definition is the same as the earlier definition. If there is one said to be sub harmonic if the following hold following conditions holds, what is that for any ball, so, you are asking for any ball. So, you have a domain here omega, you have a take any ball is not a fixed ball for any ball B of course, it should be compactly contains should not coincide ball B thing.

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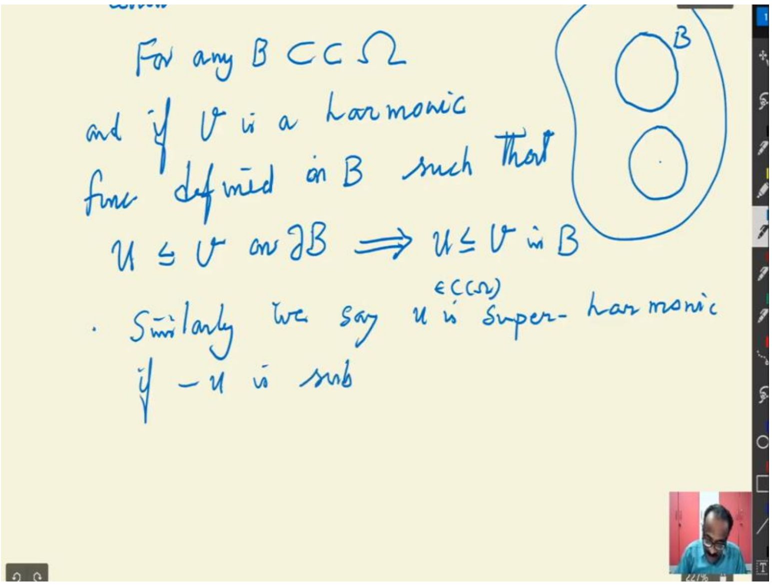

Contained in omega and if v is a harmonic function in the normal way, smooth function, v is a harmonic function, defined on v defined on omega define in omega such that u less than or equal to v on d omega implies, so basically we are demanding that maximum principle u less than or equal to the maximum principle is basically satisfied, but we write it in this way in omega. You see, so, this is what is x and v if u is sub harmonic in the classical way, you know, this result is true.

So, you are demanding that result because you do not need now the differentiability of u to define these definitions is there is one correction v is a harmonic function defined on v this should happen for B inside in B. So, if you take another ball you should satisfy because on the boundary of ball you have this in fact you do not have to even assume the continuity up to the boundary.

You do not need even that one because you are working out inside the thick that is more than enough, but eventually not our problem you will have continuity up to the boundary for this thing. You do not need a similar definition for super harmonic. Similarly, if we say u is super harmonic, if u belongs to C omega so you say that we generalise u super harmonic if - u is sub harmonic equally in others inequality.

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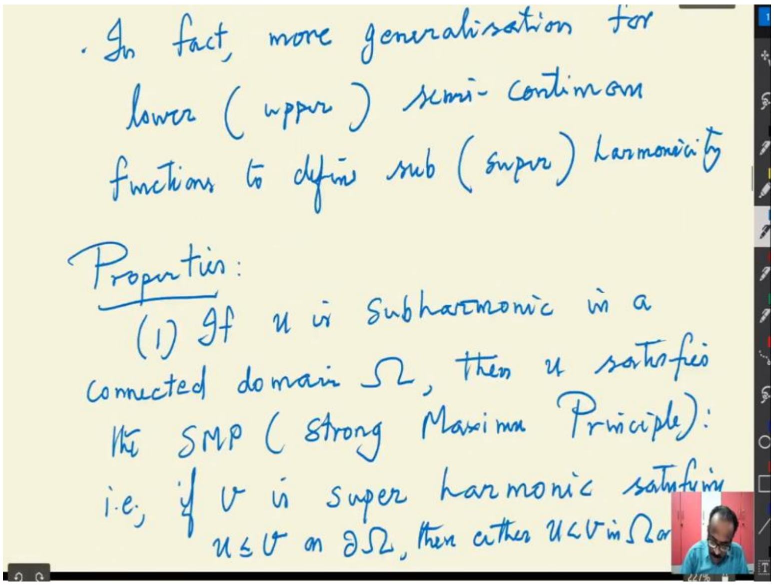

So, you can say u is sub harmonic and u belongs to so, you can even define harmonicity for a continuous function u belongs to is said to be harmonic, harmonic if both u, n - u are sub harmonic so you see you are able to define harmonicity for continuous functions. This definition actually is due to a F Rieze this definition is due to F Rieze and coincides the definition via coincides the definition why I am you can also define via mean value property. So, all definitions are coincides the definition via mean value property.

So, you will see so you can define harmonicity in various ways you can use the mean value property or use the maximum principal or if it is smooth. You have the usual definition or point weights definition of harmonicity. In fact there are more generalisations in fact more generalisations are available. More generalisations for lower semi continuous lower or upper semi continuous functions have been semi continuous functions.

Lower is used for sub harmonic for a function generalisations for lower semi-continuous definitions to define sub-harmonicity for upper semi-continuous you define super harmonicity implies you to harmonicity so you can define it is all useful you can define non define these notions for much more general than the continuous function.

So, what before finishing here today the rest of the thing we want to define few of the properties which are very important. So, we start with some properties and you just go and verify these properties we will be using again and again. So, that one of the property you can check it as an exercise the properties are or almost reveal while explain I will say that the one of the first property is which I am going to explain this immediate property.

If u is connected release u is sub harmonic in a connector bounded a domain important property and you have to remember all you have to understand these properties to understand the proof of the Perrons method connected the domain omega then u already defined to the definition is something like a strong maximum principle then u satisfies the strong maximum principle.

Once u start defines no its a new definition you already know that it is sub-harmonic you have the definition even with this generalisation you have the definition strong maximum principle the SMP strong maximum principle that is what is that means that is if u or v we wanted if v is super harmonic everything we are defining that way if u is super harmonic so we are defined the definition for harmonic functions, but the maximum principle you will get it for the super harmonic function.

Satisfying u less than or equal to v on d omega then either u less than v in omega that is interior or u identically v in omega this is the same thing what you are telling it cannot have an interior maximum kind of thing. So if u is super harmonic you know that u - v sub harmonic so it says that u - v sub-harmonic and you might that is all we needed actually u - v sub harmonic and u - v less than or equal to 0 and 0 is on the boundary.

So, you can apply that immediately say that u - v is has to be there by the definition maximum principle it cannot have an interior maximum exactly like that earlier we have done it cannot have an interior maximum is the similar proof if you want you can project it cannot have that way either it will be used to u less than v in omega or u will be identically v in omega that is the first property.

So that is the second property is more interesting which we are going to use it immediately. All the properties are which you are going to define will be using it here. That is the very thing this is what is called a harmonic lifting you to understand this very well harmonic lifting. So, you are given us a harmonic function suppose u is a harmonic sub-harmonic and you have your domain here and u omega and you have a ball here B.

So, you can restrict your u to B and then you can lift that part of u as a harmonic function using what is called the Poisson integral. So, look at here of course sub-harmonic and you consider a ball of radius let a ball of radius B are contain the omega then you define g x to be the boundary values of u, because u is a continuous function. So, the boundary value exists you can define u g x = u x for x is on the boundary of B this is on the boundary B. So, I am restricting my u g is here u restrict here and cancel this is where I differ in my g. So I define my g like this.

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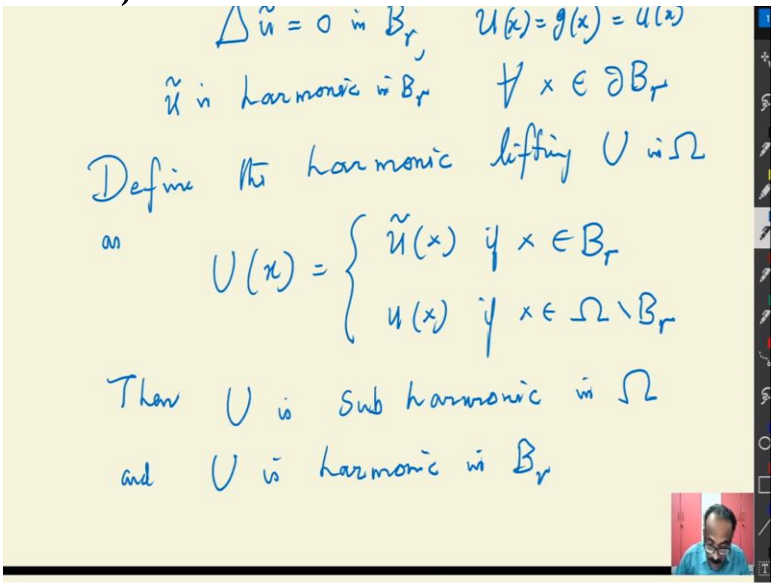

Then use g in to define a function u tilde which you can do it function u tilde in B you will see via the Poisson integral you know that so the Poisson integral that actually solves the Laplace equation if you know the boundary integral using the boundary value. So, any ball that is Poisson integral we have solved the problem Laplacian u = 0 in any ball and u = g on the boundary. So, you are using the only boundary values of u you can define u inside which will be your harmonic which will be a harmonic function.

What is that definition via Poisson integral as u tilde of x = r square - modulus of x square / n omega n r integral over the boundary of r u of y g is u here this is your g y these this is only boundary values I am using there is nothing but they are integral if I define denote K x y u of y so you have your u of y the d sigma of y this is over the boundary of the ball. So, this is your u tilde of x for all x in B r inside. So, you see, so, you have this defined u.

So, you are defined u tiled r here. So, you define u, u is defined everywhere in omega but utilise define only inside and u tilde the coincide with u on the boundary. And of course, by definition of course then u tilde source this is why do we have recall the result in fact studied in previous you u tilde solves - u tilde = 0 in B r and u tilde of y = u of g of y this is equal to u of y of x let me use x = u this is in B r u tilde of a x.

This is for mod x in boundary of this ball. That is what it is you have. So, you have a harmonic function. So, using the value of that u on the boundary, you can have your u tilde and so that is a u tilde harmonic. So that is why u tilde is harmonic in B r. So, with that we have defined now, the harmonic lifting define capital U in omega as so you can do that one immediately and what is the thing you want to define U of x = u tilde of x.

If x is in B r ball of radius r and same u of x is a continuous function, because on the boundary of B r you have the same value if x is in omega – B r. Then this you can small thing, but you can prove it then u is of course, you keep your harmonicity only at the boundary you have to verify that it is indeed sub harmonic. So, you use that story and because you have u tilde that is harmonic, so, U is sub harmonic.

So, the sub harmonicity not lost by defining the sub harmonic in omega and U is harmonic in the ball harmonic in B r and it is the same value outside of that. So, whenever your harmonic we will function is given, you can take any ball you like it on that bar you can change that function u to make it a harmonic function by redefining it without changing outside and without losing the total sub harmonicity and that is what is called the harmonic that is what we call it the harmonic lifting.

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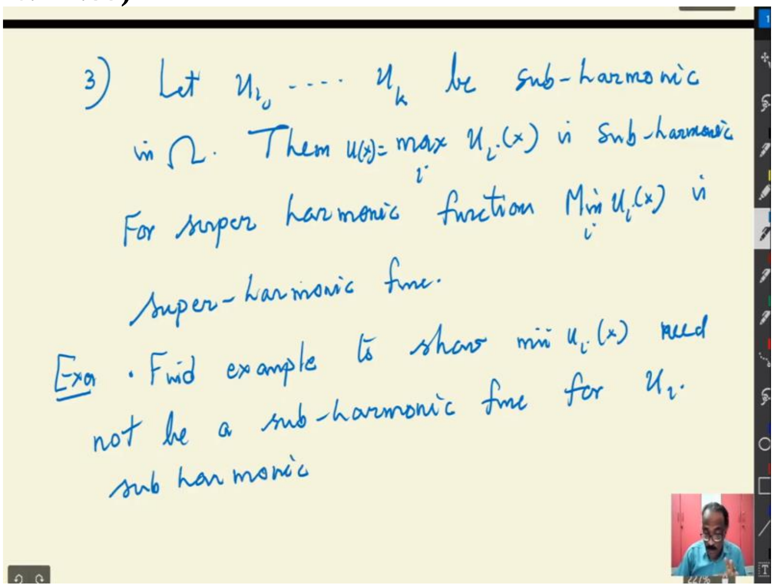

That one more property which I want to define, which will be using it here is what is called the another property the maximising property of harmonicity. So let u 1 and this is also all these are small things, which you can actually prove it B sub harmonic in omega then maximum you can define the maximum of u i, you call it the u x equal to a maximum of u x i of u i of x is sub harmonic.

So, you see the maximizer the preserves the sub harmonicity but and minimiser for super harmonics functions minimum of u i of x is super harmonic u say a harmonic function both maximising and minimising function. So, the other things need not be true you can find example harmonic functions the maximum of sub minimum or sub harmonic functions need not be sub harmonic find examples.

So, this is an exercise you can think of exercise find examples to show minimum u i of x need not be minimum you need not be need not be a sub harmonic function for u i sub harmonic. Similarly for super sub harmonic functions by minimising, so it is not immediate. So, minimizer of sub harmonic functions need not minimising function need not be sub harmonic functions for sub thing. Similarly maximum of super harmonic functions need not be super harmonic functions. So, we will more or less stop here this lecture so let me set the stage for these things.

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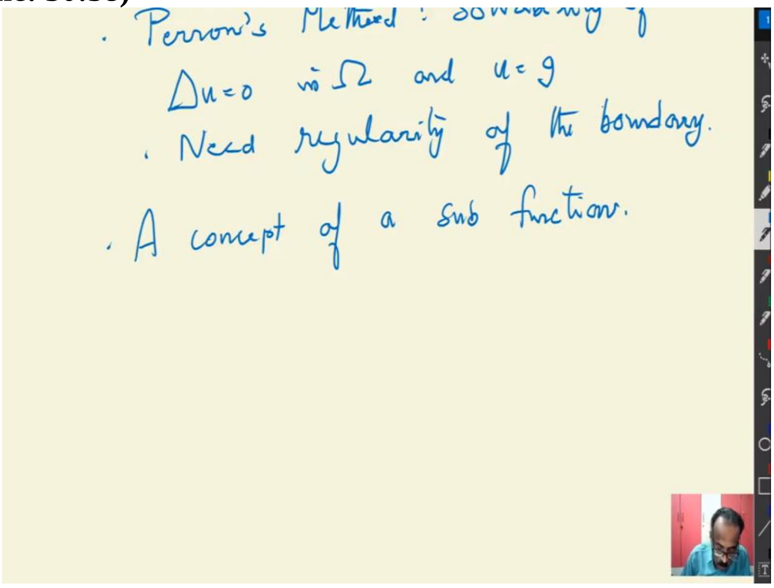

So, to prove this now, what do you want to prove is the Perrons method now, we are in the setup Perrons method which we will start next week, next class Perrons method solvability of Laplacian u = 0 in omega and u = g as I said you are we need regularity of the boundary we need regularity of the boundary. So, what do we are going to define this class the concept of sub function concept of a sub function? So, what we are going to do is that so you are going to so probably I may give the definition of the sub function and maybe recall again so let me start with the definition.

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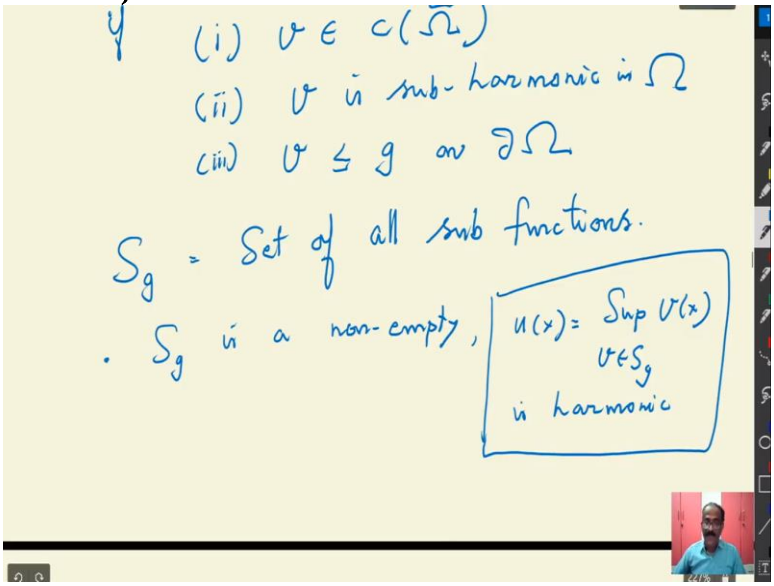

So, I can start immediately something more next class definition sub function and we will do the more things later let g belongs to c of omega is c of boundary of omega c this is always on the boundary of omega is a continuous function and we say v defined on omega defined in omega with we say v defined in omega is sub function if it satisfies the following properties. One we can do continuous v belongs to the continuous function up to the boundary and 2 your thing v is sub harmonic in omega sub harmonic.

So, it is a continuous function continuous sub harmonic function in omega and the third property is the additional property makes it a otherwise it is a standard sub harmonic function. So, sub harmonic itself for the v defined for continuous function. So, the one of 2 properties are basically the sub harmonic function third one is v less than or equal to g on d omega. So, the extracting we are defined is a so the sub function is a sub continuous sub harmonic functions whose boundary values is bounded by g.

So, it is a sub function is we say that related to g you are to write that related to g and we define is S g the set of all. Sub functions is the set of all functions so what we will next class we will prove S g is a non-empty set and eventually what we are going to state the theorem by maximising that you will get the harmonic function. So, we are going to eventually define a function u of x, what is called the supremum and of v of x where v is in S g is harmonic of course showing that it will satisfy the boundary condition.

So, the proving that it is harmonic is a bit delicate proof. So, we will do this for in the next class the main aim is to show that the maximizer over all sub functions that you get the solution of your boundary above not yet the boundary value problem at least you get that it is a harmonic function. So, we will stop here and then we will continue in the next class.